DOUBLE ETA POLYNOMIALS AND EQUIVARIANT GIAMBELLI FORMULAS

HARRY TAMVAKIS

Abstract. We use Young's raising operators to introduce and study *double eta polynomials*, which are an even orthogonal analogue of Wilson's double theta polynomials. Our double eta polynomials give Giambelli formulas which represent the equivariant Schubert classes in the torus-equivariant cohomology ring of even orthogonal Grassmannians, and specialize to the single eta polynomials of Buch, Kresch, and the author.

1. Introduction

Let k be a positive integer and $\mathrm{OG} = \mathrm{OG}(n-k, 2n)$ be the Grassmannian which parametrizes isotropic subspaces of dimension $n-k$ in the vector space $\mathbb{C}^{2n}$, equipped with an orthogonal form. The eta polynomials $H_\lambda(c)$ of Buch, Kresch, and the author [BKT2, T2] are Giambelli polynomials which represent the Schubert classes in the cohomology ring of OG. Our aim here is to define *double eta polynomials* $H_\lambda(c\,|\,t)$ which represent the equivariant Schubert classes in the equivariant cohomology ring $\mathrm{H}_T^*(\mathrm{OG})$, where T is a maximal torus of the complex even orthogonal group. The companion theory of double theta polynomials for the symplectic and odd orthogonal Grassmannians was provided in [TW]; we refer the reader there for more information, and to [T1, T2] for the solution of the equivariant Giambelli problem in general, for any isotropic partial flag variety.

The Schubert classes on $\mathrm{OG}(n-k,2n)$ are parametrized by the k-Grassmannian elements of the Weyl group $\widetilde{W}_n$ for the root system D_n. The group $\widetilde{W}_n$ is the subgroup of the hyperoctahedral group consisting of all signed permutations with an even number of sign changes. We define the embedding $\widetilde{W}_n \hookrightarrow \widetilde{W}_{n+1}$ by adjoining the fixed point $n+1$, let $\widetilde{W}_\infty := \cup_n \widetilde{W}_n$, and work initially in the latter group. An element $w = (w_1, w_2, \ldots)$ of $\widetilde{W}_\infty$ is *k-Grassmannian* if and only if

$$|w_1| < w_2 < \cdots < w_k \quad \text{and} \quad w_{k+1} < w_{k+2} < \cdots .$$

Our Giambelli formulas require the equivalent parametrization of the Schubert classes by the typed k-strict partitions of [BKT1]. An integer partition $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ is *k-strict* if no part λ_j greater than k is repeated. A *typed k-strict partition* is a pair consisting of a k-strict partition λ together with an integer $\mathrm{type}(\lambda) \in \{0,1,2\}$, which is positive if and only if $\lambda_j = k$ for some index j.

There is a bijection between the k-Grassmannian elements of $\widetilde{W}_\infty$ and typed k-strict partitions, obtained as follows. If the element w corresponds to the typed

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partition λ, then for each $j \geq 1$,

$$\lambda_j = \begin{cases} k-1+|w_{k+j}| & \text{if } w_{k+j} < 0, \\ \#\{p \leq k \,:\, |w_p| > w_{k+j}\} & \text{if } w_{k+j} > 0 \end{cases} \tag{1}$$

while $\mathrm{type}(\lambda) > 0$ if and only if $|w_1| > 1$, and in this case $\mathrm{type}(\lambda)$ is equal to 1 or 2 depending on whether $w_1 > 0$ or $w_1 < 0$, respectively. Using this bijection, we attach to any typed k-strict partition λ a finite set of pairs[1]

$$\mathcal{C}(\lambda) := \{(i,j) \in \mathbb{N} \times \mathbb{N} \mid 1 \leq i < j \ \text{ and } \ w_{k+i} + w_{k+j} < 0\} \tag{2}$$

and a sequence $\beta(\lambda) = \{\beta_j(\lambda)\}_{j \geq 1}$ defined by

$$\beta_j(\lambda) := \begin{cases} w_{k+j} + 1 & \text{if } w_{k+j} < 0, \\ w_{k+j} & \text{if } w_{k+j} > 0. \end{cases} \tag{3}$$

For example, the 3-Grassmannian element $w = (-4, 6, 8, -5, -2, -1, 3, 7)$ of $\widetilde{W}_8$ corresponds to the 3-strict partition $\lambda = (7,4,3,3,1)$ of type 2, and we have $\mathcal{C}(\lambda) = \{(1,2),(1,3),(1,4),(2,3)\}$ and $\beta(\lambda) = (-4,-1,0,3,7)$.

Let $t = (t_1, t_2, \ldots)$ be a list of commuting variables and z be a formal variable. For any integers $j \geq 0$ and $r \geq 1$, the elementary and complete symmetric polynomials $e_j(t_1,\ldots,t_r)$ and $h_j(t_1,\ldots,t_r)$ are defined by the generating series

$$\prod_{i=1}^{r}(1+t_iz) = \sum_{j=0}^{\infty} e_j(t_1,\ldots,t_r)z^j \quad \text{and} \quad \prod_{i=1}^{r}(1-t_iz)^{-1} = \sum_{j=0}^{\infty} h_j(t_1,\ldots,t_r)z^j,$$

respectively. Let $e_j^r(t) := e_j(t_1,\ldots,t_r)$, $h_j^r(t) := h_j(t_1,\ldots,t_r)$, and $e_j^0(t) = h_j^0(t) = \delta_{0,j}$, where $\delta_{0,j}$ denotes the Kronecker delta. Furthermore, if $r < 0$ then define $h_j^r(t) := e_j^{-r}(t)$. Let $b = (\widetilde{b}_k, b_1, b_2, \ldots)$ and $c = (c_1, c_2, \ldots)$ be two further families of commuting variables, and set $c_0 = b_0 = 1$ and $c_p = b_p = 0$ for any $p < 0$. These variables are related by the equations

$$c_p = \begin{cases} b_p & \text{if } p < k, \\ b_k + \widetilde{b}_k & \text{if } p = k, \\ 2b_p & \text{if } p > k. \end{cases} \tag{4}$$

For any $p, r \in \mathbb{Z}$ and for $s \in \{0,1\}$, define the polynomials c_p^r and a_p^s by

$$c_p^r := \sum_{j=0}^{p} c_{p-j}\, h_j^r(-t) \quad \text{and} \quad a_p^s := \frac{1}{2}c_p + \sum_{j=1}^{p} c_{p-j}\, h_j^s(-t).$$

Moreover, define

$$b_k^s := b_k + \sum_{j=1}^{k} c_{k-j}\, h_j^s(-t) \quad \text{and} \quad \widetilde{b}_k^s := \widetilde{b}_k + \sum_{j=1}^{k} c_{k-j}\, h_j^s(-t).$$

An integer sequence $\alpha = (\alpha_1, \alpha_2, \ldots)$ is assumed to have finite support when it appears as a subscript. For any integer sequences α and ρ, let

$$\widehat{c}_\alpha^\rho := \widehat{c}_{\alpha_1}^{\rho_1} \widehat{c}_{\alpha_2}^{\rho_2} \cdots$$

[1]The condition $w_{k+i} + w_{k+j} < 0$ in (2) is equivalent to $\lambda_i + \lambda_j \geq 2k + j - i$.

where, for each $i \geq 1$,

$$\widehat{c}^{\rho_i}_{\alpha_i} := c^{\rho_i}_{\alpha_i} + \begin{cases} (2\widetilde{b}_k - c_k)e^{\alpha_i-k}_{\alpha_i-k}(-t) & \text{if } \rho_i = k - \alpha_i < 0 \text{ and } i \text{ is odd,} \\ (2b_k - c_k)e^{\alpha_i-k}_{\alpha_i-k}(-t) & \text{if } \rho_i = k - \alpha_i < 0 \text{ and } i \text{ is even,} \\ 0 & \text{otherwise.} \end{cases}$$

The eta polynomials are defined using Young's raising operators [Y]. The basic operator R_{ij} for $i < j$ acts on an integer sequence α by the prescription

$$R_{ij}(\alpha) := (\alpha_1, \ldots, \alpha_i + 1, \ldots, \alpha_j - 1, \ldots).$$

A *raising operator* R is any finite monomial in the basic operators R_{ij}. If $R := \prod_{i<j} R_{ij}^{n_{ij}}$ is any raising operator and $m \geq 1$, denote by $\mathrm{supp}_m(R)$ the set of all indices i and j such that $n_{ij} > 0$ and $j < m$. For any typed k-strict partition λ, we consider the raising operator expression R^λ given by

$$R^\lambda := \prod_{i<j}(1 - R_{ij}) \prod_{(i,j)\in\mathcal{C}(\lambda)} (1 + R_{ij})^{-1}. \tag{5}$$

Definition 1. Let λ be a typed k-strict partition of length ℓ, let $\ell_k(\lambda)$ denote the number of parts λ_i which are strictly greater than k, let $m := \ell_k(\lambda) + 1$ and $\beta := \beta(\lambda)$. Let R be any raising operator appearing in the expansion of the power series R^λ and set $\nu := R\lambda$. If $\mathrm{type}(\lambda) = 0$, then define

$$R \star \widehat{c}^{\,\beta}_\lambda = \overline{c}^{\,\beta}_\nu := \overline{c}^{\,\beta_1}_{\nu_1} \cdots \overline{c}^{\,\beta_\ell}_{\nu_\ell}$$

where, for each $i \geq 1$,

$$\overline{c}^{\,\beta_i}_{\nu_i} := \begin{cases} c^{\beta_i}_{\nu_i} & \text{if } i \in \mathrm{supp}_m(R), \\ \widehat{c}^{\,\beta_i}_{\nu_i} & \text{otherwise.} \end{cases}$$

If $\mathrm{type}(\lambda) > 0$ and R involves any factors R_{ij} with $i = m$ or $j = m$, then define

$$R \star \widehat{c}^{\,\beta}_\lambda := \overline{c}^{\,\beta_1}_{\nu_1} \cdots \overline{c}^{\,\beta_{m-1}}_{\nu_{m-1}}\, a^{\beta_m}_{\nu_m}\, c^{\beta_{m+1}}_{\nu_{m+1}} \cdots c^{\beta_\ell}_{\nu_\ell}.$$

If R has no such factors, then define

$$R \star \widehat{c}^{\,\beta}_\lambda := \begin{cases} \overline{c}^{\,\beta_1}_{\nu_1} \cdots \overline{c}^{\,\beta_{m-1}}_{\nu_{m-1}}\, b^{\beta_m}_k\, c^{\beta_{m+1}}_{\nu_{m+1}} \cdots c^{\beta_\ell}_{\nu_\ell} & \text{if } \mathrm{type}(\lambda) = 1, \\ \overline{c}^{\,\beta_1}_{\nu_1} \cdots \overline{c}^{\,\beta_{m-1}}_{\nu_{m-1}}\, \widetilde{b}^{\beta_m}_k\, c^{\beta_{m+1}}_{\nu_{m+1}} \cdots c^{\beta_\ell}_{\nu_\ell} & \text{if } \mathrm{type}(\lambda) = 2. \end{cases}$$

Define the *double eta polynomial* $H_\lambda(c\,|\,t)$ by

$$H_\lambda(c\,|\,t) := 2^{-\ell_k(\lambda)} R^\lambda \star \widehat{c}^{\,\beta(\lambda)}_\lambda.$$

The single eta polynomial $H_\lambda(c)$ of [BKT2] is given by $H_\lambda(c) = H_\lambda(c\,|\,0)$.

Table 1 lists the double eta polynomials indexed by the 1-Grassmannian and 2-Grassmannian elements in $\widetilde{W}_3$. In the table, the symbols e^r_j and h^r_j are used to denote $e^r_j(-t)$ and $h^r_j(-t)$, respectively. As is customary, a bar over an integer is used to denote a negative sign.

