1. Results.

Let $S = \{p, q, r : p \geq 0, q \geq 0, r \geq 0, p + q + r = 1\}$. Let $\sigma$ be a distribution on $S$ such that for some $\kappa > 0$

$$\sigma(p \geq \kappa, q \geq \kappa, r \geq \kappa) = 1.$$ 

Denote $\Omega = S^\mathbb{Z}$. An element $\omega = \{(p_k, q_k, r_k)\} \in \Omega$ will be called an environment. We assume that $(p_k, q_k, r_k)$ are iid random vectors with distribution $\sigma$. Let $X_n$ be a random walk in this environment that is

$$P_\omega(X_{n+1} - X_n = \Delta|F_n) = \begin{cases} p_k & \text{if } \Delta = 1 \\ r_k & \text{if } \Delta = 0 \\ q_k & \text{if } \Delta = -1 \end{cases}.$$ 

We assume that $\sigma(\ln(p/q)) > 0$ so that due to [10] $X_n \to +\infty$ almost surely. We also assume that

(1.1) $\sigma((p/q)^2) < 1.$ 

In this case the walk satisfies a quenched Central Limit Theorem. Namely let $T_k$ be the first time the walk visits site $k$. Let

$$b_n = b_n(\omega) = \min(k : E_\omega(T_k) \geq n).$$ 

Theorem 1. [5, 9]. There is a constant $D$ such that almost surely

$$\frac{X_n - b_n(\omega)}{\sqrt{n}} \Rightarrow N(0, D^2).$$ 

In this note we obtain a local version of Theorem 1. Denote $\rho_k = E_\omega(\text{Card}(n : X_n = k))$. Let $a = E\rho$.

Theorem 2. Almost surely the following holds. For each $\varepsilon, R > 0$ there exists $n_0 = n_0(\omega)$ such that for $n \geq n_0$ uniformly for

(1.2) $|k - b_n| \leq R\sqrt{n}$

we have

$$\left| \frac{\sqrt{2\pi n D a}}{\rho_k} \exp \left[ \frac{(k - b_n)^2}{2D^2 n} \right] P_\omega(X_n = k) - 1 \right| < \varepsilon.$$
A similar result was obtained in [8] for the case \( q_k \equiv 0 \). Let us present some consequences of this result. The first application is an annealed local limit theorem.

Recall ([1]) that there is a constant \( \hat{D} \) such that

\[
    b_n - \frac{n}{a} \sqrt{n} \Rightarrow N(0, \hat{D}^2).
\]

Let \( D = \sqrt{D^2 + \hat{D}^2} \).

**Theorem 3.** For each \( \varepsilon, R > 0 \) there exists \( n_0 = n_0(\omega) \) such that for \( n \geq n_0 \) uniformly for

\[
    |k - \frac{n}{a}| \leq R\sqrt{n}
\]

we have

\[
    \left| \sqrt{2\pi n} D \exp \left[ \frac{(k - \frac{n}{a})^2}{2 \hat{D}^2 n} \right] P(X_n = k) - 1 \right| < \varepsilon.
\]

We also obtain a direct proof of the following theorem of Lalley [7].

**Theorem 4.** For almost every \( \omega \) for every continuous function \( \Phi : \Omega \to \mathbb{R} \) it holds that

\[
    E_{\omega}(\Phi(\tau_{X_n}, \omega)) \to \frac{\mu(\rho \Phi)}{a}.
\]

**2. Preliminaries.**

**Lemma 2.1.** [1, 5, 9] There exists \( \bar{D}^2 \) such that for almost every \( \omega \)

\[
    \frac{T_k - \mathbb{E}_\omega T_k}{\sqrt{k}} \Rightarrow N(0, \bar{D}^2).
\]

Let \( S \) denote the positive solution of \( \sigma(p^s/q^s) = 1 \). Due to (1.1) \( s > 2 \) (\( s \) can be equal to \( +\infty \) if the walker has positive drift with probability 1).

**Lemma 2.2.** (a) \( P(\rho > t) \leq Ct^{-s} \).

(b) For any \( \hat{u} > \frac{1}{s} \) for almost every \( \omega \) there is a constant \( C(\omega) \) such that \( \rho_k < C(\omega) k^{\hat{u}} \).

