Arakelov Theory of the Lagrangian Grassmannian

Harry Tamvakis

Abstract

Let E be a symplectic vector space of dimension $2n$ (with the standard antidiagonal symplectic form) and let G be the Lagrangian Grassmannian over $\mathrm{Spec}\mathbb{Z}$, parametrizing Lagrangian subspaces in E over any base field. Equip $E(\mathbb{C})$ with a hermitian metric compatible with the symplectic form and $G(\mathbb{C})$ with the Kähler metric induced from the natural invariant metric on the Grassmannian of n-planes in E. We give a presentation of the Arakelov Chow ring $CH(\overline{G})$ and develop an arithmetic Schubert calculus in this setting. The theory uses the $\widetilde{Q}$-polynomials of Pragacz and Ratajski [PR] and involves 'shifted hook operations' on Young diagrams. As an application, we compute the Faltings height of G with respect to its Plücker embedding in projective space.

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1 Introduction

The extension of Arakelov theory to higher dimensions by Gillet and Soulé [GS1] is an intriguing combination of arithmetic, algebraic geometry and complex differential geometry. One of the challenges of the theory is to make explicit computations in cases where the geometric picture is well understood. The difficulties lie mainly over the infinite places, where analysis provides inequalities much more often than equalities.

The pairing constructed by Arakelov [A] for arithmetic surfaces does not give a ring structure in higher dimensions unless the harmonic forms at the archimedean places are closed under wedge products. Thus there are few examples of arithmetic varieties X where an *Arakelov Chow ring* $CH(\overline{X})$ is available. In order to get a ring structure in general (with rational coefficients), Gillet and Soulé [GS1] enlarge the group of cycles to define an *arithmetic Chow ring* $\widehat{CH}(X)$, but lose much of the finite dimensionality in the construction.

For arithmetic varieties whose fiber at infinity is a homogeneous space, the presence of a group gives reason to hope for explicit formulas. This has proven to be true in the $SL(n)$ case (see [Ma] [T3] for the Grassmannian and [T2] for general flag varieties), where interesting combinatorial difficulties come into play. The goal of this paper is to analyze the analogous situation for the Lagrangian Grassmannian; this falls into the general program of extending results from classical intersection theory and enumerative geometry to the arithmetic setting (cf. [S]).

Let E be a symplectic vector space of dimension $2n$, equipped with the standard antidiagonal symplectic form (cf. §2). The Lagrangian Grassmannian over $\mathrm{Spec}\mathbb{Z}$ is an arithmetic scheme G that parametrizes Lagrangian (i.e. maximal isotropic) subspaces in E over any base field. If we equip $G(\mathbb{C})$ with the natural invariant Kähler metric (induced from the $U(2n)$-invariant metric on the Grassmannian of n-planes in E), it aquires the structure of a hermitian symmetric space. Thus we have an Arakelov Chow ring $CH(\overline{G})$; we give a presentation of this ring along the lines of [T1].

Before studying the arithmetic Schubert calculus in $CH(\overline{G})$, one must first ask how well the geometric picture for the ordinary Chow ring $CH(G)$ is known. Fortunately this has been combinatorially understood in recent years in work of Hiller and Boe [HB], Pragacz [P] and Stembridge [St]. The theory is based on Schur's Q-polynomials [Sh], which were used by him to study projective representations of the symmetric and alternating groups.

In order to describe the combinatorial nuance we encounter when working in the arithmetic setting, let us recall (from [Bor], [BGG] and [D]) the presentation

$$CH(G) = \frac{\Lambda_n}{I_n} := \frac{\mathbb{Z}[X_1, \ldots, X_n]^{S_n}}{\langle e_i(X_1^2, \ldots, X_n^2),\ 1 \leqslant i \leqslant n \rangle} \tag{1}$$

where $e_i(X_1, \ldots, X_n)$ denotes the i-th elementary symmetric polynomial, and the X_i correspond to the Chern roots of the tautological quotient bundle over G. The Arakelov Chow ring $CH(\overline{G})$ sits in a short exact sequence

