A FEM for the Square Root of the Laplace Operator

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Outline of Topics

The Square Root of the Laplace Operator

The Harmonic Extension and the Truncated Problem

The Galerkin Approximation of the Harmonic Extension

Numerical Implementation in deal.ii

Numerical Results
The Continuos Problem

The problem we shall be concerned with reads as follows: Given a smooth enough function $f$, find $u$ such that

$$\begin{cases} (-\Delta)^{1/2} u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial \Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^d$, with $d = 1, 2$ is a bounded domain with a smooth boundary $\partial \Omega$ and $(-\Delta)^{1/2}$ denotes the square root of the Laplace operator $-\Delta$ in $\Omega$ with zero boundary values on $\partial \Omega$. 
Concerning applications, nonlocal operators are of importance in a wide range of applications:

- Finance.
- Image Processing.
- Quasi-geostrophic flow models.
- Modeling hydraulic fractures and the evolution of a viscous liquid thin film.

The development of **efficient computational solution techniques for this problem is fundamental.**
Definition of the Square Root of the Laplacian

Spectral theory of the Laplacian $-\Delta$ in a smooth bounded domain $\Omega$ with zero Dirichlet boundary values. There exists a sequence of eigenvalues

$$0 < \lambda_1 < \lambda_2 \leq \cdots \lambda_k \leq \cdots \rightarrow \infty$$

and,
Definition of the Square Root of the Laplacian

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and, there exists an orthonormal basis $\{\varphi_k\}$ of $L^2(\Omega)$, where $\varphi_k \in H^1_0(\Omega)$ is an eigenfunction corresponding to $\lambda_k$:

$$\begin{cases}
-\Delta \varphi_k = \lambda_k \varphi_k & \text{in } \Omega \\
\varphi_k = 0 & \text{on } \partial \Omega,
\end{cases}$$

(1)

for $k = 1, 2, \cdots$. Regularity theory $\implies \varphi_k \in C^\infty(\bar{\Omega})$ for $k = 1, 2, \cdots$. 

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A FEM for the Square Root of the Laplace Operator
The square root of the Dirichlet Laplacian, for a smooth function $u$, is given by

$$ u = \sum_{k=1}^{\infty} c_k \varphi_k \mapsto (-\Delta)^{1/2} u = \sum_{k=1}^{\infty} c_k \lambda_k^{1/2} \varphi_k. $$

Density results $\implies (-\Delta)^{1/2} : H^1_0(\Omega) \to L^2(\Omega)$. 
Definition of the Square Root of the Laplacian

The square root of the Dirichlet Laplacian, for a smooth function $u$, is given by

$$u = \sum_{k=1}^{\infty} c_k \varphi_k \mapsto (-\Delta)^{1/2} u = \sum_{k=1}^{\infty} c_k \lambda_k^{1/2} \varphi_k.$$ 

Density results $\Rightarrow (-\Delta)^{1/2} : H^1_0(\Omega) \rightarrow L^2(\Omega)$. Then if $f \in L^2(\Omega)$, we have

$$f = \sum_{k=1}^{\infty} f_k \varphi_k \Rightarrow c_k = f_k \lambda_k^{-1/2}.$$ 

Numerical disadvantages: We need to find a sufficiently large number of eigenfunctions to obtain an accurate approximation.
On the other hand, this operator can be seen as a singular integral

\[
(-\Delta)^{1/2} u(x) = C_d \int_{\Omega} \frac{u(x) - u(y)}{|x - y|^{d+1}} dy,
\]

where \(C_d\) is a normalization constant.
Definition of the square root of the Laplacian

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\[ (-\Delta)^{1/2} u(x) = C_d \int_{\Omega} \frac{u(x) - u(y)}{|x - y|^{d+1}} dy, \]

where \( C_d \) is a normalization constant.