Let $\{e_1, \ldots, e_{2n}\}$ denote the standard orthogonal basis of $\mathbb{C}^{2n}$ and let F_i be the subspace spanned by the first i vectors of this basis, so that $F^\perp_{n-i} = F_{n+i}$ for $0 \leq i \leq n$. Let B_n denote the stabilizer of the flag $F_\bullet$ in the group $\mathrm{SO}_{2n}(\mathbb{C})$, and let T_n be the corresponding maximal torus in the Borel subgroup B_n. The T_n-equivariant cohomology ring $\mathrm{H}^*_{T_n}(\mathrm{OG}(n-k, 2n), \mathbb{Z})$ is defined as the cohomology ring of the Borel mixing space $ET_n \times^{T_n} \mathrm{OG}$. The Schubert cells in OG are the B_n-orbits, and are indexed by the typed k-strict partitions λ whose Young diagram

TABLE 1. Double eta polynomials for Grassmannian $w \in \widetilde{W}_3$

w	λ	β	$H_\lambda(c\,\vert\,t)$
123			1
213	1	$(1,3)$	$b_1 + h_1^1$
$\overline{2}\overline{1}3$	$1'$	$(0,3)$	$\widetilde{b}_1$
$\overline{1}\overline{2}3$	2	$(-1,3)$	$b_2 + \widetilde{b}_1 e_1^1$
312	$(1,1)$	$(1,2)$	$(b_1+h_1^1)(c_1+h_1^2) - (b_2 + c_1 h_1^1 + h_2^1)$
$\overline{3}\overline{1}2$	$(1,1)'$	$(0,2)$	$\widetilde{b}_1(c_1+h_1^2) - b_2$
$\overline{1}\overline{3}2$	3	$(-2,2)$	$b_3 + b_2 e_1^2 + \widetilde{b}_1 e_2^2$
$3\overline{2}\overline{1}$	$(2,1)$	$(-1,0)$	$(b_2+\widetilde{b}_1 e_1^1) b_1 - (b_3 + b_2 e_1^1)$
$\overline{3}\overline{2}1$	$(2,1)'$	$(-1,1)$	$(b_2+\widetilde{b}_1 e_1^1)(\widetilde{b}_1 + h_1^1) - (b_3 + b_2 e_1^1)$
$2\overline{3}\overline{1}$	$(3,1)$	$(-2,0)$	$(b_3 + b_2 e_1^2 + \widetilde{b}_1 e_2^2) b_1 - (b_4 + b_3 e_1^2 + b_2 e_2^2)$
$\overline{2}\overline{3}1$	$(3,1)'$	$(-2,1)$	$(b_3 + b_2 e_1^2 + \widetilde{b}_1 e_2^2)(\widetilde{b}_1 + h_1^1) - (b_4 + b_3 e_1^2 + b_2 e_2^2)$
$1\overline{3}\overline{2}$	$(3,2)$	$(-2,-1)$	$(b_3 + b_2 e_1^2 + \widetilde{b}_1 e_2^2)(b_2 + b_1 e_1^1)$ $-(b_4 + b_3 e_1^2 + b_2 e_2^2)(c_1 + e_1^1) + (b_5 + b_4 e_1^2 + b_3 e_2^2)$
132	1	2	$b_1 + h_1^2$
231	2	1	$b_2 + b_1 h_1^1 + h_2^1$
$\overline{2}3\overline{1}$	$2'$	0	$\widetilde{b}_2$
$\overline{1}3\overline{2}$	3	-1	$b_3 + \widetilde{b}_2 e_1^1$
$\overline{1}2\overline{3}$	4	-2	$b_4 + b_3 e_1^2 + \widetilde{b}_2 e_2^2$

fits in an $(n-k)\times(n+k-1)$ rectangle. Any such λ defines a Schubert cell $X^\circ_\lambda = X^\circ_\lambda(F_\bullet)$ of codimension $|\lambda| := \sum_i \lambda_i$ by the prescription

$$X^\circ_\lambda := \{\Sigma \in \mathrm{OG} \mid \dim(\Sigma\cap F_q) = \#\{j\,|\,p_j(\lambda)\leq q\} \quad \forall\, 1\leq q\leq 2n\}$$

where, for $1\leq j\leq n-k$, we have

$$p_j(\lambda) := n + \begin{cases} 1-\beta_j(\lambda) & \text{if } \beta_j(\lambda)\in\{0,1\} \text{ and } n \text{ is odd,} \\ \beta_j(\lambda) & \text{otherwise.}\end{cases}$$

The Schubert variety X_λ is the closure of the Schubert cell X°_λ. Since X_λ is stable under the action of T_n, we obtain an *equivariant Schubert class* $[X_\lambda]^{T_n} := [ET_n \times^{T_n} X_\lambda]$ in $\mathrm{H}^*_{T_n}(\mathrm{OG}(n-k,2n))$.

The natural inclusions $\widetilde{W}_n \hookrightarrow \widetilde{W}_{n+1}$ of the Weyl groups defined earlier induce surjections of graded algebras

$$\cdots \to \mathrm{H}^*_{T_{n+1}}(\mathrm{OG}(n+1-k,2n+2)) \to \mathrm{H}^*_{T_n}(\mathrm{OG}(n-k,2n)) \to \cdots$$

and the stable equivariant cohomology ring of OG, denoted by $\mathbb{H}_T(\mathrm{OG}_k)$, is the associated graded inverse limit

$$\mathbb{H}_T(\mathrm{OG}_k) := \varprojlim \mathrm{H}^*_{T_n}(\mathrm{OG}(n-k,2n)).$$

One identifies here the variables t_i with the characters of the maximal tori T_n in a compatible way, as in [BH, §2] and [IMN1, §10]. We then have that $\mathbb{H}_T(\mathrm{OG}_k)$ is a free $\mathbb{Z}[t]$-algebra with a basis of stable equivariant Schubert classes

$$\tau_\lambda := \varprojlim [X_\lambda]^{T_n},$$

one for every typed k-strict partition λ.

Consider the graded polynomial ring $\mathbb{Z}[b] := \mathbb{Z}[\widetilde{b}_k, b_1, b_2, \ldots]$, where the variable b_i has degree i for each i, and $\widetilde{b}_k$ has degree k. Let $J^{(k)} \subset \mathbb{Z}[b]$ be the homogeneous ideal generated by the relations

$$b_p b_p + \sum_{i=1}^{p} (-1)^i b_{p+i} c_{p-i} = 0 \quad \text{for } p > k, \tag{6}$$

$$b_k \widetilde{b}_k + \sum_{i=1}^{k} (-1)^i b_{k+i} b_{k-i} = 0, \tag{7}$$

where the c_i satisfy the relations (4), and define the quotient ring $B^{(k)} := \mathbb{Z}[b]/J^{(k)}$. We call the graded polynomial ring $B^{(k)}[t]$ the *ring of double eta polynomials*.

The following result establishes the precise connection between the double eta polynomials $H_\lambda(c\,|\,t)$ and the equivariant Schubert classes on OG, namely, that the former represent the latter. We regard $\mathrm{H}^*_{T_n}(\mathrm{OG}(n-k, 2n))$ as a $\mathbb{Z}[t]$-module via the natural projection map $\mathbb{Z}[t] \to \mathbb{Z}[t_1, \ldots, t_n]$.

Theorem 1. *The polynomials $H_\lambda(c\,|\,t)$, as λ runs over all typed k-strict partitions, form a free $\mathbb{Z}[t]$-basis of $B^{(k)}[t]$. There is an isomorphism of graded $\mathbb{Z}[t]$-algebras*

$$\pi : B^{(k)}[t] \to \mathbb{H}_T(\mathrm{OG}_k)$$

such that $H_\lambda(c\,|\,t)$ is mapped to τ_λ for every typed k-strict partition λ. For every $n \geq 1$, the morphism π induces a surjective homomorphism of graded $\mathbb{Z}[t]$-algebras

$$\pi_n : B^{(k)}[t] \to \mathrm{H}^*_{T_n}(\mathrm{OG}(n-k, 2n))$$

which maps $H_\lambda(c\,|\,t)$ to $[X_\lambda]^{T_n}$, if λ fits inside an $(n-k) \times (n+k-1)$ rectangle, and to zero, otherwise.

The map π_n in Theorem 1 is induced from the type D geometrization map of [IMN1, §10] and [T2, §7] (see §4.1). When all the parts λ_i of the indexing typed k-strict partition λ are greater than k, then the equality $[X_\lambda]^{T_n} = \pi_n(H_\lambda(c\,|\,t))$ is equivalent to the Chern class formula for even orthogonal degeneracy loci obtained by Kazarian [Ka] in 2001. When we set $t = 0$, Theorem 1 gives the Giambelli formula for the ordinary Schubert classes on OG from [BKT2, Thm. 1].

Our proof of Theorem 1 follows the argument of [TW], which dealt with the analogous theory of double theta polynomials $\Theta_\lambda(c\,|\,t)$ for the symplectic Grassmannians. Adapting the work of Ikeda and Matsumura [IM], we showed in [TW, §5] that the $\Theta_\lambda(c\,|\,t)$ are compatible with the action of left divided difference operators on the polynomial ring $\mathbb{Z}[c, t]$. In the type D framework of the present paper, we similarly prove that the action of $\widetilde{W}_\infty$ on $B^{(k)}[t]$ lifts to an action on $\mathbb{Z}[b, t]$, and gives rise to divided differences there. In §3.4, we introduce a family of double polynomials $\widehat{H}_\lambda(c\,|\,t)$ which are indexed by k-strict partitions. These specialize to the single polynomials $\widehat{H}_\lambda(c)$ of [BKT2, §5.2], are compatible with the divided differences on $\mathbb{Z}[b, t]$, and enjoy properties entirely parallel to those of the double theta polynomials $\Theta_\lambda(c\,|\,t)$. However, the double eta polynomials $H_\lambda(c\,|\,t)$ are more subtle: there are instances where the compatibility with divided differences is true for them only modulo the relation (7) (see Proposition 5 and Remark 1). We conclude the proof Theorem 1 by using a formula for the equivariant Schubert class of a point, which is a special case of the aforementioned result from [Ka].

It is important to note that the double eta polynomials $H_\lambda(c\,|\,t)$ defined here are new, and are not equal to the type D double Schubert polynomials of [IMN1] indexed by the k-Grassmannian elements of the Weyl group, which represent equivariant Schubert classes on the complete even orthogonal flag variety. The latter objects are really formal power series, and are expressed using a different set of variables which are not intrinsic to the Grassmannian $\mathrm{OG}(n-k,2n)$. The precise relationship between the two familes of polynomials is discussed in §4.1.

Our research on this article was influenced by three prior works: Kazarian's paper [Ka] on degeneracy locus formulas of Pfaffian type, Wilson's thesis [W], where double theta polynomials were first defined and studied, and Ikeda and Matsumura's article [IM], which exhibited the compatibility of these polynomials with left divided differences, and proved that they represent equivariant Schubert classes. We thank each of these authors for their contributions. In recent work, Anderson and Fulton [AF] have defined a family of double eta polynomials independently, and extended them further to 'multi-eta polynomials', which represent (a power of 2 times) the classes of certain degeneracy loci of even orthogonal type.

This paper is organized as follows. In Section 2, we define the type D divided difference operators on $\mathbb{Z}[b,t]$ and establish their basic properties. Section 3 proves the required compatibility of double eta polynomials with divided differences, and studies the related family of polynomials $\widehat{H}_\lambda(c\,|\,t)$. The proof of Theorem 1 is completed in Section 4, which also describes how the polynomials $H_\lambda(c\,|\,t)$ are related to the general equivariant Giambelli polynomials of [T1].

Our work on double eta polynomials was announced during the conference 'IMPANGA 15' which took place in Będlewo, Poland. It is a pleasure to thank the organizers for their hospitality and for making this stimulating event possible. I also thank the referee for comments which helped to improve the exposition.

2. Divided difference operators on $\mathbb{Z}[b,t]$

In this section we will work exclusively in the polynomial ring $\mathbb{Z}[b,t]$. We begin by defining the action of the Weyl group $\widetilde{W}_\infty$ on $\mathbb{Z}[b,t]$ by ring automorphisms and the associated family of t-divided difference operators $\{\partial_i\}_{i\geq 0}$ on $\mathbb{Z}[b,t]$.