$$0 \longrightarrow \mathrm{Harm}(G_{\mathbb{R}}) \longrightarrow CH(\overline{G}) \longrightarrow CH(G) \longrightarrow 0 \tag{2}$$

where $\mathrm{Harm}(G_{\mathbb{R}})$ is the group of harmonic real differential forms on $G(\mathbb{C})$. By choosing a $\mathbb{Z}$-basis for $CH(G)$ we can split (2), arriving at an isomorphism of abelian groups:

$$CH(\overline{G}) \cong CH(G) \oplus \mathrm{Harm}(G_{\mathbb{R}}).$$

The subgroup $\mathrm{Harm}(G_{\mathbb{R}})$ is a square zero $CH(\overline{G})$-ideal, which as a group is isomorphic to $CH(G) \otimes_{\mathbb{Z}} \mathbb{R}$.

For $f(X_1, \ldots, X_n)$ a polynomial in Λ_n, let $\widehat{f}(\widehat{X}_1, \ldots, \widehat{X}_n)$ denote its image in $CH(\overline{G})$ under the above splitting. If f belongs to the ideal I_n in (1), then $f = 0$ in $CH(G)$, but its counterpart $\widehat{f}$ does not vanish in the Arakelov Chow ring; rather, it lives as the class of a differential form in $\mathrm{Harm}(G_{\mathbb{R}})$. Thus we arrive at the combinatorial difficulty alluded to above: *a presentation of the arithmetic Schubert calculus requires a lifting of the Schubert calculus in* Λ_n/I_n *to the ring* Λ_n *of symmetric polynomials.*

In the $SL(n)$ case the analogous problems (cf. [Ma] [T2] [T3]) are solved using Schur's S-polynomials and more generally the Schubert polynomials of Lascoux and Schützenberger [LS]. The theory that seems most suitable in our setting is that of $\widetilde{Q}$-polynomials, a modification of Schur's Q-polynomials developed by Pragacz and Ratajski [PR] for studying Lagrangian and orthogonal degeneracy loci. The author was not surprised that an understanding of the relative Schubert calculus in geometry is formally analogous to the situation in Arakelov theory; this principle was also used in [T2] [T3].

The picture of the arithmetic Schubert calculus is a type C version of that in [T3], which dealt with the $SL(n)$ Grassmannian. In geometry the passage from type A to type C is combinatorially facilitated by the use of strict partitions and *shifted Young diagrams*. In Arakelov theory we need to extend slightly the class of diagrams considered (see §4.2) and use *shifted hook operations*, a type C analogue of the hook operations of [T3]. We arrive at a complete description of the multiplicative structure of $CH(\overline{G})$, which includes explicit formulas for the 'arithmetic structure constants' appearing in the formula for multiplying two arithmetic Schubert cycles. For instance there is an arithmetic version of the Pieri rule of [HB]. The height of G with respect to the canonical very ample line bundle with the induced hermitian metric is computed by applying our analysis to this particular arithmetic intersection.

This paper is organized as follows. In §2 we introduce the Arakelov Chow ring and arrive at a presentation of $CH(\overline{G})$ suitable for our purposes. Section 3 recalls some material on Young diagrams and the $\widetilde{Q}$-polynomials of Pragacz and Ratajski. We give a combinatorial 'degree formula' for these polynomials. The arithmetic Schubert calculus in $CH(\overline{G})$ is worked out in §4; there are formulas for the arithmetic structure constants involving their geometric counterparts and 'shifted hook operations'. In particular we formulate an 'arithmetic Pieri rule'. In §5 we compute the Faltings height [F] of G with respect to its Plücker embedding as an application of the theory developed. The arguments in this article are mostly algebraic and combinatorial, although some Arakelov theory and hermitian differential geometry is needed for the results of §2.