Numerical disadvantages: the integrand is singular and the matrix obtained is dense. These inconveniences complicate the numerical computation.
Harmonic Extension


Given \(u\) defined in \(\Omega\), we consider its harmonic extension \(v\) in the cylinder \(C := \Omega \times (0, \infty)\), with \(v\) vanishing on \(\partial L C := \partial \Omega \times [0, \infty)\).

\[
\begin{align*}
-\Delta v &= 0 \quad \text{in } C = \Omega \times (0, \infty), \\
v &= 0 \quad \text{on } \partial L C = \partial \Omega \times [0, \infty), \\
\frac{\partial v}{\partial \nu} &= f \quad \text{on } \Omega \times \{0\},
\end{align*}
\]

where \(\nu\) is the unit outer normal to \(C\) at \(\Omega \times \{0\}\).
Harmonic Extension


Given \(u\) defined in \(\Omega\), we consider its harmonic extension \(v\) in the cylinder \(C := \Omega \times (0, \infty)\), with \(v\) vanishing on \(\partial_L C := \partial \Omega \times [0, \infty)\).

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\end{align*}
\]

where \(\nu\) is the unit outer normal to \(C\) at \(\Omega \times \{0\}\). Then,

\[
u = \text{tr}_\Omega v := v(\cdot, 0)
\]
Spaces for $v$ and $u$

Space for $v$:

$$H_0^1(\mathcal{C}) := \{ v \in H^1(\mathcal{C}) \mid v = 0 \text{ a.e. on } \partial_L \mathcal{C} = \partial \Omega \times [0, \infty) \}. $$

Space for $u$:

$$\mathcal{V}_0(\Omega) = H_{00}^{1/2}(\Omega) = \left[ H_0^1(\Omega), L^2(\Omega) \right]_{1/2,2} = \left\{ u \in H^{1/2}(\Omega) \left| \int_{\Omega} \frac{u^2(x)}{d(x)} \, dx < +\infty \right. \right\}. $$
Truncated Problem

Numerically, it cannot be solved because $C$ is an infinite domain $\Rightarrow$ We need to consider a suitable truncated problem.
Truncated Problem

Numerically, it cannot be solved because $C$ is an infinite domain $\implies$ We need to consider a suitable truncated problem.

Why can we truncate the problem? Given $M > 0$, $v$ satisfies

$$\|\nabla v\|^2_{L^2(\Omega \times (M, \infty))} < e^{-2\sqrt{\lambda_1 M}} \|f\|^2_{V_0(\Omega)^*}.$$  

Consider $M$ adequately large and define $v^M$ in a bounded domain $C_M := \Omega \times (0, M)$, imposing a zero Dirichlet condition on $\Omega \times \{M\}$:

$$\begin{cases}
-\Delta v^M &= 0 \quad \text{in} \; C_M = \Omega \times (0, M), \\
v^M &= 0 \quad \text{on} \; \partial_L C_M := \partial \Omega \times [0, M], \\
v^M &= 0 \quad \text{on} \; \Omega \times \{M\}, \\
\frac{\partial v^M}{\partial \nu} &= f \quad \text{on} \; \Omega \times \{0\},
\end{cases}$$
Weak Formulation of the Truncated Problem

Find $\nu^M \in H^1_0(C_M)$ such that

$$\int_{C_M} \nabla \nu^M \cdot \nabla \psi = \int_{\Omega} f_{\text{tr} \Omega} \psi, \quad \text{for all } \psi \in H^1_0(C_M).$$

$$H^1_0(C_M) := \{ \nu \in H^1(C_M) | \nu = 0 \text{ a.e. on } \partial_L C_M, \quad \text{and } \nu = 0 \text{ a.e. on } \Omega \times \{M\} \}.$$
Weak Formulation of the Truncated Problem

Find $v^M \in H^1_0(C_M)$ such that

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How good is this truncated problem?

$$M > \frac{1}{\sqrt{\lambda_1}} \ln \left( \frac{2}{\epsilon^2} \right) \implies \| v - v^M \|_{H^1_0(C_M)} \leq \epsilon \| f \|_{V_0(\Omega)^*}.$$
Galerkin Approximation