**Proof.** If the distribution of \( \ln p - \ln q \) is non arithmetic then [6] gives a result which is stronger than (a), namely \( P(\rho > t) \sim \bar{c}t^{-s} \). We only need an upper bound which does not require arithmetic assumptions. Part (b) follows from part (a) and Borel-Cantelli Lemma.  \( \square \)
Lemma 2.3. \cite{[2]} (a) There exists $C > 0$ and $\theta < 1$ such that
$$
P(X \text{ visits } k \text{ after } k + m) \leq C \theta^m.$$  
(b) Accordingly, for almost every $\omega$ there is a constant $K(\omega)$ such that
$$
P(\exists k < n : X \text{ visits } k \text{ after } T_{k+\ln^2 n}) \leq K(\omega)n^{-100}.$$

Lemma 2.4. \cite{[5],Lemma 5} There exists $\epsilon_0$ such that almost surely
$$
\lim_{n \to \infty} \frac{1}{\sqrt{n}} \max_{t \leq n} |E_\omega(T_{n+t} - T_n - la)| \to 0.
$$

3. Quenched LLT.

Proof of Theorem 2. Take $\frac{1}{s} < u < \frac{1}{2}$. We claim that
$$
P_\omega(\exists k \leq n \exists m \in \mathbb{N} : X_m = k \text{ and } T_k < m - n^u) \leq \tilde{C}(\omega)n^{-100}.
$$
Indeed if $X_m = k$ and $m > T_k + n^u$ then one of the following events should happen
$$
A_1 = \{X_t \in [k - \ln^2 n, k + \ln^2 n] \text{ for all } t \in [T_k, T_k + n^u]\};
$$
$$
A_2 = \{\exists t \in [T_k, T_k + n^u] \text{ such that } X_t < m - \ln^2 n\};
$$
$$
A_3 = \{\exists t \in [T_k, T_k + n^u] \text{ such that } X_t > m + \ln^2 n \text{ and then } X \text{ backtracks to } k\}.
$$
P_\omega(A_2) \text{ and } P_\omega(A_3) \text{ are } O(n^{-100}) \text{ by Lemma 2.3. Take } \frac{1}{s} < u' < u'' < u. \text{ If } A_1 \text{ happens then there exists } \tilde{k} \in [k - \ln^2 n, k + \ln^2 n] \text{ which is visited more than } n^{u''} \text{ times. However the number of visits to } \tilde{k} \text{ has geometric distribution with mean } \rho_k < C(\omega)n^{u''}. \text{ Thus } P_\omega(A_1) \ll n^{-100} \text{ proving the claim. Combining Lemma 2.1 with the result of } [3] \text{ we see that}
$$
$$
P(T_k = m)\sqrt{2\pi k\tilde{D}} \exp \left( \frac{(m - E_\omega T_k)^2}{2\tilde{D}^2 k} \right) \to 1 \text{ as } k \to \infty
$$
uniformly for $|m - E_\omega T_k| \leq \tilde{R}\sqrt{k}$. Combining the last two estimates we see that if
$$
|m - E_\omega T_k| \leq \tilde{R}\sqrt{k}
$$
then
$$
P_\omega(X_n = k) = \left[ \sum_{j=0}^{n^u} P_\omega(T_k = n-j)P_\omega(X_j = k | X_0 = k) \right] + O(n^{-100}).
$$
For $j \in [0, n^u]$
$$
P_\omega(T_k = n-j) \sim \frac{1}{\sqrt{2\pi k\tilde{D}}} \exp \left( - \frac{(n - E_\omega T_k)^2}{2\tilde{D}^2 k} \right).$$
On the other hand
\[ \sum_{j=0}^{n} \mathbb{P}_\omega(X_j = k|X_0 = k) = \rho_k + O(n^{-100}\rho_k) = \rho_k + O(n^{-99}). \]

Thus
\[ \mathbb{P}_\omega(X_n = k) \sim \frac{\rho_k}{\sqrt{2\pi k\bar{D}}} \exp\left(-\frac{(n - \mathbb{E}_\omega T_k)^2}{2\bar{D}^2 k}\right). \]

Next we claim that given $R$ we can take $\hat{R}$ so large that (1.2) implies (3.1). Indeed we have
\[ n - \mathbb{E}_\omega T_k = (n - \mathbb{E}_\omega T_{b_n}) + (\mathbb{E}_\omega T_{b_n} - \mathbb{E}_\omega T_k). \]

Observe that by definition $\mathbb{E}_\omega T_{b_n-1} < n \leq \mathbb{E}_\omega T_{b_n}$ and by Lemma 2.4 $\mathbb{E}_\omega(T_{b_n} - T_{b_n-1}) = o(\sqrt{n})$ so that the first term in (3.3) is $o(\sqrt{n})$. Next, by Ergodic Theorem $\frac{b_n}{n} \to \frac{1}{a}$ so Lemma 2.4 implies that
\[ \mathbb{E}_\omega(T_{b_n} - T_k) = a(b_n - k) + o(\sqrt{n}). \]

