STAT400. Sample questions for midterm 1.

1. *Let A and B are sets such that $P(A) = 0.6$, $P(B) = 0.4$ and $P(AB) = 0.3$.*
(a) Compute $P(A \cup B)$.
(b) Compute $P(A|B)$.
(c) Compute $P((A \cup B)')$.

Solution. *(a)* $P(A \cup B) = P(A) + P(B) - P(AB) = 0.6 + 0.4 - 0.3 = 0.7$. *(b)* $P(A|B) = \frac{P(AB)}{P(B)} = \frac{0.3}{0.4} = 0.75$. *(c)* $P((A \cup B)') = 1 - P(A \cup B) = 1 - 0.7 = 0.3$.

2. *Let A denote the event that a customer at a certain store has visa, B denote the event that she has a master card and C denote the event that she has a discovery card. Suppose that $P(A) = 0.70$, $P(B) = 0.63$, $P(C) = 0.33$, $P(AB) = 0.50$, $P(AC) = 0.25$, $P(BC) = 0.23$, $P(A \cup B \cup C) = 0.88$. Compute the probabilities that a customer*
(a) owns none of the cards;
(b) has all three types;
(c) have exactly one type.

Solution. *(a)* $P((A \cup B \cup C)') = 1 - P(A \cup B \cup C) = 1 - 0.88 = 0.12$.

$$(b) \quad P(ABC) = P(A \cup B \cup C) - P(A) - P(B) - P(C) + P(AB) + P(AC) + P(BC) =$$
$$0.88 - (0.70 + 0.63 + 0.33) + (0.50 + 0.25 + 0.23) = 0.20.$$

(c) $P(ABC') = P(AB) - P(ABC) = 0.30$, $P(AB'C) = P(AC) - P(ABC) = 0.05$, $P(A'BC) = P(BC) - P(ABC) = 0.03$. *Hence* $P(AB'C') = P(A) - P(ABC) - P(ABC') - P(AB'C) = 0.15$, $P(A'BC') = P(B) - P(ABC) - P(ABC') - P(A'BC) = 0.10$, $P(A'B'C) = P(C) - P(ABC) - P(A'BC) - P(AB'C) = 0.05$. *Therefore the probability that the customer owns exactly one card is* $0.15 + 0.10 + 0.05 = 0.30$.

3. *How many ways are there to distribute 6 different toys and 9 differnt candies between 3 children*
(a) without any restrictions;
(b) If the first child needs to get exactly two toys and 3 candies;
(c) If all children need to get exactly two toys and 3 candies.

Solution. *(a) There are 15 objects. There are 3 possibilities for each object. So the answer is* 3^{15}.

(b) There are $\binom{6}{2}$ *ways to choose toys for the first child* $\binom{9}{3}$ *ways to choose candies for her. After the first child is served 10 objects remain so similar to part (a) there are* 2^{10} *ways to distribute them. Thus the answer is* $\binom{6}{2}\binom{9}{3}2^{10}$.

(c) There are $\begin{pmatrix} 6 \\ 2,2,2 \end{pmatrix} = \frac{6!}{(2!)^3}$ *ways todistribute the toys and* $\begin{pmatrix} 9 \\ 3,3,3 \end{pmatrix} = \frac{6!}{(2!)^3}$ *ways todistribute the candy so the answer is* $\frac{6!9!}{(2!)^3(3!)^3}$.

4. *(a) How many ways are there to divide 52 cards between four people so that each person gets 13 cards.*

(b) Find the probability that players 1 *and* 3 *have two aces each.*

(c) Find the probability that player 1 *has all his cards red (that is, they are either hearts or diamonds) and has exactly one ace.*

Solution. *(a)* $\begin{pmatrix} 52 \\ 13,13,13,13 \end{pmatrix} = \frac{52!}{(13!)^4}$. *(b) There are* $\begin{pmatrix} 4 \\ 2 \end{pmatrix}$ *to distribute the aces and* $\begin{pmatrix} 48 \\ 11,13,11,13 \end{pmatrix} = \frac{48!}{(11!)^2(13!)^2}$. *So the answer is* $\frac{48!4!}{(11!)^2(13!)^2(2!)^2} : \frac{52!}{(13!)^4} = \frac{48!4!(13!)^2}{52!(11!)^2(2!)^2}$.

(c) There are $\begin{pmatrix} 52 \\ 13 \end{pmatrix}$ *ways to choose cards for the first player. If we want him to have only red cards with exactly one ace then there are 2 ways to choose ace and* $\begin{pmatrix} 24 \\ 12 \end{pmatrix}$ *ways to choose red non ace cards. So the answer is* $2\begin{pmatrix} 24 \\ 12 \end{pmatrix} / \begin{pmatrix} 52 \\ 13 \end{pmatrix}$.

5. *The first box contains 2 red and 3 blue balls and the second box has 3 red and 2 blue balls. A ball is chosen at random from the first box and put in the second box. Then a ball is chosen at random from the second box.*

(a) Find the probability that the first ball is blue;

(b) Find the probability that both balls are blue;

(c) Find the probability that the balls have the same color.