Given a family of partitions $\mathcal{T}_k$ of the domain $\mathcal{C}_M$ into quadrilateral elements, we define for $n \geq 1$

$$\nabla^{n,0} := \{ v \in C^0(\overline{\mathcal{C}_M}) : v|_T \in Q_n(T) \ \forall \ T \in \mathcal{T}_k \} \cap H^1_0(\mathcal{C}_M),$$

Galerkin approximation for $v^M$ is given by: Find $v^M_h \in \nabla^{n,0}$ such that

$$\int_{\mathcal{C}_M} \nabla v^M_h \cdot \nabla w_h = \langle f, \text{tr}_\Omega w_h \rangle, \quad \text{for all } w_h \in \nabla^{n,0}.$$
Galerkin Approximation

Given a family of partitions $\mathcal{T}_k$ of the domain $\mathcal{C}_M$ into quadrilateral elements, we define for $n \geq 1$

$$\mathbb{V}^{n,0} := \{ \mathbf{v} \in C^0(\mathcal{C}_M) : \mathbf{v}|_T \in Q_n(T) \ \forall \ T \in \mathcal{T}_k \} \cap H^1_0(\mathcal{C}_M),$$

Galerkin approximation for $\mathbf{v}^M$ is given by: Find $\mathbf{v}^M_h \in \mathbb{V}^{n,0}$ such that

$$\int_{\mathcal{C}_M} \nabla \mathbf{v}^M_h \cdot \nabla \mathbf{w}_h = \langle f, \text{tr}_\Omega \mathbf{w}_h \rangle, \quad \text{for all } \mathbf{w}_h \in \mathbb{V}^{n,0}.$$

Standard FEM theory + truncated problem property implies

$$\| \mathbf{v} - \mathbf{v}^M_h \|_{H^1_0(\mathcal{C})} \leq C \left( \varepsilon \| f \|_{H^1_0(\Omega)} + h \| \mathbf{v} \|_{H^2(\mathcal{C}_M)} \right),$$

where $h = \max_{T \in \mathcal{T}} h_T$. 
The main function is similar to the steps discussed in class. We implement $Q_k$ adaptive refinement, $Q_k$ global refinement, and $Q_k$ exponential refinement, $k \geq 1$, in dimension $d = 2, 3$.

```cpp
int main()
{
    const unsigned int dim = 3; // We can choose dim = 2 or dim = 3.
    try{
        std::cout << "Solving with Q1 elements, adaptive refinement" << std::endl
                << "============================================" << std::endl
                << std::endl;
        LaplaceProblem<dim>
            laplace_problem ( LaplaceProblem<dim>::adaptive_refinement);
        laplace_problem.run ();
        std::cout << std::endl;
    }
    Enlarge your browser window if needed. Numerical Implementation in deal.ii
Template class LaplaceProblem. The class that does all the work.

template <int dim> class LaplaceProblem{
    public:
        enum RefinementMode{global_refinement, adaptive_refinement, exp_refinement};
        LaplaceProblem (const RefinementMode refinement_mode);
        `LaplaceProblem ();
        void run ();

    private:
        void setup_system ();
        void assemble_system ();
        void solve ();
        void refine_grid ();
        void process_solution (const unsigned int cycle);
    };

Member functions. They do what their names suggest.
Some member variables:

```cpp
triangulation; dof_handler; fe; hanging_node_constraints; sparsity_pattern; system_matrix; solution; system_rhs; refinement_mode; convergence_table;
```
LaplaceProblem::run. The code is implemented in cycles. For each kind of mesh, we consider a fixed number of cycles and we solve.