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2 The Arakelov Chow ring

In this section we will introduce the Arakelov Chow ring $CH(\overline{G})$. We refer to the foundational works of Gillet and Soulé [GS1] [GS2] and the expositions [SABK] [S] for general background.

Let k be a field, E a $2n$-dimensional vector space over k, and let $\{e_i\}_{i=1}^{2n}$ be a basis of unit coordinate vectors. Define a nondegenerate skew-symmetric bilinear form [,] on E with matrix

$$\{[e_i, e_j]\}_{i,j} = \begin{pmatrix} 0 & Id_n \\ -Id_n & 0 \end{pmatrix}.$$

We let $G = LG(n, 2n)$ denote the arithmetic scheme which parametrizes Lagrangian subspaces in E over any field k.

The variety G is smooth over $\mathrm{Spec}\mathbb{Z}$. E will also denote the trivial rank $2n$ vector bundle over G and S the tautological subbundle of E. Using the symplectic form, we can identify the quotient bundle E/S with S^*; thus there is an exact sequence

$$\mathcal{E}: \ 0 \longrightarrow S \longrightarrow E \longrightarrow S^* \longrightarrow 0$$

of vector bundles over G.

Endow the trivial bundle $E(\mathbb{C})$ over $G(\mathbb{C})$ with a (trivial) hermitian metric compatible with the symplectic form. This metric induces metrics on the bundles S, S^* and $\mathcal{E}$ becomes a sequence of *hermitian vector bundles*

$$\overline{\mathcal{E}}:\ 0 \longrightarrow \overline{S} \longrightarrow \overline{E} \longrightarrow \overline{S}^* \longrightarrow 0.$$

The Kähler form $\omega_G = c_1(\overline{S}^*)$ turns $G(\mathbb{C})$ into a hermitian symmetric space with compact presentation

$$G(\mathbb{C}) \cong Sp(n)/U(n).$$

Let $\overline{G} = (G, \omega_G)$ denote the corresponding Arakelov variety.

There are three rings attached to G: the Chow ring $CH(G)$, the ring $\mathrm{Harm}(G_{\mathbb{R}})$ of real ω_G-harmonic differential forms on $G(\mathbb{C})$, and the Arakelov Chow ring $CH(\overline{G})$. We have natural isomorphisms

$$CH(G) \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathrm{Harm}(G_{\mathbb{R}}) \cong H^*(G(\mathbb{C}), \mathbb{R}), \tag{3}$$

where the third ring $H^*(G(\mathbb{C}), \mathbb{R})$ is cohomology with real coefficients. Elements in the Arakelov Chow group $CH^p(\overline{G})$ are represented by arithmetic cycles (Z, g_Z), where Z is a codimension p cycle on G and g_Z is a *Green current* for $Z(\mathbb{C})$. More precisely, g_Z is a current of type $(p-1, p-1)$ such that the current $dd^c g_Z + \delta_{Z(\mathbb{C})}$ is represented by a harmonic form in $\mathrm{Harm}^{p,p}(G_{\mathbb{R}})$. It follows from the general theory and the fact that G has a cellular decomposition that for each p there is an exact sequence

$$0 \longrightarrow \mathrm{Harm}^{p-1,p-1}(G_{\mathbb{R}}) \stackrel{a}{\longrightarrow} CH^p(\overline{G}) \stackrel{\zeta}{\longrightarrow} CH^p(G) \longrightarrow 0, \tag{4}$$

where the maps a and ζ are defined by

$$a(\eta) = (0, \eta) \qquad \text{and} \qquad \zeta(Z, g_Z) = Z.$$

Summing (4) over all p gives the sequence

$$0 \longrightarrow \mathrm{Harm}(G_{\mathbb{R}}) \stackrel{a}{\longrightarrow} CH(\overline{G}) \stackrel{\zeta}{\longrightarrow} CH(G) \longrightarrow 0. \tag{5}$$

