

# PIECEWISE SMOOTH PERTURBATIONS OF INTEGRABLE SYSTEMS.

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ABSTRACT. We review several recent results showing that small piecewise smooth perturbations of integrable systems may exhibit unstable behavior on the set of initial condition of large measure. We also present open questions related to this subject.

## 1. INTRODUCTION.

The two most important discoveries of the theory of dynamical systems in the 20th century are stability of quasi periodic systems (Kolmogorov-Arnold-Moser theory) and possibility of stochastic behavior in deterministic systems.

Fortunately for us, most systems we encounter are stable. In the context of Hamiltonian system this means that many systems appearing in applications are either integrable or close to integrable. Some immediate examples that come to mind are the pendulum, the spring, and the  $1 + N$  body problem in celestial mechanics.

The Kolmogorov-Arnold-Moser theory guarantees that most orbits of small perturbations of integrable systems are stable, while Nekhoroshev theory says that even the orbits which wander away from the unperturbed motion do so extremely slowly.

Another manifestation of the prevalence of stable behavior is that in order to obtain a uniformly hyperbolic system—the most studied class of stochastic systems one needs either to have a manifold with complicated topology or the map itself has to be topologically complicated.

The above mentioned topological restrictions do not appear for **piecewise smooth** systems since to study a piecewise smooth system we need to cut the phase space into continuity regions killing the topological complexity. Intuitively it is also clear that systems with singularities may possess more stochasticity than smooth systems. Indeed, it is well understood now that the stochastic behavior is caused by a sensitive dependence on the initial conditions, that is, by the presence of large entries in the derivative matrix. For systems with singularities derivatives are effectively infinite near the singularities which make it easier for stochastic behavior to emerge.

The goal of this survey is to discuss the presence of stochastic behavior for small piecewise smooth perturbations of integrable systems.

## 2. EXAMPLES.

In this section we present several examples of systems which in the smooth case are stable (and their stability follows from the KAM theory), while if singularities are allowed they can exhibit stochastic behavior. We note that our examples have one and a half degrees of freedom. The reason for this is that one and a half degree of freedom systems are easier to analyze, and so more examples are available in the literature. However the phenomenon we are interested in is in no way limited to one and a half degree of freedom systems, and it would be interesting to investigate higher dimensional examples.

**2.1. Types of final motions.** In many examples below we will deal with the orbits defined on the plane or the cylinder. We will be interested in how the orbits behave at infinity. Fix some point  $p$  (for example the origin) and let  $d_n$  denote the distance between the orbit and  $p$  at time  $n$ . There are three possible behaviors at infinity.

We say that the orbit is **bounded** if  $d_n$  is bounded, that it is **oscillatory** if

$$\liminf_{n \rightarrow +\infty} d_n < +\infty \text{ but } \limsup_{n \rightarrow +\infty} d_n = +\infty$$

and **escaping** if  $\lim_{n \rightarrow +\infty} d_n = +\infty$ . We will denote the set of bounded orbits by  $\mathcal{B}$ , the set of oscillatory orbits by  $\mathcal{O}$  and the set of escaping orbits by  $\mathcal{E}$ .

**2.2. Motion in periodic potential.** Consider the following system.

$$\ddot{x} + \sum_{j=0}^{2m+1} a_j(t)x^j = 0, \text{ where } a_j(t+1) = a_j(t) \text{ and } a_{2m+1}(t) \geq c > 0.$$

**Theorem 1. (Dieckerhoff-Zehnder (1987), Laederich-Levi (1991) [11, 26])** *If the coefficients  $a_j$  are smooth then all orbits are bounded.*

The reason why this is true is that for particles having large energy, the averaging theory allows to replace  $a_j$  by their averaged values over the period, and so the KAM theorem can be applied.