```cpp
template <int dim> void LaplaceProblem<dim>::run (){
    for (unsigned int cycle=0; cycle<7; ++cycle){
        if (cycle == 0){
            // Here we generate the first mesh!
            GridGenerator::subdivided_hyper_rectangle(triangulation,
                subdivisions2,p1,p2,false);
        }else{
            refine_grid ();
        }
    }
```
We consider $\Omega = (0, 1)$ and $f(x) = \pi \sin(\pi x)$, then $C_M = (0, 1) \times (0, M)$ $u(x) = \sin(\pi x)$ and $v(x, y) = \sin(\pi x)e^{-\pi y}$.

$$\|v - v_h^M\|_{H^1_0(C)} \leq C \left( \epsilon \|f\|_{\mathcal{V}_0(\Omega)^*} + h\|v\|_{H^2(C_M)} \right), \quad \epsilon = \epsilon(M)$$
We consider $\Omega = (0, 1)$ and $f(x) = \pi \sin(\pi x)$, then $C_M = (0, 1) \times (0, M)$ $u(x) = \sin(\pi x)$ and $v(x, y) = \sin(\pi x) e^{-\pi y}$.

$$\|v - v_h^M\|_{H^1_0(C)} \leq C \left( \epsilon \|f\|_{V_0(\Omega)^*} + h \|v\|_{H^2(C_M)} \right), \quad \epsilon = \epsilon(M)$$

$M$ should change with $h$ to get $\epsilon \approx h$

$$M = -\frac{2}{\pi} \ln \left( \frac{h}{\sqrt{2}} \right)$$
Some Global Meshes

Figure: Degrees of freedom: 20, 81, 238 respectively.
Results with Global Refinement

\[ v_h^M \] with 238 degrees of freedom.

**Figure:** \[ v_h^M \] with 238 degrees of freedom.
Results with Global Refinement

Computing $L^2$ and $H^1$ error norms.

Vector<float> difference_per_cell (triangulation.n_active_cells());

VectorTools::integrate_difference (dof_handler, solution, Solution<dim>(),
    difference_per_cell, QGauss<dim>(q_points_norm ),
    VectorTools::L2_norm);

const double L2_error = difference_per_cell.l2_norm();
Results with Global Refinement

Figure: Decay of the $L^2$ and $H^1$ norms of the error.
Estimates for the Function $u$

What about $u$?
Estimates for the Function $u$

What about $u$? Trace results imply an estimate for $u$

\[
\| u - u_h^M \|_{H^{1/2}_0(\Omega)} \leq \| \nu - \nu_h^M \|_{H^1_0(C)} \\
\leq C \left( \epsilon \| f \|_{V_0(\Omega)^*} + h \| \nu \|_{H^2(C_M)} \right), \quad \epsilon = \epsilon(M)
\]

However, notice that this estimate is not optimal! Optimal estimate

\[
\| u - u_h^M \|_{H^{1/2}_0(\Omega)} \leq C \left( \epsilon \| f \|_{V_0(\Omega)^*} + h^{3/2} \| f \|_{H^1(\Omega)} \right), \quad \epsilon = \epsilon(M)
\]
Results with Global Refinement

**Figure:** Decay of the $L^2$, $H^{1/2}$ and $H^1$ norms of the error.
Exponential Refinement

We exploit the behavior of the real solution

$$v(x, y) = \sum c_k \varphi_k e^{-\sqrt{\lambda_k}y}, \quad \text{for all } (x, y) \in \mathcal{C},$$

to design an exponential mesh. **2D case:** We do global refinement in $x$ and exponential refinement in $y$. Using interpolation results we get

$$\|v - v_h^M\|^2_{H^1_0(C_M)} \leq C \sum_{k=1}^{N_y} (h_k^y)^3 e^{-\sqrt{\lambda_1}y_k} \leq CN^{-1},$$
Exponential Refinement

We exploit the behavior of the real solution

\[ v(x, y) = \sum c_k \varphi_k e^{-\sqrt{\lambda_k} y}, \quad \text{for all } (x, y) \in C, \]

to design an exponential mesh. 2D case: We do global refinement in \( x \) and exponential refinement in \( y \). Using interpolation results we get

\[
\| v - v_h^M \|_{H^1_0(C_M)}^2 \leq C \sum_{k=1}^{N_y} (h_k^y)^3 e^{-\sqrt{\lambda_1} y_k} \leq CN^{-1},
\]

and finally we obtain

\[