For each symmetric polynomial ϕ we have characteristic classes and forms associated to the vector bundles in $\mathcal{E}$. There are three different kinds: the usual classes $\phi(S)$ in $CH(G)$, the differential forms $\phi(\overline{S})$ in $\mathrm{Harm}(G_{\mathbb{R}})$ given by Chern-Weil theory, and the arithmetic classes $\widehat{\phi}(\overline{S})$ in $CH(\overline{G})$. The Chern forms and arithmetic Chern classes satisfy

$$c_i(\overline{S}^*) = (-1)^i c_i(\overline{S}), \qquad \widehat{c}_i(\overline{S}^*) = (-1)^i \widehat{c}_i(\overline{S}).$$

Let $\mathrm{x} = \{\mathrm{x}_1, \ldots, \mathrm{x}_n\}$ denote the Chern roots of S^*. We adopt the convention that symmetric functions ϕ in the formal root variables $\widehat{x} = \{\widehat{x}_1, \ldots, \widehat{x}_n\}$ and $x = \{x_1, \ldots, x_n\}$ denote arithmetic classes $\widehat{\phi}(\overline{S}^*)$ and characteristic forms $\phi(\overline{S}^*)$, respectively. The latter are identified, via the inclusion a, with elements in $CH(\overline{G})$.

The Chow ring of G has the presentation

$$CH(G) = \frac{\mathbb{Z}[c_1(S^*), \ldots, c_n(S^*)]}{\langle c(S)c(S^*) = 1\rangle} = \frac{\mathbb{Z}[\mathrm{x}_1, \ldots, \mathrm{x}_n]^{S_n}}{\langle \prod_i (1 - \mathrm{x}_i^2) = 1\rangle}. \tag{6}$$

The relation $\prod_i (1 - \mathrm{x}_i^2) = 1$ says that all non-constant elementary symmetric polynomials $e_k(\mathrm{x}^2) := e_k(\mathrm{x}_1^2, \ldots, \mathrm{x}_n^2)$ in the squares of the root variables vanish. We will give an analogous presentation for the Arakelov Chow ring $CH(\overline{G})$, following the methods of [Ma] and [T1].

Consider the abelian group

$$A = \mathbb{Z}[\widehat{x}_1, \ldots, \widehat{x}_n]^{S_n} \oplus \mathbb{R}[x_1, \ldots, x_n]^{S_n}.$$

We adopt the convention that $\widehat{\alpha}$ denotes $\widehat{\alpha} \oplus 0$, β denotes $0 \oplus \beta$ and any product $\prod \alpha_i \beta_j$ denotes $0 \oplus \prod \alpha_i \beta_j$. With this in mind we define a product $\cdot$ in A by imposing the relations $\widehat{\alpha} \cdot \beta = \alpha\beta$ and $\beta_1 \cdot \beta_2 = 0$. Consider the following two sets of relations in A:

$$\mathcal{R}_1 : \ e_k(x^2) = 0, \qquad k \geqslant 1,$$

$$\mathcal{R}_2 : \ e_k(\widehat{x}^2) = (-1)^{k-1} \mathcal{H}_{2k-1} p_{2k-1}(x), \qquad k \geqslant 1.$$

Here the *harmonic numbers* $\mathcal{H}_r$ are defined by

$$\mathcal{H}_r = 1 + \frac{1}{2} + \cdots + \frac{1}{r}$$

and $p_r(x) = \sum x_i^r$ is the *r-th power sum*. Let $\mathcal{A}$ denote the quotient of the graded ring A by the relations $\mathcal{R}_1$ and $\mathcal{R}_2$. Then we have

Theorem 1 *There is a unique ring isomorphism*

$$\Phi : \mathcal{A} \to CH(\overline{G})$$

such that

$$\Phi(e_k(\widehat{x})) = \widehat{c}_k(\overline{S}^*), \qquad \Phi(e_k(x)) = c_k(\overline{S}^*).$$