The elements of the Weyl group $\widetilde{W}_n$ of type D_n are represented as signed permutations of the set $\{1,\ldots,n\}$ with an even number of negative entries. The group $\widetilde{W}_n$ is generated by the simple transpositions $s_i=(i,i+1)$ for $1\leq i\leq n-1$ and an element $s_0:=s_0^Bs_1s_0^B$, where $s_0^B(1)=\overline{1}$ denotes the sign change. The action of $\widetilde{W}_\infty$ on $\mathbb{Z}[b,t]$ is defined as follows. The simple reflections s_i for $i>0$ act by interchanging t_i and t_{i+1} and leaving all the remaining variables fixed. The reflection s_0 maps (t_1,t_2) to $(-t_2,-t_1)$, fixes the t_j for $j\geq 3$, and satisfies the equations

$$s_0(b_p):=\begin{cases} b_p-2(t_1+t_2)c^2_{p-1} & \text{if } p<k, \\ b_p-(t_1+t_2)c^2_{p-1} & \text{if } p\geq k \end{cases}$$

and $s_0(\widetilde{b}_k):=\widetilde{b}_k-(t_1+t_2)c^2_{k-1}$. Observe that for every $p\geq 1$, we have

$$s_0(c_p)=c_p-2(t_1+t_2)c^2_{p-1}=c_p-2(t_1+t_2)\sum_{j=0}^{p-1}(-1)^j\left(\sum_{a+b=j}t_1^at_2^b\right)c_{p-1-j}.$$

It is useful to write this as an equation of generating series

$$s_0\left(\sum_{p=0}^{\infty} c_p u^p\right) = \frac{1-t_1u}{1+t_1u}\frac{1-t_2u}{1+t_2u}\cdot\sum_{p=0}^{\infty} c_p u^p \tag{8}$$

where u denotes a formal variable such that $s_i(u) = u$ for each i.

One checks that, with the above definition of s_i for $i \geq 0$, the braid relations for $\widetilde{W}_\infty$ are satisfied in $\mathbb{Z}[b,t]$, and so we obtain a well defined group action. Moreover, the action of $\widetilde{W}_\infty$ on $\mathbb{Z}[b,t]$ induces an action on the quotient ring $B^{(k)}[t]$. Define the *divided difference operators* ∂_i on $\mathbb{Z}[b,t]$ by

$$\partial_0 f := \frac{f - s_0 f}{t_1+t_2}, \qquad \partial_i f := \frac{f - s_i f}{t_{i+1}-t_i}, \quad \text{if } i \geq 1.$$

The same equations also define operators ∂_i on $B^{(k)}[t]$. These latter correspond to the left divided differences δ_i studied in [IMN1]. The previous formulas imply that

$$\partial_0(c_p) = 2c^2_{p-1} \quad \text{and} \quad \partial_0(b_k) = \partial_0(\widetilde{b}_k) = c^2_{k-1}.$$

For every $i \geq 0$, the operator ∂_i satisfies the Leibnitz rule

$$\partial_i(fg) = (\partial_i f)g + (s_i f)\partial_i g.$$

For $r < 0$ we let $t_{-r} := t_r$. We recall the following basic result from [TW, §1].

Lemma 1. *Suppose that $p, r \in \mathbb{Z}$.*

(a) *Assume that $r > 0$. Then we have*

$$c^r_p = c^{r-1}_p - t_r\, c^r_{p-1}.$$

(b) *Assume that $r \leq 0$. Then we have*

$$c^r_p = c^{r-1}_p + t_{r-1}\, c^r_{p-1}.$$

We now prove several identities satisfied by the operators ∂_i, analogous to those shown in [TW, §5]. Observe first that, for $r \geq 1$, we have

$$\sum_{p=0}^{\infty} c^r_p u^p = \left(\sum_{i=0}^{\infty} c_i u^i\right)\prod_{j=1}^{r}\frac{1}{1+t_j u} \tag{9}$$

while, for $r \leq -1$, we have

$$\sum_{p=0}^{\infty} c^r_p u^p = \left(\sum_{i=0}^{\infty} c_i u^i\right)\prod_{j=1}^{|r|}(1-t_j u). \tag{10}$$

Lemma 2. *Suppose that $i \geq 1$. We have the identities*

$$s_i(c^r_p) = \begin{cases} c^r_p & \text{if } r \neq \pm i, \\ c^{i+1}_p + t_i c^{i+1}_{p-1} & \text{if } r = i > 0, \\ c^{-i+1}_p - t_{i+1} c^{-i+1}_{p-1} & \text{if } r = -i < 0 \end{cases}$$

and

$$s_0(c^r_p) = \begin{cases} c^r_p & \text{if } |r| \geq 2, \\ c^2_p - t_1 c^2_{p-1} & \text{if } r = 1, \\ c^2_p - (t_1+t_2)c^2_{p-1} + t_1t_2 c^2_{p-2} & \text{if } r = 0, \\ c^1_p - (t_1+t_2)c^1_{p-1} + t_1t_2 c^1_{p-2} & \text{if } r = -1. \end{cases}$$

Proof. Since c_p^r is symmetric in $(t_1, \ldots, t_{|r|})$, the identity $s_i(c_p^r) = c_p^r$ for $r \neq \pm i$ is clear. If $r \geq 2$, then we apply s_0 to both sides of (9) and use (8) to show that $s_0(c_p^r) = c_p^r$ for all p; the proof when $r \leq -2$ is similar, using (10).

If $r = i > 0$, then $s_i(c_p^i) = c_p^{i+1} + t_i c_{p-1}^{i+1}$ follows from the identity

$$s_i\left(\frac{1}{1+t_i}\right) = \frac{1}{1+t_{i+1}} = \frac{1}{(1+t_i)(1+t_{i+1})} + \frac{t_i}{(1+t_i)(1+t_{i+1})}.$$

If $r = -i < 0$, then $s_i(c_p^{-i}) = c_p^{-i+1} - t_{i+1} c_{p-1}^{-i+1}$ follows from $s_i(1-t_i) = 1 - t_{i+1}$.

If $r = 1$ then equation (9) gives

$$s_0\left(\sum_{p=0}^{\infty} c_p^1 u^p\right) = \frac{1-t_1 u}{(1+t_1 u)(1+t_2 u)}\left(\sum_{p=0}^{\infty} c_p u^p\right) = (1-t_1 u)\left(\sum_{p=0}^{\infty} c_p^2 u^p\right)$$

while if $r = -1$, equation (10) gives

$$s_0\left(\sum_{p=0}^{\infty} c_p^{-1} u^p\right) = \frac{(1-t_1 u)(1-t_2 u)}{1+t_1 u}\left(\sum_{p=0}^{\infty} c_p u^p\right) = (1-t_1 u)(1-t_2 u)\left(\sum_{p=0}^{\infty} c_p^1 u^p\right).$$

The displayed formulas for $s_0(c_p^1)$ and $s_0(c_p^{-1})$ follow. Finally, we use equation (8) to compute $s_0(c_p)$. ☐

Proposition 1. *Suppose that $p, r \in \mathbb{Z}$.*

(a) *For all $i \geq 1$, we have*

$$\partial_i c_p^r = \begin{cases} c_{p-1}^{r+1} & \text{if } r = \pm i, \\ 0 & \text{otherwise.} \end{cases}$$

We have

$$\partial_0 c_p^r = \begin{cases} c_{p-1}^2 & \text{if } r = 1, \\ 2c_{p-1}^2 & \text{if } r = 0, \\ 2c_{p-1}^1 - c_{p-1} & \text{if } r = -1, \\ 0 & \text{if } |r| \geq 2. \end{cases}$$

In particular, we have

$$\partial_0 c_p^{-1} = 2a_{p-1}^1, \quad \partial_1 c_p^{-1} = 2a_{p-1}^0, \quad \text{and} \quad (\partial_0 + \partial_1) c_p^{-1} = 2c_{p-1}^1. \tag{11}$$

(b) *For all $i \geq 1$, we have*

$$\partial_i(c_p^{-i} c_q^i) = c_{p-1}^{-i+1} c_q^{i+1} + c_p^{-i+1} c_{q-1}^{i+1}. \tag{12}$$

(c) *We have*

$$\partial_0(c_p^{-1} c_q^1) = 2(a_{p-1}^1 c_q^2 + a_p^1 c_{q-1}^2), \quad \partial_1(c_p^{-1} c_q^1) = 2(a_{p-1}^0 c_q^2 + a_p^0 c_{q-1}^2), \quad \text{and} \tag{13}$$

$$(\partial_0 + \partial_1)(c_p^{-1} c_q^1) = 2(c_{p-1}^1 c_q^2 + c_p^1 c_{q-1}^2). \tag{14}$$

Proof. For part (a), observe that if $i > 0$ and $r \neq \pm i$, then the result follows from Lemma 2 immediately. If $r = i > 0$, then we compute easily that

$$\partial_i\left(\sum_{p=0}^{\infty} c_p^r u^p\right) = \left(\sum_{p=0}^{\infty} c_p u^{p+1}\right) \prod_{j=1}^{r+1} \frac{1}{1+t_j u}$$

from which the desired result follows. We work similarly when $r = -i < 0$.

To evaluate the divided difference ∂_0, note, for instance, that

$$\sum_{p=0}^{\infty} c_p^1 u^p - s_0\left(\sum_{p=0}^{\infty} c_p^1 u^p\right) = \frac{(t_1+t_2)u}{(1+t_1u)(1+t_2u)}\left(\sum_{p=0}^{\infty} c_p u^p\right)$$

and the computation of $\partial_0(c_p^1)$ follows. The rest are evaluated similarly, but we pay special attention to the third case. We compute using the Leibnitz rule that

$$\partial_0(c_p^{-1}) = \partial_0(c_p - t_1 c_{p-1}) = 2c_{p-1}^2 - c_{p-1} + 2t_2 c_{p-2}^2.$$

However, for any p, we have

$$c_p^2 + t_2 c_{p-1}^2 = c_p + \sum_{j=0}^{p-1} c_{p-1-j}(h_{j+1}^2(-t) + t_2 h_j^2(-t)) = c_p + \sum_{j=0}^{p-1} c_{p-1-j} h_{j+1}^1(-t) = c_p^1.$$

It follows that $\partial_0(c_p^{-1}) = 2c_{p-1}^1 - c_{p-1} = 2a_{p-1}^1$. Since $\partial_1(c_p^{-1}) = c_{p-1} = 2a_{p-1}^0$, the equations (11) follow.

For part (b), use the Leibnitz rule and Lemmas 1 and 2 to compute

$$\begin{aligned}
\partial_i(c_p^{-i}c_q^i) &= \partial_i(c_p^{-i})c_q^i + s_i(c_p^{-i})\partial_i(c_q^i) \\
&= c_{p-1}^{-i+1}c_q^i + (c_p^{-i+1} - t_{i+1}c_{p-1}^{-i+1})c_{q-1}^{i+1} \\
&= c_{p-1}^{-i+1}(c_q^i - t_{i+1}c_{q-1}^{i+1}) + c_p^{-i+1}c_{q-1}^{i+1} \\
&= c_{p-1}^{-i+1}c_q^{i+1} + c_p^{-i+1}c_{q-1}^{i+1}.
\end{aligned}$$

We finally establish the equations (13) and (14). An application of (12) gives

$$\partial_1(c_p^{-1}c_q^1) = c_{p-1}c_q^2 + c_p c_{q-1}^2 = 2(a_{p-1}^0 c_q^2 + a_p^0 c_{q-1}^2).$$

The Leibnitz rule implies that

$$\begin{aligned}
\partial_0(c_p^{-1}c_q^1) &= 2a_{p-1}^1 c_q^1 + s_0(c_p^{-1})c_{q-1}^2 \\
&= 2a_{p-1}^1(c_q^2 + t_2c_{q-1}^2) + (c_p^1 - (t_1+t_2)c_{p-1}^1 + t_1t_2c_{p-2}^1)c_{q-1}^2 \\
&= 2a_{p-1}^1 c_q^2 + (c_p^1 - t_1c_{p-1}^1 + t_2(c_{p-1}^1 - c_{p-1} + t_1c_{p-2}^1))c_{q-1}^2 \\
&= 2a_{p-1}^1 c_q^2 + (2c_p^1 - c_p)c_{q-1}^2 = 2(a_{p-1}^1 c_q^2 + a_p^1 c_{q-1}^2).
\end{aligned}$$

where we employed the identity $c_q^1 = c_q^2 + t_2 c_{q-1}^2$ and, in the second to last equation, the identity $c_p^1 = c_p - t_1 c_{p-1}^1$ twice. This completes the proof. □

We will require certain variations of the previous identities. Set $a_p := a_p^0 = \frac{1}{2}c_p$ for each integer p. Let f_k be a variable of degree k, which will be equal to b_k, $\widetilde{b}_k$, or a_k, depending on the context. For $s \in \{0,1\}$, define

$$f_k^s := f_k + \sum_{j=1}^{k} c_{k-j} h_j^s(-t),$$

set $\widetilde{f}_k := c_k - f_k$, and $\widetilde{f}_k^s := c_k - 2f_k + f_k^s$. For any $p, r \in \mathbb{Z}$, define $\widehat{c}_p^r$ by

$$\widehat{c}_p^r := c_p^r + \begin{cases} (2f_k - c_k)e_{p-k}^{p-k}(-t) & \text{if } r = k-p < 0, \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 3. *Suppose that $i \geq 1$. We then have the identities*

$$s_i(\widehat{c}_p^r) = \begin{cases} \widehat{c}_p^r & \text{if } r \neq \pm i, \\ \widehat{c}_p^{i+1} + t_i \widehat{c}_{p-1}^{i+1} & \text{if } r = i > 0, \\ \widehat{c}_p^{-i+1} - t_{i+1}\widehat{c}_{p-1}^{-i+1} & \text{if } r = -i < 0. \end{cases}$$

Moreover, if $|r| \geq 2$, then $s_0(\widehat{c}_p^r) = \widehat{c}_p^r$.