**Theorem 2. (Levi-You (1997) [28])** *For typical piecewise smooth  $a_j$ , there exist both oscillatory and escaping orbits.*

**Question 1.** *What is the measure of  $\mathcal{O}$  and  $\mathcal{E}$  in the above example?*

**2.3. Billiards.** Consider a particle moving smoothly inside some domain and bouncing elastically from the boundary.

Recall that a **caustic** for the billiard map is a curve  $\gamma$  which has the property that if one segment of the billiard orbit is tangent to  $\gamma$ , then all orbit segments are also tangent to it.

**Theorem 3. (Lazutkin (1973) [27])** *If boundary is smooth and strictly convex, then there exist caustics arbitrary close to the boundary.*

In particular caustics prevent billiard orbits from approaching the boundary.

**Theorem 4. (Hubacher (1987) [22])** *If the boundary has points with curvature jumps, then there exist orbits approaching the boundary.*

**Theorem 5. (Mather (1982) [31])** *If the boundary has points with zero curvature, then there exist orbits approaching the boundary.*

To see why Theorem 5 fits into our setting we note that if the boundary of the domain is smooth, but has points of zero curvature then the billiard map itself is not smooth.

**2.4. Outer billiards: approach to the boundary.** An **outer billiard map**  $F$  is defined outside a closed convex curve  $\Gamma$  in the following way (see figure 1). Let  $z$  be a point on the plane. Consider the supporting line  $L(z)$  from  $z$  to  $\Gamma$  such that  $\Gamma$  lies on the right of  $L$ . Then  $F(z)$  lies on  $L(z)$  so that the point of contact divides the segment  $[z, F(z)]$  in half. If  $\Gamma$  contains segments then  $F(z)$  is not defined if  $L(z)$  contains a segment. In this case  $F(z)$  is defined almost everywhere but it is discontinuous. In analogy to the usual (inner) billiards the invariant curves for outer billiard maps are also called (outer) caustics. We refer the reader to [37] for an introduction to outer billiards.

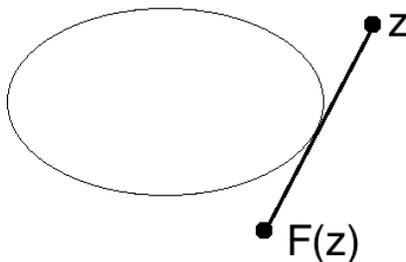


FIGURE 1. Outer billiard map

**Theorem 6.** (Moser (1973), Douady (1982) [32, 17]) *If the boundary is smooth and strictly convex, then there exist caustics arbitrary close to boundary.*

**Theorem 7.** (Boyland (1996) [2]) *If the boundary has points with curvature jumps, then there exist orbits approaching the boundary.*

**Theorem 8.** (Gutkin-Katok (1995) [21]) *If  $\Gamma$  is smooth and strictly convex curve having a point of zero curvature radius, then there are no caustics for outer billiards.*

The reader will notice that Theorems 6, 7 and 8 are outer billiard analogues of Theorems 3, 4 and 5 respectively.

We have the following estimates for the rate of approach to the boundary in Theorem 4 and 7.

**Theorem 9.** (Zhong (2010) [40]) *If the boundary is strictly convex and smooth except for finitely many curvature jumps then for both inner and outer billiards*

- (1) *For all orbits  $\liminf d(x_n, \Gamma) \geq \frac{c}{n^2}$ ;*
- (2) *There exist orbits such that  $\limsup d(x_n, \Gamma) \leq \frac{C}{n^2}$ .*

**Question 2.** *Estimate the rate of approach to the boundary in the Theorems 5 and 8.*

**2.5. Stochastic billiards.** It is well known that KAM theory provides obstructions to ergodicity. Theorems 5 and 4 show for piecewise smooth convex billiards there are no KAM obstructions near the boundary where the billiard map is near integrable. In fact, presently there are several examples of piecewise smooth convex domains with ergodic billiard maps.

One class of such billiards is given by **focusing billiards** discovered by Bunimovich (see [3]). We refer the reader to the work of Wojtkowski [41, 42] for a general approach to constructing focusing billiards with non-zero Lyapunov exponents and to Chapters 8 and 9 of [6] for discussion of their statistical properties.