Solution. *(a)* $P(B_1) = \frac{3}{5}$. *(b)* $P(B_1B_2) = P(B_1)P(B_2|B_1) = \frac{3}{5} \times \frac{3}{6} = \frac{3}{10}$.

(c) $P(R_1R_2) = P(R_1)P(R_2|R_1) = \frac{2}{5} \times \frac{4}{6} = \frac{4}{15}$. *So the answer is* $P(B_1B_2) + P(R_1R_2) = \frac{17}{30}$.

6. *In a certain city 70% of drivers are careful and 30% are agressive. Assume that a careful driver has 10% chance of getting a speed ticket independent of the past performance and an agressive driver has 30% chance of getting a speed ticket independent of the past performance.*

(a) Find the probability that a careful driver will get a ticket during the first year but not during the second year.

(b) Find the probability that an agressive driver will get a ticket during the first year but not during the second year.

(c) Given that a driver was ticket during the first year but not during the second year find the probability that he is agressive.

Solution. *Let C be the event that the driver is careful, A be the event that he is agressive, T be the event that he gets ticket during some year and F be the event that he is ticket free. Then*

(a) $P(T_1F_2|C) = P(T_1|C)P(F_2|C) = 0.1 \times 0.9 = 0.09.$
(b) $P(T_1F_2|A) = P(T_1|A)P(F_2|A) = 0.3 \times 0.7 = 0.21.$
(c) By Bayes Theorem

$$P(A|T_1F_2) = \frac{P(A)P(T_1F_2|A)}{P(A)P(T_1F_2|A) + P(C)P(T_1F_2|C)} = \frac{0.3 \times 0.21}{0.3 \times 0.21 + 0.7 \times 0.09} = 0.5.$$

7. *There are two identical urns. The first urn contains 5 balls numbered 1 to 5. The first urn contains 10 balls numbered 1 to 10. One urn is chosen at random and then 3 balls are selected from that urn without replacement. Let A be the event that the first urn is chosen, B be the event that the second urn is chosen and C be the event that the maximum number of the balls chosen is 4. Compute*

(a) $P(C|A)$ *and* $P(C|B)$;
(b) $P(C)$;
(c) $P(A|C)$.

Solution. *(a) Note that in order for maximum to be equal to 4 we need one of the three balls to have number 4 and remaining two be from the set $\{1, 2, 3\}$. So*

$$P(C|A) = \frac{\begin{pmatrix} 3 \\ 2 \end{pmatrix}}{\begin{pmatrix} 5 \\ 3 \end{pmatrix}} = \frac{3}{10}, \quad P(C|B) = \frac{\begin{pmatrix} 3 \\ 2 \end{pmatrix}}{\begin{pmatrix} 10 \\ 3 \end{pmatrix}} = \frac{1}{40}.$$

(b) $P(C) = \frac{1}{2} \times \frac{3}{10} + \frac{1}{2} \times \frac{1}{40} = \frac{13}{80}$. *(c)* $P(A|C) = \frac{P(AC)}{P(C)} = \frac{1}{2} \times \frac{3}{10} / \frac{13}{80} = \frac{12}{13}$.

8. *Consider the chain below. Suppose that each element works with probability $2/3$ independently of the others.*

(a) Find the probability that the chain works.
(b) Find the probability that the chain works given that the first element works.
(c) Find the probability that the first element works given that the chain works.

Solution. *Let A_j be event that element j works and C be the event that the whole chain works. Then*

(a) $P(C) = P(A_1A_2 \cup A_3) = P(A_1A_2) + P(A_3) - P(A_1A_2A_3) = \frac{2}{3} + \left(\frac{2}{3}\right)^2 - \left(\frac{2}{3}\right)^3 = \frac{22}{27}.$
(b) $P(C|A_1) = P(A_2 \cup A_3) = P(A_2) + P(A_3) - P(A_2A_3) = 2 \times \frac{2}{3} - \frac{4}{9} = \frac{8}{9}.$

(c) $P(A_1|C) = \frac{P(A_1)P(C|A_1)}{P(C)} = \frac{2}{3}\frac{8}{9}/\frac{22}{27} = \frac{8}{11}$.

9. *Let A, B and C be mutually independent and $P(A) = 1/2$, $P(B) = 1/3$, $P(C) = 1/4$. Let D be the event that exactly one of events A, B and C occurs.*
(a) Compute $P(D)$.
(b) Compute $P(A|D)$, $P(B|D)$, $P(C|D)$.
(c) Compute $P(A|(B \cup C))$.

Solution. (a) $P(D) = P(AB'C') + P(A'BC') + P(A'B'C) = \frac{1}{2}\times\frac{2}{3}\times\frac{3}{4} + \frac{1}{2}\times\frac{1}{3}\times\frac{3}{4} + \frac{1}{2}\times\frac{2}{3}\times\frac{1}{4} = \frac{6}{24} + \frac{3}{24} + \frac{2}{24} = \frac{11}{24}$.
(b) $P(A|D) = \frac{P(AB'C'}{P(D)} = \frac{6}{11}$, $P(B|D) = \frac{P(A'BC'}{P(D)} = \frac{3}{11}$, $P(C|D) = \frac{P(A'B'C}{P(D)} = \frac{2}{11}$.
(c) Due to independence $P(A|B \cup C) = P(A) = \frac{1}{2}$.

