Stat 410: Midterm 2 - Solutions

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Read carefully the following instructions:

- Write your name & student ID on the exam book and sign it.
- You may not use any books, notes, or calculators.
- Solve all problems. Answer all problems after carefully reading them. Start every problem on a new page.
- Show all your work and explain everything you write.
- Exam time: 75 minutes
- Good luck!

Problems: (Each problem = 10 points. Maximum total points = 60)

1. Suppose that the average number of bicycles stolen weekly in College Park is 3.6. Approximate the probability that there will be

   (a) No stolen bicycles in the next two weeks.

   Solution: The number of bicycles that are stolen, N, has a Poisson distribution with parameter $3.6t$. Here t denotes time measured in weeks. Hence

   $$P(N = i) = e^{-3.6t} \frac{(3.6t)^i}{i!}.$$

   Hence, the probability of no stolen bicycles in the next two weeks equals

   $$P(N = 0) = e^{-7.2}.$$

   (b) At least 2 stolen bicycles in the next week.

   Solution: The probability of having at least 2 stolen bicycles in the next week is one minus the probability of having no more than 2 bikes stolen. Hence, with $t = 1$:

   $$1 - P(N = 0) - P(N = 1) = 1 - e^{-3.6} - 3.6e^{-3.6} = 1 - 4.6e^{-3.6}.$$

2. Assume a random variable with $\mathrm{E}[X] = 2$ and $\mathrm{Var}[X] = 3$.

   (a) Compute $\mathrm{Var}[-2 - 5X]$.

   Solution: We know that $\mathrm{Var}[aX + b] = a^2 Var[x]$. Hence

   $$\mathrm{Var}[-2 - 5X] = 25\mathrm{Var}[X] = 25 \cdot 3 = 75.$$

   (b) Compute $\mathrm{E}[(2 + X)^2]$

   Solution: First, we note that $\mathrm{Var}[X] = \mathrm{E}(X^2) - (\mathrm{E}(X))^2$. Hence

   $$\mathrm{E}(X^2) = (\mathrm{E}(X))^2 + \mathrm{Var}[X] = 2^2 + 3 = 7.$$

   This means that

   $$\mathrm{E}[(2 + X)^2] = \mathrm{E}(2 + 4X + X^2) = \mathrm{E}(4) + 4\mathrm{E}(X) + \mathrm{E}(X^2) = 4 + 4 \cdot 2 + 7 = 19.$$

3. Let X be a continuous random variable with the density

$$f_X(x) = \begin{cases} ce^{-2x}, & x \geq 1, \\ 0, & x < 1. \end{cases}$$

(a) Find the constant c such that $f_X(x)$ is indeed a probability density function.

Solution:

$$1 = c\int_1^\infty e^{-x dx} = -\frac{c}{2}e^{-2x}\Big|_1^\infty = \frac{c}{e}e^{-2},$$

which means that $f_X(x)$ is a pdf if $c = 2e^2$.

(b) Compute, $F_X(x)$, the cumulative distribution function of X.

Solution: $F_X(x) = 0$ for $x < 1$. For $x \geq 1$:

$$\begin{aligned} F_X(x) &= \int_1^x f_X(y)dt = 2e^2\int_1^x e^{-2y}dy = 2e^2\frac{-1}{2}e^{-2y}\Big|_1^x = \\ &= -e^2(e^{-2x} - e^{-2}) = 1 - e^{2-2x}. \end{aligned}$$

(c) Find $P(X \geq 2)$.

Solution:

$$P(X \geq 2) = 1 - P(X \leq 2) = 1 - F_X(2) = e^{2-4} = e^{-2}.$$

(d) Find $\mathrm{E}[X]$ and $\mathrm{Var}[X]$. You may use the formula (for n positive integer):

$$\int x^n e^{ax}dx = \frac{e^{ax}}{a}\left(x^n - \frac{nx^{n-1}}{a} + \frac{n(n-1)x^{n-2}}{a^2} - \cdots \frac{(-1)^n n!}{a^n}\right).$$

Solution:

$$\mathrm{E}[X] = \int_1^\infty x2e^2e^{-2x}dx = 2e^2\frac{e^{-2x}}{-2}\left(x + \frac{1}{2}\right)\Big|_1^\infty = \frac{3}{2}.$$

Also

$$\mathrm{E}[x^2] = \int_1^\infty x^2 2e^2e^{-2x}dx = 2e^2\frac{e^{-2x}}{-2}\left(x^2 - \frac{2x}{-2} + \frac{2}{(-2)^2}\right)\Big|_1^\infty = \frac{5}{2}$$

Hence

$$\mathrm{Var}[x] = \mathrm{E}[x^2] - (\mathrm{E}(x))^2 = \frac{5}{2} - \left(\frac{3}{2}\right)^2 = \frac{10-9}{4} = \frac{1}{4}.$$

4. Resistors produced by a company will be defective with probability 0.05, independently of one another. The company sells the resistors in packages of 40 and offers a money-back guarantee that at most 1 of the 40 resistors is defective. What is the probability that a package of resistors will be replaced?

Solution: $X \sim \text{Bin}(40, 0.05)$. A package is replaced when $X > 1$. Hence

$$\begin{aligned} p(\text{replacement}) &= 1 - P(X=0) - P(X=1) \\ &= 1 - \binom{40}{0}(0.05)^0(0.95)^{40} + \binom{40}{1}(0.05)^1(0.95)^{39} \\ &= 1 - (0.95)^{40} - 40 \cdot 0.05 \cdot (0.95)^{39}. \end{aligned}$$

Alternatively, the problem can be solved assuming that X has a Poisson distribution with $\lambda = np = 40 \cdot 0.05 = 2$.

5. When coin #1 is flipped, it lands on heads with probability 0.3; when coin #2 is flipped, it lands on heads with probability 0.6. One of these coins is randomly chosen and flipped 12 times.

   (a) What is the probability that the coin lands on heads on exactly 8 of the 12 flips?

   Solution: Let X denote the number of Heads in 12 flips. Let Y_i denote the event that coin i is chosen. Then

$$\begin{aligned} p(X=8) &= p(X=8 \mid Y_1)p(Y_1) + p(X=8 \mid Y_2)p(Y_2) \\ &= \frac{1}{2}\binom{12}{8}0.3^8 0.7^4 + \frac{1}{2}\binom{12}{8}0.6^8 0.4^4. \end{aligned}$$

   (b) Given that the first two flips out of these 12 flips lands heads, what is the conditional probability that exactly 8 of the 12 flips land on heads?

   Solution: Let Z be the event that the first 2 flips are H. Then

$$\begin{aligned} p(X \mid Z) &= \frac{p(X \cap Z)}{p(Z)} = \frac{p(X \cap Z \mid Y_1)p(Y_1) + p(X \cap Z \mid Y_2)p(Y_2)}{p(Z \mid Y_1)p(Y_1) + p(Z \mid Y_2)p(Y_2)} = \\ &= \frac{\frac{1}{2}0.3^2\binom{10}{6}0.3^6 0.7^4 + \frac{1}{2}0.6^2\binom{10}{6}0.6^6 0.4^4}{\frac{1}{2}0.3^2 + \frac{1}{2}0.6^2} \end{aligned}$$

6. If independent trials, each resulting in a success with probability p, are performed, what is the probability of r successes occurring before m failures?

Solution: We know that the trials should end with a success. Hence that leaves $r-1$ successes prior to the last trial. The number of failures can vary between 0 and m. Hence, the desired probability is:

$$\sum_{n=r}^{r+m-1} \binom{n-1}{r-1} p^r (1-p)^{n-r}.$$