This implies (3.1) and shows moreover that
\[ \frac{(n - \mathbb{E}_\omega T_k)^2}{k} \sim \frac{a^3(b_n - k)^2}{n}. \]

Combining this with (3.2)
\[ \mathbb{P}_\omega(X_n = k) \sim \frac{\sqrt{a}\rho_k}{\sqrt{2\pi n\bar{D}}} \exp\left(-\frac{(k - b_n)^2 a^3}{2D^2 n}\right) = \frac{\rho_k}{\sqrt{2\pi nDa}} \exp\left(-\frac{(k - b_n)^2 a^3}{2D^2 n}\right) \]
where $D = \bar{D}/a^{3/2}$. \hfill \Box

4. Annealed LLT.

Proof of Theorem 3. The result would be immediate if $b_n$ and $\rho_k$ were independent. This is not the case. However, they are almost independent. Namely by Lemma 2.1
\[ b_n = b_{n-n^{3/4}} + \frac{n^{3/4}}{a} + \varepsilon_n \]
where $\mathbb{P}(|\varepsilon_n| \geq n^{(3/8)+\eta}) \to 0$ for each $\eta > 0$. Also by (1.3) we have
\[ \mathbb{P}\left(\left|b_{n-n^{3/4}} - \frac{n-n^{3/4}}{a}\right| \geq n^{(1/2)+\eta}\right) \to 0 \text{ for each } \eta > 0. \]

Hence we can approximate $b_n$ by $\tilde{b}_n = \min(b_{n-n^{3/4}}, \frac{n}{a} - n^{5/8}) + n^{3/4}/a$. Note that if (1.4) holds then $\tilde{b}_n$ and $\rho_k$ are independent since the former depends only on the environment to the left of $\frac{n}{a} - n^{5/8}$ while the later depends only on environment to the right of $k$. Indeed if the walker is at $k - 1$ then he visits $k$ with probability 1 so $\rho_k$ is determined by the
probability that the walker starting from \( k + 1 \) does not return to \( k \).)

Thus,

\[
\sqrt{2\pi n} P(X_n = k) = \sqrt{2\pi n} E(\omega(X_n = k)) \sim E \left( \frac{1}{D} \exp \left( -\frac{(k - \tilde{b}_n)^2}{2nD^2} \rho_k \right) \right)
\]

\[
= E \left( \frac{\rho_k}{a} \right) E \left( \frac{1}{D} \exp \left( -\frac{(k - \tilde{b}_n)^2}{2nD^2} \right) \right).
\]

The first factor equals to 1 while due to (1.3) the second factor is asymptotic to

\[
\frac{1}{D} \exp \left( -\frac{(k - \frac{n}{a})^2}{2D^2n} \right).
\]

The result follows. \( \square \)

5. Environment as seen from the particle.

Proof of Theorem 4. Due to the properties of the product topology it suffices to consider the case where \( \Phi \) depends only on \( \{(p_j, q_j, r_j)\}_{|j| \leq M} \).

We consider the case where \( M = 0 \), the general case is completely similar except for notational complications. So we assume that \( \Phi(\omega) = \phi(p_0, q_0, r_0) \). Denote \( \phi_k = \phi(p_k, q_k, r_k) \). We have

\[
E_\omega(\tau_{X_n} \omega) = \sum_{k=-\infty}^{\infty} P_\omega(X_n = k) \phi_k.
\]

By Theorem 2 given \( \varepsilon \) we can find \( R \) and \( n = n(\omega) \) such that for \( n \geq n(\omega) \)

\[
\left| E_\omega(\tau_{X_n} \omega) - \sum_{|k-b_n| \leq R\sqrt{n}} P_\omega(X_n = k) \phi_k \right| \leq \varepsilon.
\]

Divide \([b_n - R\sqrt{n}, b_n + R\sqrt{n}]\) into intervals \( I_j \) of length \( n^u \) for some \( \frac{1}{2} < u < \frac{1}{2} \). Let \( k_j \) be the center of \( I_j \). Theorem 2 allows us to approximate \( E_\omega(\Phi(\tau_{x} \omega)) \) by

\[
\frac{1}{\sqrt{2\pi nDa}} \sum_j \left[ \exp \left( -\frac{(k_j - n)^2}{2D^2n} \right) \right] \sum_{k \in I_j} \rho_k \phi_k.
\]