Proof. If $p > k$, then we have that

$$\widehat{c}_p^{k-p} = c_p^{k-p} + (2f_k - c_k)e_{p-k}^{p-k}(-t).$$

Since $s_i((2f_k - c_k)e_{p-k}^{p-k}(-t)) = (2f_k - c_k)s_i(e_{p-k}^{p-k}(-t))$, the identity $s_i(\widehat{c}_p^{p-k}) = \widehat{c}_p^{p-k}$ for $p-k \neq \pm i$ is clear. If $p - k = -i$, then $s_i(\widehat{c}_p^{-i}) = \widehat{c}_p^{-i+1} - t_{i+1}\widehat{c}_{p-1}^{-i+1}$ follows from the corresponding identity of Lemma 2 and the calculation

$$s_i(e_{p-k}^i(-t)) = e_{p-k}^{i-1}(-t) - t_{i+1}e_{p-1-k}^{i-1}(-t).$$

We have $\widehat{c}_p^{k-p} = c_p^{k-p} + (-1)^{p-k}(2f_k - c_k)t_1 \cdots t_{p-k}$. If $p - k \geq 2$, then s_0 leaves all terms on the right hand side invariant, and hence $s_0(\widehat{c}_p^{k-p}) = \widehat{c}_p^{k-p}$. The remaining equalities follow from Lemma 2. □

Proposition 2. *Suppose that $p \in \mathbb{Z}$ and $p > k$.*

(a) *For all $i \geq 1$, we have*

$$\partial_i \widehat{c}_p^{k-p} = \begin{cases} \widehat{c}_{p-1}^{k-p+1} & \text{if } i = p-k \geq 2, \\ 2f_k & \text{if } i = p-k = 1, \\ 0 & \text{otherwise.} \end{cases}$$

We have

$$\partial_0 \widehat{c}_p^{k-p} = \begin{cases} 2\widetilde{f}_k^1 & \text{if } k-p = -1, \\ 0 & \text{if } k-p < -1. \end{cases}$$

In particular, we have

$$\partial_0 \widehat{c}_{k+1}^{-1} = 2\widetilde{f}_k^1, \quad \partial_1 \widehat{c}_{k+1}^{-1} = 2f_k, \quad \textit{and} \quad (\partial_0 + \partial_1)\widehat{c}_{k+1}^{-1} = 2c_k^1. \tag{15}$$

(b) *For all $i \geq 2$, we have*

$$\partial_i(\widehat{c}_p^{-i} c_q^i) = \widehat{c}_{p-1}^{-i+1} c_q^{i+1} + \widehat{c}_p^{-i+1} c_{q-1}^{i+1}. \tag{16}$$

(c) *We have*

$$\partial_0(\widehat{c}_{k+1}^{-1} c_q^1) = 2(\widetilde{f}_k^1 c_q^2 + a_{k+1}^1 c_{q-1}^2), \tag{17}$$

$$\partial_1(\widehat{c}_{k+1}^{-1} c_q^1) = 2(f_k c_q^2 + a_{k+1}^0 c_{q-1}^2), \quad \textit{and} \tag{18}$$

$$(\partial_0 + \partial_1)(\widehat{c}_{k+1}^{-1} c_q^1) = 2(c_k^1 c_q^2 + c_{k+1}^1 c_{q-1}^2). \tag{19}$$

Proof. Recall that

$$\widehat{c}_p^{k-p} = c_p^{k-p} + (2f_k - c_k)e_{p-k}^{p-k}(-t).$$

For part (a), observe that if $p - k \neq i \geq 1$, then the result follows from Lemma 2. If $i = p - k \geq 2$, then we compute that

$$\partial_i\left((2f_k - c_k)e_{p-k}^{p-k}(-t)\right) = (2f_k - c_k)\partial_i e_{p-k}^i(-t) = (2f_k - c_k)e_{p-1-k}^{i-1}(-t)$$

from which the desired result follows. For $p = k+1$, we have

$$\widehat{c}_{k+1}^{-1} = c_{k+1}^{-1} + t_1(c_k - 2f_k)$$

and we compute that

$$\partial_1(\widehat{c}_{k+1}^{-1}) = c_k - (c_k - 2f_k) = 2f_k.$$

The fact that $\partial_0 \widehat{c}_p^{k-p} = 0$ for $k-p < -1$ follows immediately from Lemma 3. Since Proposition 1(a) gives $\partial_0(c_{k+1}^{-1}) = 2c_k^1 - c_k$, we deduce that

$$\partial_0(\widehat{c}_{k+1}^{-1}) = 2c_k^1 - 2f_k = 2\widetilde{f}_k^1. \tag{20}$$

This completes the proof of part (a).

Part (b) follows from the Leibnitz rule and Lemmas 1 and 3, exactly as in the proof of (12). For part (c), use (13) to compute that

$$\begin{aligned}
\partial_1(\widehat{c}_{k+1}^{-1} c_q^1) &= \partial_1(c_{k+1}^{-1}c_q^1 + (c_k - 2f_k)t_1 c_q^1) \\
&= \partial_1(c_{k+1}^{-1}c_q^1) + (c_k - 2f_k)\partial_1(t_1 c_q^1) \\
&= (c_k c_q^2 + c_{k+1}c_{q-1}^2) - (c_k - 2f_k)c_q^2 \\
&= 2f_k c_q^2 + c_{k+1}c_{q-1}^2.
\end{aligned}$$

We similarly have

$$\begin{aligned}
\partial_0(\widehat{c}_{k+1}^{-1} c_q^1) &= \partial_0(c_{k+1}^{-1}c_q^1 + (c_k - 2f_k)t_1 c_q^1) \\
&= \partial_0(c_{k+1}^{-1}c_q^1) + (c_k - 2f_k)\partial_0(t_1 c_q^1) \\
&= 2(a_k^1 c_q^2 + a_{k+1}^1 c_{q-1}^2) + (c_k - 2f_k)c_q^2 \\
&= (2c_k^1 - 2f_k)c_q^2 + 2a_{k+1}^1 c_{q-1}^2.
\end{aligned}$$

□

Proposition 3. *Suppose that $p, q \in \mathbb{Z}$. We have*

$$\partial_0(c_p^{-1}a_q^0) = 2(a_{p-1}^1 c_q^2 + a_p^1 c_{q-1}^2) - 2a_{p-1}^1 a_q^1 \tag{21}$$

$$\partial_0(c_p^{-1}\widetilde{f}_k^0) = 2(a_{p-1}^1 c_k^2 + a_p^1 c_{k-1}^2) - 2a_{p-1}^1 f_k^1 \tag{22}$$

$$\partial_0(\widehat{c}_{k+1}^{-1}a_q^0) = 2(\widetilde{f}_k^1 c_q^2 + a_{k+1}^1 c_{q-1}^2) - 2\widetilde{f}_k^1 a_q^1, \quad \textit{and} \tag{23}$$

$$\partial_0(\widehat{c}_{k+1}^{-1}\widetilde{f}_k^0) = 2(\widetilde{f}_k^1 c_k^2 + a_{k+1}^1 c_{k-1}^2) - 2\widetilde{f}_k^1 f_k^1. \tag{24}$$

We also have

$$\partial_1(c_p^{-1}a_q^1) = 2(a_{p-1}^0 c_q^2 + a_p^0 c_{q-1}^2) - 2a_{p-1}a_q \tag{25}$$

$$\partial_1(c_p^{-1}f_k^1) = 2(a_{p-1}^0 c_k^2 + a_p^0 c_{k-1}^2) - 2a_{p-1}\widetilde{f}_k \tag{26}$$

$$\partial_1(\widehat{c}_{k+1}^{-1}a_q^1) = 2(f_k^0 c_q^2 + a_{k+1}^0 c_{q-1}^2) - 2f_k a_q, \quad \textit{and} \tag{27}$$

$$\partial_1(\widehat{c}_{k+1}^{-1}f_k^1) = 2(f_k^0 c_k^2 + a_{k+1}^0 c_{k-1}^2) - 2f_k \widetilde{f}_k. \tag{28}$$

Proof. To prove equation (21), we use (13), the Leibnitz rule, and the observation that $\partial_0(a_q^1) = 0$, to compute

$$\partial_0(c_p^{-1}a_q^0) = \partial_0(c_p^{-1}c_q^1) - \partial_0(c_p^{-1}a_q^1) = 2(a_{p-1}^1 c_q^2 + a_p^1 c_{q-1}^2) - 2a_{p-1}^1 a_q^1.$$

Since $\partial_0(f_k^1) = \partial_0(c_k^1 - \widetilde{f}_k) = c_{k-1}^2 - c_{k-1}^2 = 0$, we similarly have that

$$\partial_0(c_p^{-1}\widetilde{f}_k^0) = \partial_0(c_p^{-1}c_k^1) - \partial_0(c_p^{-1}f_k^1) = 2(a_{p-1}^1 c_k^2 + a_p^1 c_{k-1}^2) - 2a_{p-1}^1 f_k^1.$$

We compute using (17) that

$$\partial_0(\widehat{c}_{k+1}^{-1}a_q^0) = \partial_0(\widehat{c}_{k+1}^{-1}c_q^1) - \partial_0(\widehat{c}_{k+1}^{-1}a_q^1) = 2(\widetilde{f}_k^1 c_q^2 + a_{k+1}^1 c_{q-1}^2) - 2\widetilde{f}_k^1 a_q^1.$$

We similarly have

$$\partial_0(\widetilde{c}_{k+1}^{\,-1}\widetilde{f}_k^0) = \partial_0(\widetilde{c}_{k+1}^{\,-1}c_k^1) - \partial_0(\widetilde{c}_{k+1}^{\,-1}f_k^1) = 2(\widetilde{f}_k^1 c_k^2 + a_{k+1}^1 c_{k-1}^2) - 2\widetilde{f}_k^1 f_k^1.$$

The proof of equations (25)–(28) is analogous, applying (13) and (18). □

3. Double eta polynomials

3.1. **A basis theorem.** For the rest of this paper, we will sometimes write equalities that hold only in the ring $B^{(k)}[t]$, where we have imposed the relations (6) and (7) on the generators b_p. Whenever these relations are needed, we will emphasize this by noting that the equalities are true in $B^{(k)}[t]$ rather than in $\mathbb{Z}[b,t]$.

