AMSC 466: Midterm 2 - SOLUTIONS

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Read carefully the following instructions:

- Write your name & student ID on the exam book and sign it.
- You may not use any books, notes, or calculators.
- Solve all problems. Answer all problems after carefully reading them. Start every problem on a new page.
- Show all your work and explain everything you write.
- Exam time: 75 minutes
- Good luck!

1. (a) **(10 points).** Define a spline of degree k on $[a, b]$. Prove that if $S(x)$ is a spline of degree k on $[a, b]$ then $S'(x)$ is a spline of degree $k-1$ on $[a, b]$.

Solution:

A spline of degree k on $[a, b]$ with knots $a < t_0 < t_1 < \ldots t_n < b$ is a piecewise polynomial function, $S(x)$, with the following properties:

i. $S(x)$ is a polynomial of degree k on each interval $[t_i, t_{i+1})$, $i = 0, \ldots, n-1$.

ii. The function and its derivatives up to order $k-1$ are continuous in $[a, b]$.

The derivative of a Spline, $S'(x)$, will therefore be a polynomial of degree $k-1$ on each interval $[t_i, t_{i+1}]$. It will also have $k-2$ continuous derivatives in $[a, b]$. Hence $S'(x)$ is a spline of degree $k-1$.

(b) **(10 points).** Determine the coefficients a, b, c, d such that

$$S(x) = \begin{cases} S_0(x), & 0 \le x \le 1, \\ S_1(x), & 1 \le x \le 2, \end{cases} = \begin{cases} x^2 + x^3, & 0 \le x \le 1, \\ a + bx + cx^2 + dx^3, & 1 \le x \le 2, \end{cases}$$

is a cubic spline that satisfies $S_1'''(x) = 12$.

Solution: The condition $S_1'''(x) = 12$ implies that $d = 2$. The continuity of S, S', and S'' at $x = 1$ implies:

$$\begin{cases} a + b + c + d = 2 \\ b + 2c + 3d = 5 \\ 2c + 6d = 8 \end{cases}$$

The solution of this linear system is: $c = -2, b = 3$, and $a = -1$.

2. (a) **(10 points).** Use $f(x-2h), f(x), f(x+4h)$ to write an approximation for $f''(x)$. What is the order of this approximation?

Solution:

We write the Taylor expansions of the above quantities, centered at x:

$$f(x+4h) = f(x) + 4hf'(x) + \frac{1}{2}(4h)^2 f''(x) + \frac{1}{6}(4h)^3 f'''(\xi_1).$$

$$f(x-2h) = f(x) - 2hf'(x) + \frac{1}{2}(2h)^2 f''(x) - \frac{1}{6}(2h)^3 f'''(\xi_2).$$

We now consider a linear combination:

$$Af(x+4h) + Bf(x) + Cf(x-2h).$$

To approximated the second derivative, $f''(x)$, we require

$$A + B + C = 0,$$
$$4hA - 2hC = 0,$$
$$\frac{1}{2}(4h)^2 A + \frac{1}{2}(2h)^2 C = 1.$$

The solution of this system is

$$A = \frac{1}{12h^2}, \quad B = -\frac{1}{4h^2}, \quad C = \frac{1}{6h^2}.$$

The approximation is of order $O(h)$, since the error term is:

$$A\frac{1}{6}(4h)^3 f'''(\xi_1) + C\frac{1}{6}(2h)^3 f'''(\xi_2) =$$
$$= \frac{1}{12h^2}\frac{1}{6}(4h)^3 f'''(\xi_1) - \frac{1}{6h^2}\frac{1}{6}(2h)^3 f'''(\xi_2) = O(h).$$

(b) **(10 points).** What is the most accurate approximation you can write for $f'(x)$ using the same three values, $f(x-2h), f(x), f(x+4h)$? What is the order of this approximation?

Solution:

Once again we consider a linear combination of the form

$$Af(x+4h) + Bf(x) + Cf(x-2h).$$

Only this time, we are asked to approximate the first derivative $f'(x)$. Hence, we require

$$A + B + C = 0,$$
$$4hA - 2hC = 1.$$

Since we have three unknowns and only two equations, we can add an additional equation, and increase the order of accuracy of the approximation:

$$\frac{1}{2}(4h)^2 A + \frac{1}{2}(2h)^2 C = 0.$$

This time, the solution is

$$A = \frac{1}{12h}, \quad B = \frac{1}{4h}, \quad C = -\frac{1}{3h},$$

and the approximation will be second-order, $O(h^2)$.

3. (a) **(6 points).** Find the first two orthogonal polynomials, $P_0(x), P_1(x)$ with respect to the weight $w(x) = \sqrt{x}$ on the interval $[0, 1]$. Do not normalize them.

Solution: Set $P_0 = 1$, and $P_1 = x - c$. To compute c we require orthogonality, i.e.,

$$0 = \langle P_0, P_1 \rangle_w = \int_0^1 1 \cdot (x - c)\sqrt{x}dx = \frac{2}{5} - \frac{2}{3}c.$$

Hence $c = 3/5$, i.e. $P_1(x) = x - \frac{3}{5}$.

(b) **(4 points).** Normalize $P_0(x)$.

Solution: Denote the normalized $P_0(x)$ by $\tilde{P}_0(x)$. Then $\tilde{P}_0(x) = cP_0(x) = c$. Hence

$$1 = \left\langle \tilde{P}_0, \tilde{P}_0 \right\rangle_w = \int_0^1 c \cdot c \cdot \sqrt{x}dx = \frac{2}{3}c^2.$$

Hence $c = \sqrt{\frac{3}{2}}$, i.e., $\tilde{P}_0(x) = \sqrt{\frac{3}{2}}$.

(c) **(6 points).** Let $Q_1^*(x) = a_0P_0(x) + a_1P_1(x)$. What should a_0, a_1 satisfy so that $Q_1^*(x)$ minimizes

$$\int_0^1 (x - Q_1(x))^2\sqrt{x}dx.$$

over all linear polynomials $Q_1(x)$. Express a_0 and a_1 as integrals. Do not explicitly compute these integrals quite yet.

Solution: Note that $Q_1(x) = x$ is the solution of this least squares problem. However, we are explicitly asked to find a_0 and a_1:

$$a_0 = \frac{\langle x, P_0 \rangle_w}{\langle P_0, P_0 \rangle_w} = \frac{\int_0^1 x \cdot 1 \cdot \sqrt{x}dx}{\int_0^1 1 \cdot 1 \cdot \sqrt{x}dx} = \frac{\int_0^1 x^{3/2}dx}{\int_0^1 x^{1/2}dx}.$$

$$a_1 = \frac{\langle x, P_1 \rangle_w}{\langle P_1, P_1 \rangle_w} = \frac{\int_0^1 x\left(x - \frac{3}{5}\right)\sqrt{x}dx}{\int_0^1 \left(x - \frac{3}{5}\right)^2\sqrt{x}dx}.$$

Here, I chose to use the non-normalized polynomials, $P_0(x)$ and $P_1(x)$.

(d) **(4 points).** Find a_0.

Solution: In solving this question we are using the expression from Part (c). If a_0 was written as the coefficient of the normalized $\tilde{P}_0$, then the answer would have been different.

$$a_0 = \frac{\int_0^1 x^{3/2} dx}{\int_0^1 x^{1/2} dx} = \frac{\frac{2}{5} x^{5/2} \Big|_0^1}{\frac{2}{3} x^{3/2} \Big|_0^1} = \frac{\frac{2}{5}}{\frac{2}{3}} = \frac{3}{5}.$$