AMSC 466: Midterm 2

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Read carefully the following instructions:

- Write your name & student ID on the exam book and sign it.
- You may not use any books, notes, or calculators.
- Solve all problems. Answer all problems after carefully reading them. Start every problem on a new page.
- Show all your work and explain everything you write.
- Exam time: 75 minutes
- Good luck!
1. (a) **(10 points).** Define a spline of degree \( k \) on \([a, b]\). Prove that if \( S(x) \) is a spline of degree \( k \) on \([a, b]\) then \( S'(x) \) is a spline of degree \( k - 1 \) on \([a, b]\).

(b) **(10 points).** Determine the coefficients \( a, b, c, d \) such that

\[
S(x) = \begin{cases} 
S_0(x), & 0 \leq x \leq 1, \\
S_1(x), & 1 \leq x \leq 2,
\end{cases}
\]

\[
= \begin{cases} 
x^2 + x^3, & 0 \leq x \leq 1, \\
a + bx + cx^2 + dx^3, & 1 \leq x \leq 2,
\end{cases}
\]

is a cubic spline that satisfies \( S'''_1(x) = 12 \).

2. (a) **(10 points).** Use \( f(x - 2h), f(x), f(x + 4h) \) to write an approximation for \( f''(x) \). What is the order of this approximation?

(b) **(10 points).** What is the most accurate approximation you can write for \( f'(x) \) using the same three values, \( f(x - 2h), f(x), f(x + 4h) \)? What is the order of this approximation?

3. (a) **(6 points).** Find the first two orthogonal polynomials, \( P_0(x), P_1(x) \) with respect to the weight \( w(x) = \sqrt{x} \) on the interval \([0, 1]\). Do not normalize them.

(b) **(4 points).** Normalize \( P_0(x) \).

(c) **(6 points).** Let \( Q_1^*(x) = a_0 P_0(x) + a_1 P_1(x) \). What should \( a_0, a_1 \) satisfy so that \( Q_1^*(x) \) minimizes

\[
\int_0^1 (x - Q_1(x))^2 \sqrt{x} \, dx.
\]

over all linear polynomials \( Q_1(x) \). Express \( a_0 \) and \( a_1 \) as integrals. Do not explicitly compute these integrals quite yet.

(d) **(4 points).** Find \( a_0 \).