We begin this section with a basis theorem for the $\mathbb{Z}[t]$-algebra $B^{(k)}[t]$. For any typed k-strict partition λ of length ℓ, with $m := \ell_k(\lambda)+1$, we define $b_\lambda \in \mathbb{Z}[b]$ as follows. If $\mathrm{type}(\lambda)=0$, then set $c_\lambda := c_{\lambda_1}\cdots c_{\lambda_\ell}$, while if $\mathrm{type}(\lambda)>0$, define

$$c_\lambda := \begin{cases} c_{\lambda_1}\cdots c_{\lambda_{m-1}}\, b_k\, c_{\lambda_{m+1}}\cdots c_{\lambda_\ell} & \text{if } \mathrm{type}(\lambda)=1, \\ c_{\lambda_1}\cdots c_{\lambda_{m-1}}\, \widetilde{b}_k\, c_{\lambda_{m+1}}\cdots c_{\lambda_\ell} & \text{if } \mathrm{type}(\lambda)=2. \end{cases}$$

Finally, define $b_\lambda := 2^{-\ell_k(\lambda)}\, c_\lambda$.

Proposition 4. *The monomials b_λ, the single eta polynomials $H_\lambda(c)$, and the double eta polynomials $H_\lambda(c\,|\,t)$ form three $\mathbb{Z}[t]$-bases of $B^{(k)}[t]$, as λ runs over all typed k-strict partitions.*

Proof. It follows from [BKT1, Thm. 3.2] that the elements b_λ and the single eta polynomials $H_\lambda(c)$ for λ a typed k-strict partition form two $\mathbb{Z}$-bases of $B^{(k)}$. We deduce that these two families are also $\mathbb{Z}[t]$-bases of $B^{(k)}[t]$. By expanding the raising operator definition of $H_\lambda(c\,|\,t)$, we obtain that

$$H_\lambda(c\,|\,t) = b_\lambda + \sum_\mu a_{\lambda\mu}\, b_\mu$$

where $a_{\lambda\mu} \in \mathbb{Z}[t]$ and the sum is over typed k-strict partitions μ with $\mu \succ \lambda$ in dominance order or $|\mu|<|\lambda|$. Therefore, the $H_\lambda(c\,|\,t)$ for λ typed and k-strict form another $\mathbb{Z}[t]$-basis of $B^{(k)}[t]$. □

3.2. **The left weak Bruhat order on $\widetilde{W}_\infty$.** The length of an element w in $\widetilde{W}_\infty$ is denoted by $\ell(w)$. It follows, for example, from [BB, p. 253] that

$$\ell(w) = \#\{i<j \mid w_i > w_j\} + \sum_{i\,:\, w_i<0} (|w_i|-1)$$

for each $w \in \widetilde{W}_\infty$. We deduce the following lemma.

Lemma 4. *Suppose that w is a k-Grassmannian element of $\widetilde{W}_\infty$.*

(a) *We have $\ell(s_0 w) < \ell(w)$ if and only if $w = (\cdots \overline{2} \cdots)$.*

(b) *Assume that $i \geq 1$. We have $\ell(s_i w) < \ell(w)$ if and only if w has one of the following four forms:*

$$(\cdots i+1 \cdots i \cdots) \quad (\cdots i \cdots \overline{i+1} \cdots) \quad (\overline{i} \cdots \overline{i+1} \cdots) \quad (\cdots \overline{i+1} \cdots i \cdots).$$

Let λ and μ be two typed k-strict partitions, set $w := w_\lambda$, $w' := w_\mu$, $\beta := \beta(\lambda)$, $\beta' := \beta(\mu)$, and assume that $w = s_i w'$ holds for some simple reflection $s_i \in \widetilde{W}_\infty$. It follows that $\mu \subset \lambda$, so μ is obtained by removing a single box from λ, and hence $\mu_p = \lambda_p - 1$ for some $p \geq 1$ and $\mu_j = \lambda_j$ for all $j \neq p$. Moreover, we must have $\mathrm{type}(\lambda) + \mathrm{type}(\mu) \neq 3$.

Using Lemma 4, we distinguish seven possible cases for w, discussed below. In each case, the properties listed follow immediately from equations (1), (2), and (3). First, we consider the four cases with $i \geq 1$:

(a) $w = (\cdots i+1 \cdots i \cdots)$. In this case $\mathcal{C}(\lambda) = \mathcal{C}(\mu)$, $\beta_p = i$, $\beta'_p = i+1$, while $\beta_j = \beta'_j$ for all $j \neq p$.

(b) $w = (\cdots i \cdots \overline{i+1} \cdots)$. In this case $\mathcal{C}(\lambda) = \mathcal{C}(\mu)$, $\beta_p = -i$, $\beta'_p = -i+1$, and $\beta_j = \beta'_j$ for all $j \neq p$.

(c) $w = (\bar{i} \cdots \overline{i+1} \cdots)$. In this case $w_1 = \bar{i}$, $\mathrm{type}(\lambda) = 2$ if $i \geq 2$, $\mathcal{C}(\lambda) = \mathcal{C}(\mu)$, $\beta_p = -i$, $\beta'_p = -i+1$, and $\beta_j = \beta'_j$ for all $j \neq p$.

(d) $w = (\cdots \overline{i+1} \cdots i \cdots)$. We distinguish two subcases here: Case (d1): $w_1 \neq \overline{i+1}$. Then $\mathcal{C}(\lambda) = \mathcal{C}(\mu) \cup \{(p,q)\}$, where $w_{k+p} = \overline{i+1}$ and $w_{k+q} = i$. It follows that $\beta_p = -i$, $\beta_q = i$, $\beta'_p = -i+1 = \beta_p + 1$, and $\beta'_q = i+1 = \beta_q + 1$, while $\beta_j = \beta'_j$ for all $j \notin \{p,q\}$. Case (d2): $w_1 = \overline{i+1}$ and we have $w^{-1}(i) > k$. In this case $\mathrm{type}(\lambda) = 2$, $\mathcal{C}(\lambda) = \mathcal{C}(\mu)$, $\beta_p = i$, $\beta'_p = i+1$, and $\beta_j = \beta'_j$ for all $j \neq p$.

Next, we consider the three cases where $i = 0$.

(e) $w = (\widehat{1} \cdots \overline{2} \cdots)$. In this case $\mathcal{C}(\lambda) = \mathcal{C}(\mu)$, $\beta_p = -1$, and $\beta'_p = 0$ if $w_1 = 1$, while $\beta'_p = 1$ if $w_1 = \overline{1}$. We also have $\beta_j = \beta'_j$ for all $j \neq p$.

(f) $w = (\overline{2} \cdots \widehat{1} \cdots)$. In this case $\mathcal{C}(\lambda) = \mathcal{C}(\mu)$, $\beta'_p = 2$, and $\beta_p = 0$ if $w_{k+p} = \overline{1}$, while $\beta_p = 1$ if $w_{k+p} = 1$. We also have $\beta_j = \beta'_j$ for all $j \neq p$.

(g) $w = (\cdots \overline{2}\overline{1} \cdots)$, with $|w_1| > 2$. In this case $\mathcal{C}(\lambda) = \mathcal{C}(\mu) \cup \{(p,p+1)\}$, where $w_{k+p} = \overline{2}$ and $w_{k+p+1} = \overline{1}$. It follows that $\lambda_p = k+1$, $\lambda_{p+1} = k$, $\beta_p = -1$, $\beta_{p+1} = 0$, while $\mu_p = \mu_{p+1} = k$, $\beta'_p = 1$, $\beta'_{p+1} = 2$, and $\beta_j = \beta'_j$ for all $j \notin \{p, p+1\}$.

3.3. **Double eta polynomials and divided differences.** We are now ready to establish the fundamental result about the compatibility of the polynomials $H_\lambda(c\,|\,t)$ with left divided differences.

Proposition 5. *Let λ and μ be typed k-strict partitions such that $|\lambda| = |\mu| + 1$ and $w_\lambda = s_i w_\mu$ for some simple reflection $s_i \in \widetilde{W}_\infty$. Then we have*

$$\partial_i H_\lambda(c\,|\,t) = H_\mu(c\,|\,t)$$

in $B^{(k)}[t]$.

Proof. Let $w = w_\lambda$ and $w' = w_\mu$, where λ and μ are typed and such that $w = s_i w'$ holds. We are in the situation of §3.2, hence $\mu_p = \lambda_p - 1$ for some $p \geq 1$ and $\mu_j = \lambda_j$ for all $j \neq p$. Set $\beta = \beta(\lambda)$. Let ϵ_j denote the j-th standard basis vector in $\mathbb{Z}^\ell$. We now distinguish the following cases.

Case 1. $\mathrm{type}(\lambda) = \mathrm{type}(\mu) = 0$.

Note that we have $|w_1| = |w'_1| = 1$, and hence $i \geq 2$ and $\ell_k(\lambda) = \ell_k(\mu)$. We must be in one among cases (a), (b), or (d1) of §3.2. In cases (a) or (b), it follows from Propositions 1 and 2 and the Leibnitz rule that for any integer sequence $\alpha = (\alpha_1, \ldots, \alpha_\ell)$, we have

$$\begin{aligned}\partial_i \widehat{c}_\alpha^{\beta(\lambda)} &= \widehat{c}^{(\beta_1,\ldots,\beta_{p-1})}_{(\alpha_1,\ldots,\alpha_{p-1})}\left(\partial_i(\widehat{c}^{\beta_p}_{\alpha_p})\widehat{c}^{(\beta_{p+1},\ldots,\beta_\ell)}_{(\alpha_{p+1},\ldots,\alpha_\ell)} + s_i(\widehat{c}^{\beta_p}_{\alpha_p})\partial_i(\widehat{c}^{(\beta_{p+1},\ldots,\beta_\ell)}_{(\alpha_{p+1},\ldots,\alpha_\ell)})\right) \\ &= \widehat{c}^{(\beta_1,\ldots,\beta_{p-1})}_{(\alpha_1,\ldots,\alpha_{p-1})}\left(\widehat{c}^{\beta_p+1}_{\alpha_p-1}\widehat{c}^{(\beta_{p+1},\ldots,\beta_\ell)}_{(\alpha_{p+1},\ldots,\alpha_\ell)} + s_i(\widehat{c}^{\beta_p}_{\alpha_p})\cdot 0\right) = \widehat{c}^{(\beta_1,\ldots,\beta_p+1,\ldots,\beta_\ell)}_{(\alpha_1,\ldots,\alpha_p-1,\ldots,\alpha_\ell)} = \widehat{c}^{\beta(\mu)}_{\alpha-\epsilon_p}.\end{aligned}$$

Since $\lambda - \epsilon_p = \mu$, it follows that if R is any raising operator, then

$$\partial_i(R\star\widehat{c}^{\beta(\lambda)}_\lambda) = \partial_i(\overline{c}^{\beta(\lambda)}_{R\lambda}) = \overline{c}^{\beta(\mu)}_{R\lambda-\epsilon_p} = R\star\widehat{c}^{\beta(\mu)}_\mu.$$

As $R^\lambda = R^\mu$, we deduce that

$$\partial_i H_\lambda(c\,|\,t) = 2^{-\ell_k(\lambda)}\partial_i(R^\lambda\star\widehat{c}^{\beta(\lambda)}_\lambda) = 2^{-\ell_k(\mu)}R^\mu\star\widehat{c}^{\beta(\mu)}_\mu = H_\mu(c\,|\,t).$$

In case (d1), for any integer sequence $\alpha = (\alpha_1, \ldots, \alpha_\ell)$, we compute that

$$\begin{aligned}\partial_i\widehat{c}^{\beta(\lambda)}_\alpha &= \partial_i\widehat{c}^{(\beta_1,\ldots,-i,\ldots,i,\ldots,\beta_\ell)}_{(\alpha_1,\ldots,\alpha_p,\ldots,\alpha_q,\ldots,\alpha_\ell)} \\ &= \widehat{c}^{(\beta_1,\ldots,-i+1,\ldots,i+1,\ldots,\beta_\ell)}_{(\alpha_1,\ldots,\alpha_p-1,\ldots,\alpha_q,\ldots,\alpha_\ell)} + \widehat{c}^{(\beta_1,\ldots,-i+1,\ldots,i+1,\ldots,\beta_\ell)}_{(\alpha_1,\ldots,\alpha_p,\ldots,\alpha_q-1,\ldots,\alpha_\ell)} = \widehat{c}^{\beta(\mu)}_{\alpha-\epsilon_p} + \widehat{c}^{\beta(\mu)}_{\alpha-\epsilon_q}.\end{aligned}$$

This follows from the Leibnitz rule, as in the proof of Proposition 1(b). If R is any raising operator, then since $i \geq 2$ we must have $q > \ell_k(\lambda)$ and hence $q \notin \mathrm{supp}_m(RR_{pq})$, where $m = \ell_k(\mu)+1$. As $\lambda - \epsilon_p = \mu$, we deduce that

$$\partial_i(R\star\widehat{c}^{\beta(\lambda)}_\lambda) = \partial_i(\overline{c}^{\beta(\lambda)}_{R\lambda}) = \overline{c}^{\beta(\mu)}_{R\lambda-\epsilon_p} + \overline{c}^{\beta(\mu)}_{R\lambda-\epsilon_q} = R\star\widehat{c}^{\beta(\mu)}_\mu + RR_{pq}\star\widehat{c}^{\beta(\mu)}_\mu.$$

Since $R^\lambda + R^\lambda R_{pq} = R^\mu$, it follows that $\partial_i H_\lambda(c\,|\,t) = H_\mu(c\,|\,t)$.