Lemma 2.2 allows us to cutoff the last expression as follows

\[
E_\omega(\Phi(\tau_{x} \omega)) \sim \frac{1}{\sqrt{2\pi nDa}} \sum_j \left[ \exp \left( -\frac{(k_j - n)^2}{2D^2n} \right) \right] \sum_{k \in I_j} \tilde{\rho}_k \phi_k
\]

\[
\tilde{\rho}_k = \rho_k \exp \left( -\frac{(k - \tilde{b}_n)^2}{2nD^2} \right).
\]
where \( \bar{\rho}_k = \rho_k 1_{\rho_k < n^v} \) for some \( \frac{1}{s} < \bar{u} < u \). Let \( A = E(\rho_0 \phi_0) \). By Borel-Cantelli Lemma it suffices to prove that
\[
P\left( \sum_{k=n}^{n+N} [\bar{\rho}_k \phi_k - A] \geq \varepsilon N \right) \leq \frac{C}{N^{100}}
\]
where \( N \) is of order \( n^u \). By stationarity we may assume that \( n = 0 \). Pick \( v \) such that \( 2v < u < sv \) and split \( \bar{\rho}_k = \rho^h_k + \rho^l_k \) where \( \rho^h_k = \rho_k 1_{n^v < \rho_k \leq n^w} \), \( \rho^l_k = \rho_k 1_{\rho_k \leq n^v} \). Denote \( Z = \sum_{k=0}^{N} [\bar{\rho}_k \phi_k - A] \) then \( Z = Z^l + Z^h + N \tilde{A} \) where
\[
Z^l = \sum_{k=0}^{N} [\rho^l_k \phi_k - A^l], \quad Z^h = \sum_{k=0}^{N} \rho^h_k \phi_k, \quad A^l = E(\rho^l_0 \phi_0), \quad \tilde{A} = A - A^l.
\]
Note that \( \tilde{A} \to 0 \) so it is enough to show that
\[
P(|Z^l| > \varepsilon N) = O(N^{-100}), \quad P(|Z^h| > \varepsilon N) = O(N^{-100}).
\]
We need

**Lemma 5.1.** (cf [4]) For each \( d \) there is \( K \) such that if \( k_1, k_2 \ldots k_d \) satisfy
\[
|k_{i1} - k_{i2}| > K \ln N
\]
we have
\[
P(\rho_{k_i} > n^v \text{ for } i = 1 \ldots d) \leq \frac{C}{n^{ud}}.
\]
Note that if \( |Z^h| > \varepsilon N \) then there are \( k_1, k_2 \ldots k_d \) satisfying (5.1) such that \( \rho_{k_i} > n^v \). By Lemma 5.1 the probability of such an event is \( O(n^{ud - svd}) \) which can be made less than \( N^{-100} \) if \( d \) is large enough since \( sv > u \).

It remains to handle \( Z^l \). Split \( [0, N] \) into segments \( J_j \) of length \( n^w \) where \( w \ll 1 \). Let
\[
Z^l_j = \sum_{k \in J_j} [\rho^l_k \phi_k - A^l], \quad Z^l_{\text{odd}} = \sum_{j - \text{odd}} Z^l_j, \quad Z^l_{\text{even}} = \sum_{j - \text{even}} Z^l_j.
\]
It suffices to show that
\[
P(|Z^l_{\text{odd}}| > \varepsilon N) = O(N^{-100}), \quad P(|Z^l_{\text{even}}| > \varepsilon N) = O(N^{-100}).
\]
We shall prove the first inequality, the second one is similar. Lemma 2.3 easily implies that
\[
P\left( Z^l_j - Z^l_j^* > \frac{1}{N} \right) = O(N^{-100})
\]
where
\[ Z^*_j = \sum_{k \in J_j} [\rho^*_k \phi_k - A^*], \quad \rho^*_k = \hat{\rho}_k 1_{\hat{\rho}_k < n^u}, \]
\(\hat{\rho}_k\) is expected number to visits to \(k\) before \(T_{k,j+1}\) and \(A^* = E(\rho_0^* \phi_0)\).

Since \(\{Z^*_j\}_{j-odd}\) are iid random variables satisfying
\[ E(Z^*_j) = 0, \quad |Z^*_j| \leq C n^{v+w} \]
we have
\[ E \left( \left( \sum_{j-odd} Z^*_j \right)^{2d} \right) \leq C n^{2vd+2wd} n^{ud} = C n^{(2v+u+2w)d}. \]

By Markov inequality
\[ P \left( \left| \sum_{j-odd} Z^*_j \right| > \varepsilon N \right) \leq C(\varepsilon) n^{(2v+2w-u)d} \]
which is less than \(N^{-100}\) if \(w\) is small enough and \(d\) is large enough since \(2v < u\). (5.2) follows and hence Theorem 4 is proven. \(\square\)

References