10. *An urn contains 4 red and 6 blue balls. Balls are taken at random without replacement until both colors are present.*
(a) Find the probability that two balls are enough.
(b) Find the probability that three balls are not enough.
(c) Find the probability that exactly three balls are needed.

Solution. *Let X be the number of trials needed.*

(a) There are $\binom{10}{2}$ ways to choose two balls. If $X = 2$ then one ball should be red (4 possibilities) and the other blue (6 possibilites). Thus $P(X = 2) = \frac{6\times 4}{\binom{10}{2}} = \frac{8}{15}$.

(b) If $X > 3$ then the first three ball should be all red or all blue. Thus

$$P(X > 2) = \frac{\binom{4}{3} + \binom{6}{3}}{\binom{10}{3}} = \frac{3}{15}.$$

(c) $P(X \leq 3) = 1 - P(X > 3) = \frac{12}{15}$. $P(X = 3) = P(X \leq 3) - P(X = 2) = \frac{4}{15}$.

11. *An urn has 5 red, 5 green and 5 blue balls. 3 balls are chosen at random (without replacement). Let X be the number of different colors chosen.*
(a) Compute the probability mass function of X.
(b) Compute the cumulative distribution function of X.
(c) Compute EX and VX.

Solution. (a) There are $\binom{15}{3}$ ways to chose the balls. If they are of the same color then we need (I) to choose color (3 possibilities) and (II) chose the balls of that color ($\binom{5}{3}$

possibilities). Thus $P(X=1)=\dfrac{3\times\begin{pmatrix}5\\3\end{pmatrix}}{\begin{pmatrix}15\\3\end{pmatrix}}=\frac{6}{91}$. *If the balls are of the two colors then we need to choose (I) colors of the pair, the single ball and the void color (*$3!$ *possibilities), choose a pair (*$\begin{pmatrix}5\\3\end{pmatrix}$ *possibilities) and choose a single ball (5 possibilities). So* $P(X=2)=\dfrac{3\times\begin{pmatrix}5\\2\end{pmatrix}\times 5}{\begin{pmatrix}15\\3\end{pmatrix}}=\frac{60}{91}$. *If all balls are of different colors there are* 5^3 *ways to choose the balls since there 5 possibilities for each color, so* $P(X=3)=\dfrac{5^3}{\begin{pmatrix}15\\3\end{pmatrix}}=\frac{25}{91}$. *Thus the probability mass function takes form*

X	*1*	*2*	*3*
p	$\frac{6}{91}$	$\frac{60}{91}$	$\frac{25}{91}$

.

(b) Since $\frac{6}{91}+\frac{60}{91}=\frac{66}{91}$ *and* $\frac{66}{91}+\frac{25}{91}=1$ *we have* $F(x)=\begin{cases}0 & \text{if } x<1\\ \frac{6}{91} & \text{if } 1\le x<2\\ \frac{66}{91} & \text{if } 2\le x<3\\ 1 & \text{if } x\ge 3\end{cases}$.

(c) $EX=1\times\frac{6}{91}+2\times\frac{60}{91}+3\times\frac{25}{91}=\frac{201}{91}\approx 2.21$ $EX^2=1\times\frac{6}{91}+4\times\frac{60}{91}+9\times\frac{25}{91}=\frac{471}{91}\approx 5.18$ $VX=EX^2-(EX)^2=\frac{2460}{8281}\approx 0.29$.

12. *The probability mass function of X is given in the following table*

X	*0*	*1*	*2*	*3*
p	*0.1*	*0.2*	*0.3*	*0.4*

.
Let $Y=(X-1)^2$.
(a) Compute the probability mass function of Y.
(b) Compute the cumulative distribution function of Y.
(c) Compute EY and VY.

Solution. *(a)* $P(Y=0)=P(X=1)=0.2$, $P(Y=1)=P(X=0)+P(X=2)=0.1+0.3=0.4$. $P(Y=4)=P(X=3)=0.4$. *So the probability mass function of Y takes form*

X	*0*	*1*	*4*
p	*0.2*	*0.4*	*0.4*

.

(b) Since $0.2 + 0.4 = 0.6$, $0.6 + 0.4 = 1$ *we have* $F(x) = \begin{cases} 0 & \textit{if } x < 0 \\ 0.2 & \textit{if } 0 \le x < 1 \\ 0.6 & \textit{if } 1 \le x < 4 \\ 1 & \textit{if } x \ge 4 \end{cases}$.

(c) $EY = 0 \times 0.2 + 1 \times 0.4 + 4 \times 0.4 = 2$. $EY^2 = 0 \times 0.2 + 1 \times 0.4 + 16 \times 0.4 = 6.8$ $VY = 6.8 - 2^2 = 2.8$.