A different class of billiards exhibiting stochastic properties is given by polygonal billiards.

**Theorem 10.** (Kerckhoff-Masur-Smillie (1986) [23]) *Polygons with ergodic billiard flows form a dense  $G_\delta$  set in the space of all polygons.*

The following remains one of the most challenging questions in the billiard theory.

**Question 3.** *Do polygons with ergodic billiard flows constitute a positive measure set in the space of all polygons?*

Outer billiards with non-zero exponents are much less studied. In fact only one example is known so far. To describe it we need to recall the construction of tables with a given caustic (see the left part of figure 2). Let  $S$  be a convex curve on the plane. We want to construct a table  $\Gamma$  such that  $S$  is an invariant curve for the outer billiard dynamics of  $\Gamma$ . Fix a parameter  $a$  and consider all segments which cut domains of the area  $a$  from  $S$ . Let  $\Gamma(a)$  be the set of midpoints of those segments. Then  $S$  is a caustic for the outer billiard on  $\Gamma(a)$ .

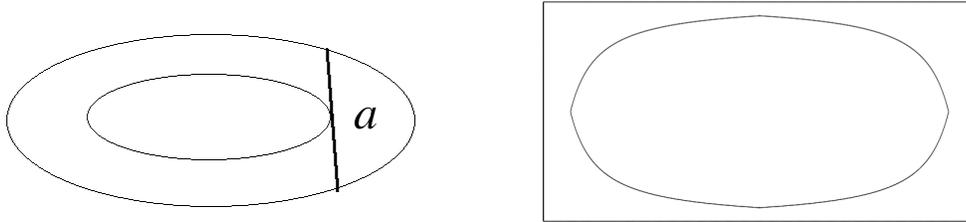


FIGURE 2. Left: Area construction. Right: Genin table.

**Theorem 11. (Genin (2006) [18])** *If  $S$  is a rectangle and  $a$  is sufficiently small then the outer billiard on  $\Gamma(a)$  has non-zero Lyapunov exponents.*

**Question 4.** *Prove ergodicity and mixing and investigate the rate correlation decay for the above table.*

**Question 5.** *Is the same result true if the rectangle is replaced by other convex polygons?*

**Question 6.** *Find analogues of Theorem 11 for inner billiards.*

## 2.6. Outer billiards: unbounded orbits.

**Theorem 12. (Moser (1973), Douady (1982) [32, 17])** *If  $\Gamma$  is smooth and strictly convex then all orbits are bounded.*

**Question 7.** (Moser (1978) [33])

- (1) *What happens if  $\Gamma$  is only piecewise smooth?*
- (2) *What happens if  $\Gamma$  contains flat points?*

A lot of research on this subject was devoted to the case when  $\Gamma$  is a polygon since in this case the plane can be divided into finitely many pieces so that on each piece the outer billiard map is the reflection about one of the vertices.

**Theorem 13. (Kolodziej (1989) [25])** *If all vertices of  $P$  are rational then all orbits are periodic.*

In fact Theorem 13 applies to a wider class of **quasi-regular polygons** which includes both rational polygons and the regular polygons. We refer the reader to [25] for the definition of quasi-regular polygons.

The first example of a polygon with unbounded outer billiard orbits was constructed in [35]. Note that Theorem 13 implies that outer billiards on triangles have bounded orbits, since all triangles are affine equivalent to each other and in particular to the equilateral right triangle, and the outer billiard map commutes with affine transformations. Thus the

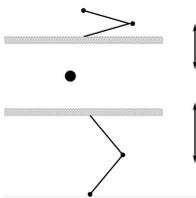


FIGURE 3. Ulam ping-pong

simplest polygons which may exhibit unbounded orbits are quadrilaterals. Given  $A \in \mathbb{R}^+$  let  $K(A)$  be the kite. That is,  $K(A)$  is the quadrilateral with the vertices  $(0, 1)$ ,  $(-1, 0)$ ,  $(0, -1)$ , and  $(A, 0)$ . Then Theorem 13 implies that all orbits are bounded if  $A$  is rational.