Case 2. $\mathrm{type}(\lambda) = 0$ and $\mathrm{type}(\mu) > 0$.

In this case, we have $|w_1| = 1$ and $|w_1'| > 1$, so $i \in \{0,1\}$. We must be in one of cases (b), (c), or (e) of §3.2, hence $\mathcal{C}(\lambda) = \mathcal{C}(\mu)$. We also have $\lambda_p = k+1$ and $\lambda_{p+1} < k$, so $(p,p+1) \notin \mathcal{C}(\lambda)$, $\beta_p(\lambda) = -1$, $\beta_p(\mu) \in \{0,1\}$, and $\ell_k(\lambda) = \ell_k(\mu)+1$.

Observe that, for any integer sequence $\alpha = (\alpha_1, \ldots, \alpha_\ell)$, we have

$$\begin{aligned}\partial_i\widehat{c}^{\beta(\lambda)}_\alpha &= \widehat{c}^{(\beta_1,\ldots,\beta_{p-1})}_{(\alpha_1,\ldots,\alpha_{p-1})}\left(\partial_i(\widehat{c}^{-1}_{\alpha_p})\widehat{c}^{(\beta_{p+1},\ldots,\beta_\ell)}_{(\alpha_{p+1},\ldots,\alpha_\ell)} + s_i(\widehat{c}^{-1}_{\alpha_p})\partial_i(\widehat{c}^{(\beta_{p+1},\ldots,\beta_\ell)}_{(\alpha_{p+1},\ldots,\alpha_\ell)})\right) \\ &= \widehat{c}^{(\beta_1,\ldots,\beta_{p-1})}_{(\alpha_1,\ldots,\alpha_{p-1})}\partial_i(\widehat{c}^{-1}_{\alpha_p})\widehat{c}^{(\beta_{p+1},\ldots,\beta_\ell)}_{(\alpha_{p+1},\ldots,\alpha_\ell)}.\end{aligned}$$

We now compute using Propositions 1 and 2(a) that

$$\partial_1\widehat{c}^{-1}_q = \begin{cases} 2a^0_{q-1} & \text{if } q \neq k+1 \\ 2f_k & \text{if } q = k+1. \end{cases}$$

Proposition 1(a) and equation (20) give

$$\partial_0\widehat{c}^{-1}_q = \begin{cases} 2a^1_{q-1} & \text{if } q \neq k+1 \\ 2\widetilde{f}^1_k & \text{if } q = k+1. \end{cases}$$

The rest is straightforward from the definitions, arguing as in Case 1.

Case 3. $\mathrm{type}(\lambda) > 0$ and $\mathrm{type}(\mu) = 0$.

We have $|w_1| > 1$ and $|w_1'| = 1$, so $i \in \{0,1\}$, and we are in one of cases (a), (d2), or (f) of §3.2, hence $\mathcal{C}(\lambda) = \mathcal{C}(\mu)$. We also have $\lambda_p = k$, $\beta_p(\lambda) \in \{0,1\}$, $\beta_p(\mu) = 2$, and $\ell_k(\lambda)=\ell_k(\mu)$. Recall that $\widehat{c}^r_p=c^r_p$ whenever $p\leq k$, $b^1_k=c^1_k-\widetilde{b}_k$, $\widetilde{b}^1_k=c^1_k-b_k$, and $a^s_p=c^s_p-\frac{1}{2}c_p$. We deduce the calculations

$$\partial_0 b_k=\partial_0\widetilde{b}_k=\partial_1 b^1_k=\partial_1\widetilde{b}^1_k=c^2_{k-1}$$
$$\partial_0 a^0_p=\partial_1 a^1_p=c^2_{p-1}.$$

As in the previous cases, it follows that $\partial_i H_\lambda(c\,|\,t)=H_\mu(c\,|\,t)$.

Case 4. $\text{type}(\lambda)=\text{type}(\mu)>0$.

We have $|w_1|>1$ and $|w'_1|>1$. If $i\geq 2$, then we must be in one of cases (a), (b), (c), or (d1) of §3.2, and the result is proved by arguing as in Case 1. It remains to study (i) case (d1) with $w=(\cdots\overline{2}1\cdots)$ and $i=1$, or (ii) case (g) with $w=(\cdots\overline{21}\cdots)$ and $i=0$. In both of these subcases, we have $\mathcal{C}(\lambda)=\mathcal{C}(\mu)\cup\{(p,p+1)\}$, $\ell_k(\lambda)=\ell_k(\mu)+1$, $\lambda_p=k+1$, $\lambda_{p+1}=\mu_p=\mu_{p+1}=k$, $\beta_p(\lambda)=-1$, $\beta_{p+1}(\mu)=2$, and $\beta_j(\lambda)=\beta_j(\mu)$ for all $j\notin\{p,p+1\}$. In subcase (i), we have $\beta_{p+1}(\lambda)=1$ and $\beta_p(\mu)=0$, while in subcase (ii), we have $\beta_{p+1}(\lambda)=0$ and $\beta_p(\mu)=1$.

To deal with subcase (i), we argue as in Case 1 (d1), this time applying the identities (25)–(28). There is now an added complication: we must show that the total contribution from the four residual terms that appear with a negative sign in equations (25)–(28) vanishes. To prove this, we may assume that λ has length $p+1$, and consider the effect of the raising operators R in the expansion of R^λ which involve only basic operators R_{ij} with $i=p$ or $j=p+1$.

An integer sequence $\alpha=(\alpha_1,\ldots,\alpha_\ell)$ is a *composition* if $\alpha_i\geq 0$ for all i. For any composition α, let $|\alpha|:=\sum_i\alpha_i$ and $\#\alpha$ denote the number of non-zero components α_i. The relevant raising operator expression is

$$\begin{aligned}\Psi &:= \left(\prod_{i=1}^{p-1}\frac{1-R_{ip}}{1+R_{ip}}\right)\left(\prod_{i=1}^{p-1}\frac{1-R_{i,p+1}}{1+R_{i,p+1}}\right)\frac{1-R_{p,p+1}}{1+R_{p,p+1}}\\ &= \sum_{\alpha',\alpha,d\geq 0}(-1)^{|\alpha'|+|\alpha|+d}\,2^{\#(\alpha',\alpha,d)}\left(\prod_{i=1}^{p-1}R_{ip}^{\alpha'_i}\,R_{i,p+1}^{\alpha_i}\right)R^d_{p.p+1}\end{aligned}$$

where the sum is over all compositions $\alpha'=(\alpha'_1,\ldots,\alpha'_{p-1})$, $\alpha=(\alpha_1,\ldots,\alpha_{p-1})$, and integers $d\geq 0$. If $\nu=(\nu_1,\ldots,\nu_{p-1})$ is a fixed integer vector, then $(-1/2)$ times the total residual term in the expansion of $\partial_1(\Psi\,\overline{c}^{(\beta_1,\ldots,\beta_{p+1})}_{(\nu,k+1,k)})$ is equal to

$$S_\nu:=\sum_{\alpha',\alpha,d\geq 0}(-1)^{|\alpha'|+|\alpha|+d}\,2^{\#(\alpha',\alpha,d)}\,\overline{c}^{(\beta_1,\ldots,\beta_{p-1})}_{\nu+\alpha'+\alpha}\,\overline{a}_{k-|\alpha'|+d}\,\overline{a}_{k-|\alpha|-d}. \tag{29}$$

The two factors $\overline{a}_q$ in each summand of (29) are equal to a_q, f_k, or $\widetilde{f}_k$, according to the equations (25)–(28) and depending on the choice of α', α, and d, as in Definition 1. We now make a change of variables in the sum (29) by setting $\rho:=\alpha'+\alpha$ and $r:=|\alpha|+d$, to obtain

$$S_\nu=\sum_{\rho\geq 0}(-1)^{|\rho|}\,\overline{c}^{(\beta_1,\ldots,\beta_{p-1})}_{\nu+\rho}\sum_{r=0}^{k}T(\rho,r) \tag{30}$$

where the first sum is over all compositions ρ and

$$T(\rho,r):=(-1)^r\,\overline{a}_{k-|\rho|+r}\,\overline{a}_{k-r}\sum_{\substack{0\leq\alpha\leq\rho\\ |\alpha|\leq r}}(-1)^{|\alpha|}\,2^{\#(\rho-\alpha,\alpha,r-|\alpha|)}.$$

We compute that

$$\sum_{r=0}^{k} T(0,r) = f_k\widetilde{f}_k + 2\sum_{r=1}^{k}(-1)^r a_{k+r}a_{k-r} = b_k\widetilde{b}_k + \sum_{r=1}^{k}(-1)^r b_{k+r}b_{k-r} \in J^{(k)}.$$

It follows that the sum of the terms in (30) with $\rho = 0$ vanishes in $B^{(k)}[t]$. We claim that the sum of all the remaining terms in (30) is identically zero.

Lemma 5. *Let ρ be a non-zero composition. If $r > |\rho|$, then $T(\rho, r) = 0$, while if $0 \leq r \leq |\rho|$, then $T(\rho, r) + T(\rho, |\rho| - r) = 0$.*

Proof. The argument is based on the elementary identity

$$\sum_{i=0}^{s}(-1)^i\, 2^{\#(s-i,i)} = \delta_{s,0}. \tag{31}$$

By multiplying together a finite number of equations of the form (31), we obtain

$$\sum_{0\leq\alpha\leq\rho}(-1)^{|\alpha|}\, 2^{\#(\rho-\alpha,\alpha)} = \delta_{\rho,0} \tag{32}$$

for any composition ρ, where the sum is over all compositions α with $\alpha \leq \rho$.

Assume now that $\rho \neq 0$. If $r > |\rho|$, then using (32) gives

$$T(\rho,r) = (-1)^r\, a_{k-|\rho|+r}\, a_{k-r}\cdot 2\sum_{0\leq\alpha\leq\rho}(-1)^{|\alpha|}\, 2^{\#(\rho-\alpha,\alpha)} = 0.$$

If $0 < r < |\rho|$, then

$$T(\rho,r) = (-1)^r a_{k-|\rho|+r}\, a_{k-r}\sum_{\substack{0\leq\alpha\leq\rho\\ |\alpha|\leq r}}(-1)^{|\alpha|}\, 2^{\#(\rho-\alpha,\alpha,r-|\alpha|)} \tag{33}$$

and the substitution $\alpha' := \rho - \alpha$ gives

$$T(\rho,|\rho|-r) = (-1)^r a_{k-r}\, a_{k-|\rho|+r}\sum_{\substack{0\leq\alpha'\leq\rho\\ |\alpha'|\geq r}}(-1)^{|\alpha'|}\, 2^{\#(\rho-\alpha',\alpha',|\alpha'|-r)}. \tag{34}$$

Adding (33) to (34) and applying (32) gives $T(\rho, r) + T(\rho, |\rho| - r) = 0$.