y_{k+1} = y_k + \frac{1}{k} N^{-2/3} e^{\sqrt{\lambda_1}/3y_k}.
\]
Some Exponential Meshes

**Figure:** Degrees of freedom: 54, 170, 627 respectively.
Results with Exponential Refinement

Figure: Decay of the $L^2$, $H^{1/2}$ and $H^1$ norms of the error.
Adaptive Refinement

The estimate

$$\| \mathbf{v} - \mathbf{v}_h^M \|_{H^1_0(C)} \leq C \left( \varepsilon \| f \|_{V_0(\Omega)} + h \| \mathbf{v} \|_{H^2(C_M)} \right),$$

is not computable and provides only asymptotic information. We create a mesh adapted to the function $\mathbf{v}$. Basic ingredient:

$$\| \mathbf{v} - \mathbf{v}_h^M \|_{H^1_0(C)} \leq C_1 E_T(\mathbf{v}_h^M) \leq C_2 \left( \| \mathbf{v} - \mathbf{v}_h^M \|_{H^1_0(C)} + \text{osc}_T(\mathbf{v}_h^M) \right)$$
The estimate

\[ \| \nu - \nu_h^M \|_{H^1_0(\Omega)} \leq C \left( \epsilon \| f \|_{V_0(\Omega)} + h \| \nu \|_{H^2(\Omega)} \right), \]

is not computable and provides only asymptotic information. We create a mesh adapted to the function \( \nu \). Basic ingredient:

\[ \| \nu - \nu_h^M \|_{H^1_0(\Omega)} \leq C_1 \mathcal{E}_T(\nu_h^M) \leq C_2 \left( \| \nu - \nu_h^M \|_{H^1_0(\Omega)} + \text{osc}_T(\nu_h^M) \right) \]

Error Estimator Implemented in deal.ii

\[ \mathcal{E}^2_T(\nu_h^M, T) = \frac{h_T}{24} \int_{\partial T} \left[ \frac{\partial \nu_h^M}{\partial \nu} \right] \]
3D Numerical Example

We consider $\Omega = (0, 1) \times (0, 1)$ and $f(x) = \sqrt{2\pi} \sin(\pi x) \sin(\pi y)$, then $u(x) = \sin(\pi x) \sin(\pi y)$ and $v(x, y) = \sin(\pi x) \sin(\pi y) e^{-\sqrt{2\pi} y}$.

We have optimal estimates for every refinement: adaptive, exponential and global. We show the results obtained using Adaptivity.

**Figure**: Degrees of freedom: 13435.
## Convergence Table for $v$

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<th>$H^1$-error</th>
<th>$L^2$-error</th>
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<td>4.790e-02</td>
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<tr>
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<td>6.252e-01</td>
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<tr>
<td>6</td>
<td>5.983e-02</td>
<td>6.091e-04</td>
</tr>
</tbody>
</table>

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A FEM for the Square Root of the Laplace Operator
Results with Adaptive Refinement

Figure: Decay of the $L^2$, $H^{1/2}$ and $H^1$ norms of the error.
Results with Adaptive Refinement

Figure: $u_h^M$ and $v_h^M$ with 13435 degrees of freedom.
We consider the following numerical example. Given a smooth function $f(x, y) = \sqrt{2\pi} \sin(\pi x) \sin(\pi y)$, find $u$ such that

$$
\begin{cases}
(\Delta)^{1/2}u = f & \text{in } \Omega, \\
u = 0 & \text{on } \partial\Omega,
\end{cases}
$$

where $\Omega = (-1, 1)^2 - \text{disk}(0,0)(0.5)$. 

GridGenerator::hyper_cube_with_cylindrical_hole (triangulation, inner_radius, outer_radius, extension, repetition, true);

**Figure:** Meshes for $z = 0$ and $z = 4$.  

Outline

1. The Square Root of the Laplace Operator
2. The Harmonic Extension and the Truncated Problem
3. The Galerkin Approximation of the Harmonic Extension
4. Numerical Implementation in deal.ii
5. Numerical Results

Figure: Degrees of freedom: 22492.

Figure: $u_h^M$ computed with 22492 degrees of freedom.
Questions?