Let  $L$  denote the set of all points whose  $y$  coordinate is even integer. Note that this set is invariant by outer billiard dynamics.

**Theorem 14. (Schwartz (2007) [35, 36])** *If  $A \notin \mathbb{Q}$  then there are unbounded orbits in  $L$ . Moreover*

- (1) *almost every orbit in  $L$  is periodic;*
- (2) *every orbit is either periodic or erratic in the sense that*

$$\liminf d(x_n, K) = 0, \quad \limsup d(x_n, K) = 0;$$

- (3) *the set of erratic orbits has positive Hausdorff dimension.*

**Question 8.** *Is it true that typical  $n$ -gons with  $n \geq 4$  have unbounded orbits?*

**Question 9.** *Is it true that for all polygons the set of unbounded orbits has measure 0?*

Another example of curve with unbounded outer billiard orbits is given by the semicircular boundary.

**Theorem 15. (Dolgopyat-Fayad (2009) [14])** *If  $\Gamma$  is a semicircle then  $\text{mes}(\mathcal{E}) = \infty$ . E.g.  $x_n^2 + y_n^2 \rightarrow \infty$  if  $|x_0 - 1500.25| < 0.01$ ,  $|y_0 - 1.75| < 0.01$ .*

**Question 10.** *Do unbounded orbits exist for other circular caps? If so what is the speed of escape? It is known that for any curve  $x_n^2 + y_n^2 < Cn$ . Do circular caps have orbits with  $x_n^2 + y_n^2 \sim vn$ ? If so how  $v$  behaves as the cap approaches the circle?*

**Question 11.** *Is  $\mathcal{E}$  nonempty for the following tables*

- (1) *union of two circular arcs*
- (2) *curve which is strictly convex except for one point of zero curvature?*

**Theorem 16.** *(See e.g. [13]) For the tables from Question 11*

- (a)  $x_n^2 + y_n^2 \ll n$ ;
- (b)  $\text{mes}(\mathcal{E}) = 0$ .

**Question 12.** *Do the tables from Question 11 or circle caps possess oscillatory orbits?*

**2.7. Ulam ping-pong.** Consider a ball bouncing between two periodically moving infinitely heavy plates.

**Theorem 17. (Pustynnikov (1977), Douady (1982), Laederich-Levi (1991) [34, 17, 26])** *If the motion of the wall is smooth then all ping-pong trajectories are bounded.*

**Theorem 18. (Zharnitsky (1998) [39])** *There is an open set of piecewise smooth wall motions for which there exist unbounded trajectories.*

Suppose that one of the walls is fixed and the velocity of the second wall has a single discontinuity at 0. Let  $\ell(t)$  denote the distance between the walls at time  $t$ . Set

$$\Delta = \ell(0)(\dot{\ell}(0^+) - \dot{\ell}(0^-)) \int_0^1 \frac{ds}{\ell^2(s)}.$$

**Theorem 19. (de Simoi-Dolgopyat (2012) [9])**

- (1) *If  $\Delta \in (0.5, 4)$  then  $\text{mes}(\mathcal{E}) = \infty$*
- (2) *If  $\Delta < 0$  or  $\Delta > 4$  then  $\text{mes}(\mathcal{E}) = 0$  but  $HD(\mathcal{E}) = 2$ .*

**Conjecture 13.**  *$\text{mes}(\mathcal{E}) = \infty$  for all  $\Delta \in (0, 4)$ .*

Thus in case  $\Delta \notin [0, 4]$  most orbits can not accelerate indefinitely. In fact almost every orbit eventually drops energy below a fixed threshold.

**Theorem 20. (de Simoi-Dolgopyat (2012) [9])**

- (1)  *$\Delta \notin [0, 4]$  then there exists a constant  $C$  such that almost every orbit enters the region  $v < C$ .*
- (2) *If  $\Delta \in (0, 4)$  and a non-degeneracy condition is satisfied then there is  $C > 0$  such that for each  $\bar{v}$ , there exists an orbit such that for all  $n$ , we have*

$$\bar{v}/C < v_n < C\bar{v}.$$

The results of [9] show that pingpongs with  $\Delta \in (0, 4)$  and pingpongs with  $\Delta \notin [0, 4]$  have very different behaviors. This is also clear from looking at the phase portraits.