Finally, we have

$$T(\rho,0) = 2^{\#\rho}\, a_{k-|\rho|}\widetilde{f}_k$$

while

$$\begin{aligned} T(\rho,|\rho|) &= (-1)^{|\rho|}\overline{a}_k\overline{a}_{k-|\rho|}\sum_{0\leq\alpha\leq\rho}(-1)^{|\alpha|}\, 2^{\#(\rho-\alpha,\alpha,|\rho|-|\alpha|)} \\ &= 2^{\#\rho}\, f_k a_{k-|\rho|} + 2a_k a_{k-|\rho|}\sum_{\substack{0\leq\alpha\leq\rho\\ \alpha\neq\rho}}(-1)^{|\rho|-|\alpha|}\, 2^{\#(\rho-\alpha,\alpha)}. \end{aligned}$$

Since $f_k + \widetilde{f}_k = 2a_k$, adding the previous equations and applying (32) again shows that $T(\rho, 0) + T(\rho, |\rho|) = 0$. ☐

Using Lemma 5 in equation (30) proves the claim, and completes the argument in subcase (i). The proof for subcase (ii) is similar, this time using the equations (21)–(24) and the relation

$$\widetilde{f}^1_k f^1_k + 2\sum_{r=1}^{k}(-1)^r a^1_{k+r}a^1_{k-r} = 0$$

in $B^{(k)}[t]$, which is easily checked. $\square$

Remark 1. The proof of Proposition 5 establishes that the equality $\partial_i H_\lambda(c\,|\,t) = H_\mu(c\,|\,t)$ holds in $\mathbb{Z}[b,t]$ in all cases of §3.2 except case (d1) with $i=1$ or case (g) with $i=0$. In each of the latter two cases, we need to use the relation (7) exactly once. The basic example that illustrates this is the equality

$$\partial_i H_{(k+1,k)}(c\,|\,t) = H_{(k,k)}(c\,|\,t), \tag{35}$$

where $i \in \{0,1\}$ and both of the indexing partitions have the same (positive) type. Equation (35) is true in $B^{(k)}[t]$, but fails in $\mathbb{Z}[b,t]$.

3.4. **The polynomials $\widehat{H}_\lambda(c\,|\,t)$.** In this subsection we define and study a closely related family of polynomials $\widehat{H}_\lambda(c\,|\,t)$ indexed by k-strict partitions λ. The polynomials $\widehat{H}_\lambda(c) := \widehat{H}_\lambda(c\,|\,0)$ were studied in [BKT2, §5.2]. As explained in op. cit., $\widehat{H}_\lambda(c)$ represents the cohomology class of a certain Zariski closed subset Y_λ of $\mathrm{OG}(n-k,2n)$, which is either a Schubert variety or a union of two Schubert varieties. The double polynomials $\widehat{H}_\lambda(c\,|\,t)$ similarly represent the T_n-equivariant cohomology class $[Y_\lambda]^{T_n}$ in $\mathrm{H}^*_{T_n}(\mathrm{OG})$, under the geometrization map π_n defined in §4.1; this follows immediately from their definition below and Theorem 1.

If λ is any k-strict partition, define the finite set of pairs

$$\mathcal{C}(\lambda) := \{(i,j) \in \mathbb{N}\times\mathbb{N} \mid 1 \le i < j \ \text{ and } \ \lambda_i + \lambda_j \ge 2k + j - i\}$$

and the sequence $\overline{\beta}(\lambda) = \{\overline{\beta}_j(\lambda)\}_{j\ge 1}$ by

$$\overline{\beta}_j(\lambda) := k - \lambda_j + \#\{i<j \mid (i,j) \notin \mathcal{C}(\lambda)\} + \begin{cases} 1 & \text{if } \lambda_j \le k \\ 0 & \text{if } \lambda_j > k, \end{cases} \quad \text{for all} \ \ j \ge 1.$$

We have $\mathcal{C}(\lambda) = \mathcal{C}(\overline{\lambda})$ and $\overline{\beta}(\lambda) = \beta(\overline{\lambda})$, where $\overline{\lambda}$ denotes the unique typed k-strict partition which has the same shape as λ, and with the property that $\beta_j(\overline{\lambda}) \neq 0$, for each $j \ge 1$. For comparison with [BKT2], we note that $\overline{\beta}_j(\lambda) = \overline{p}_j(\lambda) - n$, where $\overline{p}_j(\lambda)$ is the function defined in the introduction of op. cit.

If $\lambda_i = k$ for some index i, then we agree that $H_\lambda(c\,|\,t)$ and $H'_\lambda(c\,|\,t)$ denote the double eta polynomials indexed by λ of type 1 and 2, respectively; otherwise, $H_\lambda(c\,|\,t)$ denotes the associated double eta polynomial indexed by λ of type zero. We define the raising operator expression R^λ by equation (5), as before.

Definition 2. For any k-strict partition λ, let $m := \ell_k(\lambda)+1$ and $\overline{\beta} := \overline{\beta}(\lambda)$. If R is any raising operator appearing in the expansion of the power series R^λ and $\nu := R\lambda$, define

$$R \star \widehat{c}_\lambda^{\,\overline{\beta}(\lambda)} := \overline{c}_{\nu_1}^{\overline{\beta}_1} \cdots \overline{c}_{\nu_\ell}^{\overline{\beta}_\ell}$$

where for each $i \ge 1$,

$$\overline{c}_{\nu_i}^{\overline{\beta}_i} := \begin{cases} c_{\nu_i}^{\overline{\beta}_i} & \text{if } i \in \mathrm{supp}_m(R), \\ \widehat{c}_{\nu_i}^{\overline{\beta}_i} & \text{otherwise.} \end{cases}$$

The polynomial $\widehat{H}_\lambda(c\,|\,t)$ is defined by

$$\widehat{H}_\lambda(c\,|\,t) := 2^{-\ell_k(\lambda)} R^\lambda \star \widehat{c}_\lambda^{\,\overline{\beta}(\lambda)} = \begin{cases} H_\lambda(c\,|\,t) + H'_\lambda(c\,|\,t) & \text{if } \lambda_i = k \text{ for some } i, \\ H_\lambda(c\,|\,t) & \text{otherwise.} \end{cases}$$

Table 2 lists the double eta hat polynomials associated to the Grassmannian elements in $\widetilde{W}_3$. We have retained the negative powers of 2 in this table for clarity.

TABLE 2. Double eta hat polynomials for Grassmannian $w \in \widetilde{W}_3$

w	λ	$\overline{\beta}$	$\widehat{H}_\lambda(c\,\|\,t)$
123			1
$213,\ \overline{21}3$	1	$(1,3)$	$c_1+h_1^1$
$\overline{12}3$	2	$(-1,3)$	$\frac{1}{2}(c_2+2\widetilde{b}_1e_1^1)$
$312,\ \overline{31}2$	$(1,1)$	$(1,2)$	$(c_1+h_1^1)(c_1+h_1^2)-(c_2+c_1h_1^1+h_2^1)$
$\overline{13}2$	3	$(-2,2)$	$\frac{1}{2}(c_3+c_2e_1^2+2\widetilde{b}_1e_2^2)$
$3\overline{21}, \overline{321}$	$(2,1)$	$(-1,1)$	$\frac{1}{2}((c_2+2\widetilde{b}_1e_1^1)(c_1+h_1^1)-2(c_3+c_2e_1^1))$
$2\overline{31},\ \overline{231}$	$(3,1)$	$(-2,1)$	$\frac{1}{2}((c_3+c_2e_1^2+2\widetilde{b}_1e_2^2)(c_1+h_1^1)-2(c_4+c_3e_1^2+c_2e_2^2))$
$1\overline{32}$	$(3,2)$	$(-2,-1)$	$\frac{1}{4}((c_3+c_2e_1^2+2\widetilde{b}_1e_2^2)(c_2+2b_1e_1^1)$
			$-2(c_4+c_3e_1^2+c_2e_2^2)(c_1+e_1^1)+2(c_5+c_4e_1^2+c_3e_2^2))$
132	1	2	$c_1+h_1^2$
$231,\ \overline{2}3\overline{1}$	2	1	$c_2+c_1h_1^1+h_2^1$
$\overline{1}3\overline{2}$	3	-1	$\frac{1}{2}(c_3+2\widetilde{b}_2e_1^1)$
$\overline{1}2\overline{3}$	4	-2	$\frac{1}{2}(c_4+c_3e_1^2+2\widetilde{b}_2e_2^2)$

Proposition 6. *Let λ and μ be two k-strict partitions with $|\lambda| = |\mu| + 1$. Assume that there exist a simple reflection $s_i \in \widetilde{W}_\infty$ and a choice of type assigned to λ and μ such that $w_\lambda = s_i w_\mu$ in $\widetilde{W}_\infty$. Then the following assertions hold in $\mathbb{Z}[b,t]$.*

(i) *If* $\mathrm{type}(\lambda) = \mathrm{type}(\mu) = 0$*, then $i \geq 2$ and*
$$\partial_i \widehat{H}_\lambda(c\,|\,t) = \widehat{H}_\mu(c\,|\,t).$$
(ii) *If* $\mathrm{type}(\lambda) = 0$ *and* $\mathrm{type}(\mu) > 0$*, then $i \in \{0,1\}$ and*
$$(\partial_0 + \partial_1)\widehat{H}_\lambda(c\,|\,t) = \widehat{H}_\mu(c\,|\,t).$$
(iii) *If* $\mathrm{type}(\lambda) > 0$ *and* $\mathrm{type}(\mu) = 0$*, then $i \in \{0,1\}$ and*
$$\partial_0 \widehat{H}_\lambda(c\,|\,t) = \partial_1 \widehat{H}_\lambda(c\,|\,t) = \widehat{H}_\mu(c\,|\,t).$$
(iv) *If* $\mathrm{type}(\lambda) = \mathrm{type}(\mu) > 0$*, then*
$$\partial_i \widehat{H}_\lambda(c\,|\,t) = \widehat{H}_\mu(c\,|\,t),$$
if $i \geq 2$, and
$$(\partial_0 + \partial_1)\widehat{H}_\lambda(c\,|\,t) = \widehat{H}_\mu(c\,|\,t),$$
if $i \in \{0,1\}$.

Proof. We will only give the outline of the proof of claims (i)–(iv) here, as the argument is very similar to the proof of Proposition 5, only easier, because the relation (7) is never used. Recall the seven possible cases (a)–(g) for w_λ from §3.2.

For claim (i), or claim (iv) when $i \geq 2$, we must be in one among cases (a), (b), (c), or (d1) of §3.2, and the proof is exactly as in Proposition 5. We are left with examining the claims (ii), (iii), and (iv) when $i \in \{0,1\}$. For claim (ii), we must be in one of cases (b), (c), or (e), and we use equations (11) and (15). For claim (iii), we are in one of cases (a), (d2), or (f), and use the computation
$$\partial_0 c_p^1 = \partial_1 c_p^1 = c_{p-1}^2.$$
Finally, for claim (iv) we must be in case (d1) with $w = (\cdots \overline{2}1 \cdots)$ and $i = 1$, or in case (g) with $w = (\cdots \overline{21} \cdots)$ and $i = 0$. The result follows as in claim (i), case (d1), but now using equations (14) and (19). $\square$

4. The proof of Theorem 1

4.1. The geometrization map. Let

$$0 \to E' \to E \to E'' \to 0$$

denote the universal exact sequence of vector bundles over $\mathrm{OG}(n-k,2n)$, with E the trivial bundle of rank $2n$ and E' the tautological subbundle of rank $n-k$. For $0 \leq j \leq 2n$, define the subbundles F_j of E as in the introduction. Let $OM_n := ET_n \times^{T_n} \mathrm{OG}$ denote the Borel mixing space for the action of the torus T_n on OG. The T_n-equivariant vector bundles E', E, E'', F_j over OG induce vector bundles over OM_n, and their equivariant Chern classes in $\mathrm{H}^*(OM_n,\mathbb{Z}) = \mathrm{H}^*_{T_n}(\mathrm{OG}(n-k,2n))$ are denoted by $c_p^T(E')$, $c_p^T(E)$, $c_p^T(E'')$, and $c_p^T(F_j)$.

