Stat 410: Midterm 1 – Solutions

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Read carefully the following instructions:

- Write your name & student ID on the exam book and sign it.
- You may not use any books, notes, or calculators.
- Solve all problems. Answer all problems after carefully reading them. Start every problem on a new page.
- Show all your work and explain everything you write.
- Exam time: 75 minutes
- Good luck!

Problems: (Each problem = 10 points. Maximum total points = 50)

1. (a) How many different numbers can be obtained by arranging the digits contained in the number 5,751,542,269 ?

   Solution: There are 10 digits in the given number. The digit 2 repeats twice, and the digit 5 repeats 3 times. This means that the answer is:

   $$\binom{10}{2,3} = \frac{10!}{2!3!}.$$

   (b) How many of these numbers have the digit 9 followed by the three 5's?

   Solution: When the digit 9 is followed by 555, the sequence 9555 shows as a group in the 10-digit number. There are seven locations for this sequence (starting from the first digit, through starting with the seventh digit). Once we fix a location for 9555, there are 6 remaining digits to arrange (out of which the digit 2 repeats twice). Hence the solution is

   $$7\frac{6!}{2!}.$$

2. An American citizenship test is given in a class with 10 Mexican nationals, 8 Saudi nationals, and 3 French nationals. What is the probability that the first three to finish the test are all Mexican?

   Solution:

   There are 10+8+3=21 students taking the exam. Accounting for the repetitions, there are $\binom{21}{10,8,3} = \frac{21!}{10!8!3!}$ options for the order of finishing the exam.

   If the first three to finish the exam are all Mexican, there are 21-3=18 students remaining, out of which, 7 are Mexican, 8 Saudi, and 3 French. This means that the desired probability is

   $$P = \frac{\binom{18}{7,8,3}}{\binom{21}{10,8,3}} = \frac{\frac{18!}{7!8!3!}}{\frac{21!}{10!8!3!}} = \frac{8 \cdot 9 \cdot 10}{19 \cdot 20 \cdot 21} = \frac{12}{133}.$$

   Alternatively, we can choose 3 out of 10 for the first 3 to finish the exam, and divide by the number of options for the first 3 to finish the exam, as they can be selected from the 21 exam takers, we have

   $$P = \frac{\binom{10}{3}}{\binom{21}{3}}.$$

3. A box contains 7 fair dice and 3 unfair dice. For the unfair dice, the probability that 6 comes out is 1/4. Suppose a die is randomly selected and tossed 5 times.

   (a) Compute the probability that the first toss was a 6.

   Solution:
   Let F denote the event that a fair dice was chosen. Let E denote the event that the first toss was a 6. Then

$$P(E) = P(E|F)P(F) + P(E|F^c)P(F^c) = \frac{1}{6} \cdot \frac{7}{10} + \frac{1}{4} \cdot \frac{3}{10} = \frac{23}{60}.$$

   (b) Compute the probability that the dice is unfair given that all 5 times it landed on 6.

   Solution:
   Let G denote the event that all 5 tosses were 6. Then

$$P(F^c|G) = \frac{P(F^c \cap G)}{P(G)} = \frac{P(G|F^c)P(F^c)}{P(G|F^c)P(F^c) + P(G|F)P(F)} = \frac{\left(\frac{1}{4}\right)^5 \frac{3}{10}}{\left(\frac{1}{4}\right)^5 \frac{3}{10} + \left(\frac{1}{6}\right)^5 \frac{7}{10}}.$$

4. There are 5 boxes. Each of them contains 4 red balls, 3 green balls, and i blue balls ($i = 1, \ldots, 5$). (The first box contains 4 red balls, 3 green balls, and one blue ball. The second box contains 4 red balls, 3 green balls, and two blue balls, etc.). Carly randomly selects a box and then removes two balls from that box (without replacement).

   (a) What is the probability that the two balls are one red and one blue?

   Solution:
   Let E_i be the event that box i was chosen. Let B be the event that the two chosen balls are one red and one blue. Then

$$P(B) = \sum_{i=1}^{5} P(B|E_i)P(E_i) = \frac{1}{5}\sum_{i=1}^{5} P(B|E_i) = \frac{1}{5}\sum_{i=1}^{5} \frac{\binom{i}{1}\binom{4}{1}}{\binom{7+i}{2}}.$$

   (b) Assume that the two chosen balls are one red and one blue. What is the probability that the i-th box has been selected, for each $i = 1, \ldots, 5$?

   Solution:

$$P(E_i|B) = \frac{P(E_i \cap B)}{P(B)} = \frac{\frac{\frac{1}{5}\binom{i}{1}\binom{4}{1}}{\binom{7+i}{2}}}{P(B)},$$

   where $P(B)$ is the value computed in part (a).

5. Bernie and Ted choose 3 rocks each from a box containing 7 red and 7 blue rocks.

   (a) Find the probability that Bernie gets 1 red and 2 blue rocks.

   Solution:
   Let B be the event that Bernie chooses 1 red rock and 2 blue rocks. We are asked to compute $P(B)$. Bernie has to choose 1 red rock out of 7 rocks and two blue rocks out of 7 rocks. This can be done in $\binom{7}{1}\binom{7}{2}$ ways. The total number of ways to choose 3 rocks out of 14 rocks is $\binom{14}{3}$. Hence the probability is

$$P(B) = \frac{\binom{7}{1}\binom{7}{2}}{\binom{14}{3}}.$$

   Here, if we want to multiply the numerator by $\binom{11}{3}$ for Ted, then we should do the same in the denominator. Note that

$$\binom{6}{3}\binom{14}{6} = \binom{14}{3}\binom{11}{3}.$$

   The LHS means that we choose 6 out of 14, and then have to split them into two groups of 3 between Ted and Bernie, while the RHS means that we choose 3 out of 14 for Bernie and 3 out of the remaining 11 for Ted.

   (b) Find the probability that Bernie gets 1 red and 2 blue rocks given that Ted got 2 red and 1 blue rocks.

   Solution: Let T be the event that Ted chooses 2 red rocks and 1 blue rocks. Given the choice of Ted, Bernie is left with 2 less red and 1 less blue to choose from. Hence

$$P(B|T) = \frac{\binom{5}{1}\binom{6}{2}}{\binom{11}{3}}.$$

   (c) Find the probability that Bernie gets 1 red and 2 blue rocks given that Ted did not get 2 red and 1 blue rocks. (Hint: use the law of total probability).

   Solution:
   According to the law of total probability:

$$P(B) = P(B|T^c)P(T^c) + P(B|T)P(T).$$

   We are asked to find $P(B|T^c)$. Now $P(B)$ is the answer to part (a). $P(B|T)$ is the answer to part (b). $P(T) = P(B)$, and $P(T^c) = 1 - P(T)$. From this we can compute $P(B|T)$. Alternatively

$$P(B|T^c) = \frac{P(B \cap T^c)}{P(T^c)} = \frac{P(T^c|B)P(B)}{1 - P(T)} = \frac{(1 - P(T|B))P(B)}{1 - P(T)}.$$