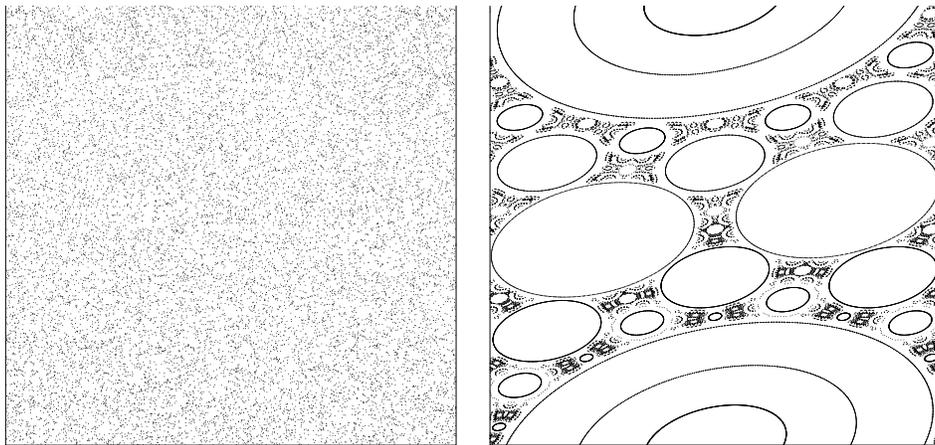


FIGURE 4. On the left: phase portrait of a single orbit of the map  $F$  from definition 1 (see below) for  $\Delta = -0.3$ . On the right: phase portrait of selected orbits of the map  $F$  for  $\Delta = 0.32$ .

**Example.** We now consider pingpongs with piecewise linear velocity. That is, we assume that

$$\ell_{a,b}(t) = b + a((t \bmod 1) - 0.5)^2.$$

This is one of the cases which have been numerically investigated in [38]. Later numerical and heuristic analysis of this system can be found in [7, 4, 29]. We can scale the space so

that  $b = 1$ . Then  $\ell(t) \geq 0$  for all  $t$  iff  $a > -4$ . In this case  $\Delta$  can be computed explicitly. Namely,  $\Delta(a) = -2a(1 + a/4)J(a)$  where

$$J(a) = \frac{2}{a+4} + \begin{cases} (|a|^{-1/2}/2) \log \frac{2+|a|^{1/2}}{2-|a|^{1/2}} & \text{if } -4 < a \leq 0 \\ |a|^{-1/2} \arctan(|a|^{1/2}/2) & \text{if } a > 0. \end{cases}$$

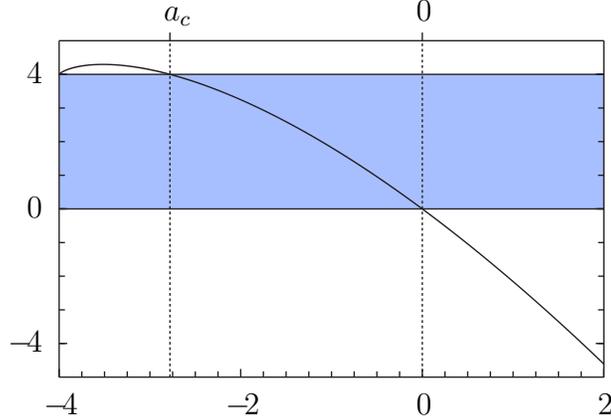


FIGURE 5. Graph of  $\Delta$  as a function of  $a$ . The shaded area denotes the elliptic regime  $\Delta \in (0, 4)$ . We have  $a_c \approx -2.77927$