The class $c_p^T(E-E'-F_j)$ for $p \geq 0$ is defined by the total Chern class equation

$$c^T(E-E'-F_j) := c^T(E)c^T(E')^{-1}c^T(F_j)^{-1}.$$

Let $\mathrm{t}_i := -c_1^T(F_{n+1-i}/F_{n-i})$ for $1 \leq i \leq n$. Following [IMN1, §10] and [T2, §7], we define the *geometrization map* π_n as the $\mathbb{Z}[t]$-algebra homomorphism

$$\pi_n : B^{(k)}[t] \to \mathrm{H}^*_{T_n}(\mathrm{OG}(n-k,2n))$$

determined by setting

$$\pi_n(b_p) := \begin{cases} c_p^T(E-E'-F_n) & \text{if } p<k, \\ \frac{1}{2}c_p^T(E-E'-F_n) & \text{if } p>k, \end{cases}$$
$$\pi_n(b_k) := \frac{1}{2}(c_k^T(E-E'-F_n)+c_k^T(E_n-E')),$$
$$\pi_n(\widetilde{b}_k) := \frac{1}{2}(c_k^T(E-E'-F_n)-c_k^T(E_n-E')),$$
$$\pi_n(t_i) := \begin{cases} \mathrm{t}_i & \text{if } 1 \leq i \leq n, \\ 0 & \text{if } i>n. \end{cases}$$

Here E_n denotes a maximal isotropic subbundle of the (pullback of) E to the complete flag variety, which is in the same family as F_n. Note that the images of the elements b_p, $\widetilde{b}_k$, c_p in the ring of type D Billey-Haiman Schubert polynomials [BH] are given in [BKT2, §5] by the power series $\eta_p(x\,;y)$, $\eta'_k(x\,;y)$, $\vartheta_p(x\,;y)$, respectively, and, using this, the equations defining π_n are derived in [T2, §7.4]. For more information on the image of the double eta polynomials $H_\lambda(c\,|\,t)$ in the ring of type D double Schubert polynomials of [IMN1], see [T3, §4.5].

The above equations imply that $\pi_n(c_p) = c_p^T(E-E'-F_n)$ for all $p \geq 0$. Since $\mathrm{t}_1\ldots,\mathrm{t}_r$ are the (equivariant) Chern roots of F_{n+r}/F_n for $1 \leq r \leq n$, it follows that

$$\pi_n(c_p^r) = \sum_{j=0}^{p} c_{p-j}^T(E-E'-F_n)h_j^r(-\mathrm{t}) = c_p^T(E-E'-F_{n+r}) \tag{36}$$

for $-n \leq r \leq n$. Equation (36) can be extended to any $r \in \mathbb{Z}$ if we set $F_j = F_{2n} = E$ for $j>2n$ and $F_j = 0$ for $j<0$. Moreover, for $s := p-k > 0$, we have

$$\pi_n(\widehat{c}_p^{\,-s}) = \pi_n(c_p^{-s} + (2f_k - c_k)e_s^s(-t)) = c_p^T(E-E'-F_{n-s}) \pm e^T(E',F_{n-s}),$$

where the sign depends on the choice of $f_k \in \{b_k, \widetilde{b}_k\}$, as above, and the equivariant Euler class $e^T(E', F_{n-s})$ is given by

$$e^T(E', F_{n-s}) := c_p^T(E_n/E' + F_n/F_{n-s}) = c_k^T(E_n - E')c_s^T(F_n - F_{n-s}).$$

The embedding of $\widetilde{W}_n$ into $\widetilde{W}_{n+1}$ defined in the introduction induces maps of equivariant cohomology rings $\mathrm{H}^*_{T_{n+1}}(\mathrm{OG}(n+1-k, 2n+2)) \to \mathrm{H}^*_{T_n}(\mathrm{OG}(n-k, 2n))$ which are compatible with the morphisms π_n. We therefore obtain an induced $\mathbb{Z}[t]$-algebra homomorphism

$$\pi : B^{(k)}[t] \to \mathbb{H}_T(\mathrm{OG}_k).$$

The above map π is the one that appears in Theorem 1, and we proceed to show that it has the properties listed there.

4.2. **Proof of Theorem 1.** The argument is similar to the one found in [TW, §6.3], but we include the details here for completeness. Fix a rank n and let

$$\lambda_0 := (n+k-1, n+k-2, \ldots, 2k)$$

be the typed k-strict partition associated to the k-Grassmannian element of maximal length in $\widetilde{W}_n$. Definition 1 gives

$$H_{\lambda_0}(c\,|\,t) = 2^{k-n}\, R^{\lambda_0}_{n-k} \star c_{\lambda_0}^{(1-n,2-n,\ldots,-k)} \tag{37}$$

where

$$R^{\lambda_0}_{n-k} := \prod_{1\leq i<j\leq n-k} \frac{1-R_{ij}}{1+R_{ij}}\,.$$

Using (37) and the equations of §4.1, one checks that $\pi_n(H_{\lambda_0}(c\,|\,t))$ agrees with a known formula of Kazarian [Ka] for the cohomology class of the degeneracy locus which correponds to $[X_{\lambda_0}]^{T_n}$. Although the final result in [Ka, App. D] is expressed as a Pfaffian, this is not required for the application here. (The equivalence of the two formulas is a consequence of some formal Pfaffian algebra from [Ka, Kn]; for a detailed discussion of this, see [AF, App. A]). It follows that

$$\pi_n(H_{\lambda_0}(c\,|\,t)) = [X_{\lambda_0}]^{T_n}. \tag{38}$$

We have shown in Proposition 4 that the $H_\lambda(c\,|\,t)$ for λ a typed k-strict partition form a $\mathbb{Z}[t]$-basis of $B^{(k)}[t]$. Let $\widetilde{\mathcal{P}}(k,n)$ denote the set of all typed k-strict partitions whose diagrams fit inside a rectangle of size $(n-k)\times(n+k-1)$. The elements of $\widetilde{\mathcal{P}}(k,n)$ correspond to the k-Grassmannian elements of $\widetilde{W}_n$ under the bijection described in the introduction. Let w_λ denote the element of $\widetilde{W}_n$ associated to λ under this bijection.

Following [TW, §6.3], for any typed k-strict partition $\lambda \in \widetilde{\mathcal{P}}(k,n)$, write $w_\lambda w_{\lambda_0} = s_{a_1}\cdots s_{a_r}$ as a product of simple reflections s_{a_j} in $\widetilde{W}_n$, with $r = |\lambda_0| - |\lambda|$. Since $w_{\lambda_0}^2 = 1$, Proposition 5 implies that

$$H_\lambda(c\,|\,t) = \partial_{a_1}\circ\cdots\circ\partial_{a_r}(H_{\lambda_0}(c\,|\,t)) \tag{39}$$

holds in $B^{(k)}[t]$.

The left divided differences δ_i on $\mathrm{H}^*_{T_n}(\mathrm{OG}(n-k,2n))$ from [IMN1, §2.5] correspond to the operators ∂_i on $B^{(k)}[t]$, and are compatible with the geometrization map $\pi_n : B^{(k)}[t] \to \mathrm{H}^*_{T_n}(\mathrm{OG}(n-k,2n))$. Moreover, it is known by [IMN1, Prop.

2.3] that $\delta_i([X_\lambda]^{T_n}) = [X_\mu]^{T_n}$ whenever $|\lambda| = |\mu|+1$ and $w_\lambda = s_i w_\mu$ for some simple reflection s_i. It follows from this and equations (38) and (39) that

$$\pi_n(H_\lambda(c\,|\,t)) = [X_\lambda]^{T_n}.$$

The vanishing property for equivariant Schubert classes (see, for example, [IMN1, Prop. 7.7]) now implies that $\pi_n(H_\lambda(c\,|\,t)) = 0$ whenever $\lambda \notin \widetilde{\mathcal{P}}(k,n)$ (or equivalently $w_\lambda \notin \widetilde{W}_n$). The induced map $\pi : B^{(k)}[t] \to \mathbb{H}_T(\mathrm{OG}_k)$ satisfies $\pi(H_\lambda(c\,|\,t)) = \tau_\lambda$ for all typed k-strict partitions λ, and is a $\mathbb{Z}[t]$-algebra isomorphism because the $H_\lambda(c\,|\,t)$ and τ_λ for λ k-strict and typed form $\mathbb{Z}[t]$-bases of the respective algebras.

4.3. **A splitting theorem for $H_\lambda(c\,|\,t)$.** In this subsection, following [TW, Cor. 2], we apply Theorem 1 to compare the double eta polynomials $H_\lambda(c\,|\,t)$ of the present paper with the general degeneracy locus formulas of [T1, §6].

The symmetric group S_n is the subgroup of $\widetilde{W}_n$ generated by the transpositions s_i for $1 \leq i \leq n-1$; we let $S_\infty := \cup_n S_n$ be the corresponding subgroup of $\widetilde{W}_\infty$. For every permutation $u \in S_\infty$, let $\mathfrak{S}_u(t)$ denote the type A Schubert polynomial of Lascoux and Schützenberger [LS] indexed by u (our notation follows [T2, §5]). The $\mathfrak{S}_u(t)$ for $u \in S_\infty$ form a free $\mathbb{Z}$-basis of the polynomial $\mathbb{Z}[t]$. We deduce from Proposition 4 that the products $H_\mu(c)\mathfrak{S}_u(-t)$ where μ ranges over all typed k-strict partitions and $u \in S_\infty$ form a free $\mathbb{Z}$-basis of $B^{(k)}[t]$. The following result gives the unique expansion of (the class of) the double eta polynomial $H_\lambda(c\,|\,t)$ in $B^{(k)}[t]$ as a $\mathbb{Z}$-linear combination of this product basis.

We say that a factorization $w_\lambda = uv$ in $\widetilde{W}_\infty$ is reduced if $\ell(w_\lambda) = \ell(u) + \ell(v)$. In any such factorization, the right factor $v = w_\mu$ is also k-Grassmannian for some typed k-strict partition μ.

Corollary 1. *Let λ be any typed k-strict partition. Then we have*

$$H_\lambda(c\,|\,t) = \sum_{uw_\mu = w_\lambda} H_\mu(c)\mathfrak{S}_{u^{-1}}(-t) \tag{40}$$

in the ring $B^{(k)}[t]$, where the sum is over all reduced factorizations $uw_\mu = w_\lambda$ with $u \in S_\infty$.

Proof. As a special case of the splitting and degeneracy locus formulas of [T1, §6], we deduce that the polynomial on right hand side of (40) represents the stable equivariant Schubert class τ_λ in $\mathbb{H}_T(\mathrm{OG}_k)$ under the geometrization map π. The result is therefore a direct consequence of Theorem 1. □

It is tempting to view Corollary 1 as a separation of the variables b and t in $H_\lambda(c\,|\,t)$. However equation (40) does not hold in the polynomial ring $\mathbb{Z}[b,t]$ for a general λ, as it depends on the relations (6) and (7) among the b_p.

4.4. **The Grassmannian $\mathrm{OG}(n,2n)$.** We conclude this paper with a short discussion of the situation when $k=0$, so that $\mathrm{OG} = \mathrm{OG}(n,2n)$ parametrizes one connected component of the space of all isotropic subspaces of $\mathbb{C}^{2n}$ of maximal dimension n. One knows that this variety is isomorphic (in fact, projectively equivalent) to the odd orthogonal Grassmannian $\mathrm{OG}(n-1,2n-1)$. Moreover, one can arrange that this isomorphism is torus-equivariant, and hence induces an isomorphism of equivariant cohomology rings (see e.g. [IMN2, §3.5]). It follows that the double theta polynomials $\Theta_\lambda(c\,|\,t)$ of [TW] times the appropriate negative power of

2, which represent the equivariant Schubert classes on $\mathrm{OG}(n-1, 2n-1)$, also serve as equivariant Giambelli polynomials for $\mathrm{OG}(n, 2n)$ (compare with [IMN2]).

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University of Maryland, Department of Mathematics, 1301 Mathematics Building, College Park, MD 20742, USA

E-mail address: harryt@math.umd.edu