MATH642 Homework problems.

1. *Let Z be the space of continuous maps $\alpha : \mathbb{R} \to \mathbb{R}$ such that $\alpha(x) - x$ is 1-periodic, that is*

$$\alpha(x+1) = \alpha(x) + 1.$$

Introduce the distance on Z by

$$d(\alpha_1, \alpha_2) = \sup_{x \in \mathbb{R}} |\alpha_1(x) - \alpha_2(x)| = \sup_{x \in [0,1]} |\alpha_1(x) - \alpha_2(x)|.$$

Show that (Z, d) is a complete metric space.

2. *Consider the map of $\mathbb{T}^d$ given by $f(x) = A(x)$ mod $\mathbb{Z}^d$ where A is a matrix with integer entries such that $\det A \neq 0$. Show that f preserves the Lebesgue measure on $\mathbb{T}^d$.*

3. *Consider the map of $\mathbb{T}^2$ given by*

$$f(x) = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} x \text{ mod } \mathbb{Z}^2.$$

(a) Let e_s be the eigenvector with eigenvalue less than 1. Show that if p is a periodic point then for each c the set of limit points of the forward orbit of $p + ce_s$ consists of the orbit of p.

(b) Show that there exists a point whose forward orbit is not dense and the set of its limit points is **not** *a periodic orbit.*

4. *Let $G(x) = \{1/x\}$ be the Gauss map. Suppose that x satisfies $Ax^2 + Bx + C = 0$. Then $x_n = G^n(x)$ satisfies*

$$A_n x_n^2 + B_n x_n + C_n = 0$$

where

$$A_n = Ap_{n-1}^2 + Bp_{n-1}q_{n-1} + Cq_{n-1}^2,$$
$$B_n = 2Ap_n p_{n-1} + B(p_n q_{n-1} + q_n p_{n-1}) + 2Cq_n q_{n-1},$$
$$C_n = Ap_n^2 + Bp_n q_n + Cq_n^2.$$

Show that

$$B_n^2 - 4A_n C_n = B^2 - 4AC.$$

5. *Let $X = \mathbb{R} \cup \infty$ with natural topology (that is opens set containing ∞ contain $\{|x| > M\}$ for some M). Let $f(x) = \frac{ax+b}{cx+d}$. Describe the limit points of the orbits of f.*

Hint. The answer depends on how many fixed points f has.

6. *Let $\frac{p_n}{q_n}$ be the n-th partial convergent to x. That is if $G^n(x) = x_n$ then*

$$x = \frac{p_{n-1}x_n + p_n}{q_{n-1}x_n + q_n}.$$

Suppose that $q_n = f_n$-the n-th Fibonacci number. Find x.

7. *Let f be the circle rotation on angle α and let $S = [0, \alpha)$. Show that the induced map on S is conjugated to rotation on angle $\{1/\alpha\}$.*

8. *Show that no isometry of infinite compact metric space is expansive.*

9. *Show that expanding maps of the circe and linear hyperbolic automorphisms of $\mathbb{T}^2$ are expansive.*

10. *Consider a linear automorphism of $\mathbb{T}^2$ given by $f(x) = A(x) \bmod 1$. Show that if the spectrum of A lies on the unit circle, then $h_{top}(f) = 0$.*

11. *Let v be the eigenvector of matrix A with eigenvalue λ. Show that the following are equivalent*

(i) v is the only generalized eigenvector of A with eignvalue λ and the rest of the $Sp(A)$ is contained inside a disc of radius strictly smaller than $|\lambda|$;

and

(ii) There is a neighbourhood $U(v)$ such that for each $u \in U$

$$\angle(A^n u, v) \to 0.$$

12. *Let A be the linear map of $\mathbb{R}^d$ defined by the condition*

$$A(e_i) = e_{(i+1) \bmod d}$$

where $\{e_i\}$ is the standard basis in $\mathbb{R}^d$. Show that

$$Sp(A) = \{e^{2\pi ki/d}\}_{k=0}^{d-1}.$$

13. *Show that Hilbert distance satisfies the triangle inequality.*

14. *Let ω be a fixed point of a primitive substitution. Prove that for each word W there is a limiting frequency p_W such that if $N(\omega, W, n_1, n_2)$ be the number of appearances of W in ω between places n_1 and n_2 then*

$$\lim_{n_2 - n_1 \to +\infty} \frac{N(\omega, W, n_1, n_2)}{n_2 - n_1} = p_W.$$

15. *Prove that there are only countably many isomorphism classes of sofic shifts.*

16. *(a) Show that finite unions of Lebesgue spaces is a Lebesgue space.*

(b) Show that finite products of Lebesgue spaces is a Lebesgue space.

17. *Show that a compact manifold equipped with a smooth measure is a Lebesgue space.*

18. *Consider an equation $\dot{x} = X(x)$ where X is a smooth complete divergence free vector field. Define*

$$B^+ = \{x : Orb^+(x) \text{ is bounded}\}, \quad B^- = \{x : Orb^-(x) \text{ is bounded}\}$$

$$E^+ = \{x : Orb^+(x) \text{ tends to infinity}\}, \quad E^- = \{x : Orb^-(x) \text{ tends to infinity}\}$$
$$O^\pm = \mathbb{R}^d - (B^\pm \cup E^\pm).$$

Show that $B^- \cap O^+$, $B^+ \cap O^-$, $E^- \cap O^+$, *and* $E^+ \cap O^-$ *have zero measure.*

19. *Let* T *be a measurable map preserving a probability measure* μ. *Let* f *be a measurable function such that* $\mu(x : f(x) \neq 0) > 0)$. *Suppose that for almost every* x *we have* $f(Tx) = \lambda f(x)$ *for some constant* λ. *Show that* $|\lambda| = 1$.