**Theorem 21.** (de Simoi-Dolgopyat (2012) [9]) *If  $\Delta \notin [0, 4]$  let  $\mathbf{T}$  denote the first time the velocity falls below  $C$ , where  $C$  is the constant from Theorem 20. Fix the initial velocity  $v_0 \gg 1$ , and let the initial phase be uniformly distributed on  $[0, 1]$ .*

*As  $v \rightarrow \infty$ ,  $\frac{\mathbf{T}}{v_0^2}$  converges to a stable random variable of index  $1/2$ , i.e., there exists a constant  $\bar{D}$  such that*

$$P(\mathbf{T} > \bar{D}v_0^2t) \rightarrow \int_t^\infty \frac{e^{-1/2x}}{\sqrt{2\pi x^3}} dx \text{ as } v_0 \rightarrow \infty.$$

Moreover consider the process

$$\mathbf{B}^{v_0}(t) = \begin{cases} \frac{v(v_0^2t)}{v_0} & \text{if } v_0^2t \text{ is an integer} \\ \frac{v(n)(n+1-v_0^2t)+v(n+1)(v_0^2t-n)}{v_0} & \text{if } v_0^2t \in (n, n+1) \text{ for some integer } n. \end{cases}$$

*Stop  $B^{v_0}$  at time  $t = \frac{\mathbf{T}}{v_0^2}$ . Then, as  $v_0 \rightarrow \infty$ ,  $\mathbf{B}^{v_0}(t)$  converges to  $W(t)$  where  $W$  is a Brownian Motion started from 1 and stopped when it reaches 0.*

The second part of the last theorem implies the first part since the time the Brownian Motion drops 1 unit has stable distribution of index  $1/2$ .

**Conjecture 14.** *The stopping is not necessary for convergence to the Brownian Motion.*

The difficulty in the last theorem is that if the particle has low energy, there is no scale separation between the wall motion and the particle motion, so the system is not close to integrable, and we have little control on the dynamics. One situation where we have better understanding of the dynamics is the case of piecewise convex wall motion discussed below.

Let us summarize the results about the existence of various types of orbits. In the case  $\Delta \in (0, 4)$ , we know that there is an infinite measure set of bounded orbits, and we believe (see Question 13) that there is an infinite set of escaping orbit as well.

**Conjecture 15.** *Oscillatory orbits exist for all  $\Delta \in (0, 4)$ .*

By contrast, the orbits with different behaviors are easily constructed in case  $\Delta \notin [0, 4]$ .

**Theorem 22.** (de Simoi-Dolgopyat (2012) [9]) *If  $\Delta \notin [0, 4]$  then*

$$HD(\mathcal{E}) = HD(\mathcal{O}) = HD(\mathcal{B}) = 2.$$

**Conjecture 16.** *If  $\Delta \notin [0, 4]$  then the oscillatory behavior is prevalent in the sense that the complement to  $\mathcal{O}$  has finite measure.*

Currently we are working on proving this result under the (strong) additional assumption that

$$\ddot{\ell} \geq c > 0. \tag{*}$$

Note that in this case  $\dot{\ell}$  is increasing on  $[0, 1]$  so  $\dot{\ell}(0+) < \dot{\ell}(0-)$  and hence  $\Delta < 0$ .

**Definition 1.** *We denote by  $F$  the Poincare map corresponding to taking the first collision of the wall with the moving wall after passing the singularity (that is, if there are several collision on the interval  $[m, m + 1)$  for some  $m$  then we skip all collision except for the first one).*

**Theorem 23.** (de Simoi-Dolgopyat (2013) [10]) *If Assumption (\*) holds then  $F$  has a positive Lyapunov exponent in the sense that*

$$\lambda(x) = \liminf_{n \rightarrow \infty} \frac{\ln \|dF^n(x)\|}{n}$$

*is positive for almost  $x$ .*

Recall that in view of Theorem 20 the first return map  $R$  to the region  $\{v < C\}$  is well defined if  $C$  is large enough.

**Conjecture 17.** *If Assumption (\*) holds then  $R$  is ergodic.*

In view of Theorem 23 and the general theory of piecewise hyperbolic maps developed by Chernov, Sinai, Liverani, Wojtkowski and others (see [6, 30]) in order to prove Conjecture 17 one needs to check certain non-degeneracy conditions on the dynamics of singularities. Thus, under Assumption (\*) we have a strong evidence in favor of ergodicity.

On the other hand ergodicity implies a positive answer to Conjecture 16. Indeed by Theorem 20 almost all orbits visit  $\{v < C\}$ . Next fix any  $\bar{v}$ . Due to measure preservation and the above mentioned recurrence we have that, conversely, there is a positive set of orbits in  $\{v < C\}$  which visit  $\{v > \bar{v}\}$  before the next return to  $\{v < C\}$ . By ergodicity, this set of orbits passing eventually through  $\{v > \bar{v}\}$  has full measure. Thus, in the piecewise convex case we are close to proving a stronger version of Conjecture 16. Namely we expect that in that case almost all orbits are oscillatory. On the other hand, without Assumption (\*), Conjecture 16 seems much more difficult.

The results presented above deal with the case where the velocity of the wall has jump.

**Question 18.** *What happens when the wall velocity is continuous but the wall acceleration has a jump?*

## 3. THEORY

3.1. **Normal form.** Consider the following map of the annulus

$$f(I, \phi) = (I + I^k a(\phi) + \dots, \phi + I^m b(\phi) + \dots), \quad b > 0, \quad I \ll 1$$

where  $a(\phi)$  and  $b(\phi)$  are piecewise smooth and  $\dots$  denote the higher order terms. Let  $D$  be the fundamental domain bounded by  $\gamma$  and  $f\gamma$ , where  $\gamma$  is a vertical curve. Let  $F : D \rightarrow D$  be the first return map.

**Theorem 24.** (cf [14]) *In suitable coordinates,  $F$  can be represented as a composition of maps of the form*

$$G(J, \psi) = (\tilde{J}, \tilde{\psi}) \text{ where } \tilde{\psi} = \psi + J, \quad \tilde{J} = \left\{ \tilde{J} + A\tilde{\psi} + B \right\}.$$

Here

$$(A, B) \rightarrow \begin{cases} \infty & k < m + 1 & (i) \\ \text{Const} & k = m + 1 & (ii) \\ (Id, 0) & k > m + 1 & (iii) \end{cases}$$

as  $I \rightarrow 0$ .

Thus we have several universality classes, depending on which alternative of Theorem 24 holds for our system. In case (ii), we have to further distinguish the cases where the linear part of the normal form is hyperbolic or elliptic (see Figure 4). For example, particles in piecewise smooth potential, inner and outer billiards with curvature jump (for orbits near the boundary) belong to class (i), outer billiards with segments and pingpongs with velocity jump belong to class (ii), and outer billiards without segments and pingpongs with continuous velocities belong to class (iii). Class (i) is well studied in physics literature under the name of *antiintegrable limit* (see [1, 8] and references therein). Class (ii) is quite well understood in the hyperbolic case ([5]). In the elliptic case we have to deal with piecewise isometries. Paper [19] contains a review of this subject. Finally for class (iii) much less is known. A formal perturbation theory suitable for class (iii) is discussed below.

3.2. **Formal perturbation theory.** The last section describes the normal forms for piecewise smooth integrable maps with small twist. In the case of large twist the dynamics is much less understood. In this subsection we present some questions related to formal perturbation theory for such maps. Consider the map.

$$F(r, \phi) = (r + \varepsilon P(r, \phi), \phi + \alpha(r) + \varepsilon R(r, \phi)).$$

Then

$$r_n = r_0 + \varepsilon \sum_{j=0}^{n-1} P(r_0, \phi_0 + j\alpha) + HOT = r_0 + \varepsilon \sum_{j=0}^{n-1} A(\phi_0 + j\alpha) + HOT$$

where  $A(\phi) = P(r_0, \phi)$ . The starting point of the perturbation theory in the smooth case is the fact that if  $A$  is smooth and has zero mean then for almost every  $\alpha$  the sum

$$\sum_{j=0}^{n-1} A(\phi + \alpha j)$$

is bounded, in fact it can be written as  $B_\alpha(\phi + n\alpha) - B_\alpha(\phi)$  for a suitable function  $B_\alpha$ . Then we can make the change of variables  $\tilde{\phi} = \phi + \varepsilon B_\alpha(\phi)$  reducing the perturbation to a higher

order. We want to see how this sum behaves for piecewise smooth  $A$ . It turns out that the result depends only on the discontinuity set of  $A$ , and so to simplify the formulas we shall consider the case of indicator. Let  $A = \chi_\Omega$ . Denote

$$D_n(\Omega, \phi, \alpha) = \left( \sum_{j=0}^{n-1} \chi_\Omega(\phi + j\alpha) \right) - n \text{Vol}(\Omega).$$

The one dimensional case was analyzed by Kesten who proved the following result for the case when  $\Omega$  is a segment.

**Theorem 25. (Kesten (1960) [24])** *Suppose  $\phi, \alpha$  are independent and uniformly distributed on  $\mathbb{T}^2$ . Then  $\frac{D_n}{\ln n}$  converges as  $n \rightarrow \infty$  to a Cauchy distribution, that is, there is a function  $c(l)$  such that*

$$\text{Prob} \left( \frac{D_n(\Omega, \phi, \alpha)}{c(|\Omega|) \ln n} \leq t \right) \rightarrow \frac{\tan^{-1} t}{\pi} + \frac{1}{2}.$$

Moreover  $c(l)$  does not depend on  $l$  if  $l$  is irrational.

An interesting problem is to extend this result to higher dimensions. The first question is which sets should one consider. The least restrictive assumption is that  $\Omega$  is semialgebraic, that is, it is defined by a finite set of algebraic inequalities.

**Conjecture 19.** *If  $\Omega$  is semialgebraic then there exists a sequence  $a_n = a_n(\Omega)$  such that for translation of a random torus by a random vector, the sequence  $D_n/a_n$  has a limiting distribution.*

Here random translation of a random torus means that we consider the sequence  $x_n = x_0 + n\alpha$  on the torus  $\mathbb{R}^d/L$ , where  $L = A\mathbb{Z}^d$ , and we suppose that the triple  $(x_0, \alpha, A)$  has a smooth density with compact support.

Jointly with Bassam Fayad, we have verified this conjecture in two cases described below.

**Theorem 26. (Dolgopyat-Fayad (2013) [15])** *Let  $d \geq 2$ ,  $\Omega$  be strictly convex,  $\phi, \alpha$  and  $r$  have smooth densities then*

$$\frac{D_n(r\Omega, \phi, \alpha)}{r^{\frac{d-1}{2}} n^{\frac{d-1}{2d}}}$$

*has limiting distribution.*

**Theorem 27. (Dolgopyat-Fayad (2013) [15])** *If  $\Omega$  is a  $d$  dimensional cube then  $D_n/\ln^d n$  converges to a Cauchy distribution.*

Theorems 25, 26 and 27 describe the growth of the first term in the formal perturbation theory.

**Question 20.** *Compute higher order terms.*

#### 4. CONCLUSION

We saw that, in contrast with the smooth case, small piecewise smooth perturbations of integrable systems may exhibit stochastic behavior on a large set of initial conditions. This behavior is **universal** (that is, it is common for a diverse class of examples) due to the fact that different systems may have a common normal form. However, in contrast with the smooth case, the dynamics of those normal forms is not well understood. In the cases where we have some results about the dynamics of the normal form, an extra effort

is need to transfer the results to the actual system. Some methods to do so are developed in, for example, [9, 12, 14, 16] but more work is needed in this direction. Finally, as it was mentioned before, almost nothing is known in higher dimensional cases.

To summarize, the study of piecewise smooth perturbations of integrable systems is an active area of research which already led to the discovery of several surprising phenomena but more interesting results can be expected in the future.

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