Proof. The proof of the theorem is similar to that in [Ma], Theorem 4.0.5 and [T1], Theorem 6, so we will give a sketch of the essential points. The inclusion and projection morphisms

$$\mathbb{R}[x_1, \ldots, x_n]^{S_n} \xrightarrow{i} A \xrightarrow{\pi} \mathbb{Z}[\widehat{x}_1, \ldots, \widehat{x}_n]^{S_n}$$

induce an exact sequence of abelian groups:

$$0 \longrightarrow \mathbb{R}[x_1, \ldots, x_n]^{S_n}/(\mathcal{R}_1) \xrightarrow{i} \mathcal{A} \xrightarrow{\pi} \mathbb{Z}[\widehat{x}_1, \ldots, \widehat{x}_n]^{S_n}/(\widehat{\mathcal{R}}_2) \longrightarrow 0 \qquad (7)$$

where the relations $\widehat{\mathcal{R}}_2$ are defined by

$$\widehat{\mathcal{R}}_2 : \; e_k(\widehat{x}^2) = 0, \qquad k \geqslant 1.$$

To show that Φ is an isomorphism one uses the isomorphisms (3) and (6) and the five lemma to identify the short exact sequences (5) and (7) (as in loc. cit.). The multiplication $\cdot$ reflects the $CH(\overline{G})$-module structure of the square zero ideal $\mathrm{Harm}(G_{\mathbb{R}}) \hookrightarrow CH(\overline{G})$ (cf. loc. cit. or [GS1]). The new relation $\mathcal{R}_2$ comes from the equation

$$\widehat{c}(\overline{S}) \cdot \widehat{c}(\overline{S}^*) = 1 + \widetilde{c}(\overline{\mathcal{E}}). \qquad (8)$$

Here $\widetilde{c}(\overline{\mathcal{E}})$ is the image in $CH(\overline{G})$ of the *Bott-Chern form* of the exact sequence $\overline{\mathcal{E}}$ for the total Chern class (cf. [BC] [GS2]). This differential form is the 'natural' solution η to the equation

$$c(\overline{S})c(\overline{S}^*) - 1 = dd^c\eta.$$

Proposition 3 of [T1] provides the calculation

$$\widetilde{c}_i(\overline{\mathcal{E}}) = (-1)^{i-1}\mathcal{H}_{i-1}p_{i-1}(\overline{S}^*)$$

for all i (of course this vanishes when i is odd). If we express the two previous equations using root notation we obtain

$$e_k(\widehat{x}_1^2, \ldots, \widehat{x}_n^2) = (-1)^{k-1}\mathcal{H}_{2k-1}p_{2k-1}(x)$$

for all $k \geqslant 1$, which is relation $\mathcal{R}_2$. This completes the argument. □

Remark. As in [T1] §8, the relations $\mathcal{R}_1$ and $\mathcal{R}_2$ may be expressed in the form

$$\mathcal{R}_1' : \prod_{i=1}^{n}(1 - x_i^2 t^2) = 1,$$

$$\mathcal{R}_2' : \prod_{i=1}^{n}(1 - \widehat{x}_i^2 t^2) \cdot (1 + q_a(x,t)) = 1,$$

where t is a formal variable (note that $\mathcal{R}_2'$ uses the multiplication in A). Here $q_a(x,t)$ is the even part of the function $p_a(t)$ in loc. cit., namely

$$\begin{aligned} q_a(x,t) &= \frac{t}{2}\sum_{i=1}^{n}\left(\frac{\log(1+x_i t)}{1+x_i t} - \frac{\log(1-x_i t)}{1-x_i t}\right) \\ &= p_1(x)t^2 + \frac{11}{6}p_3(x)t^4 + \frac{137}{60}p_5(x)t^6 + \cdots \end{aligned}$$

In the next section we discuss the algebraic and combinatorial tool of $\widetilde{Q}$-polynomials. They allow one to express symmetric functions in the variables $\widehat{x}_i$ in a canonical form, which facilitates computations modulo the relations $\mathcal{R}_1$ and $\mathcal{R}_2$. We will use them to give a complete description of the ring structure of $CH(\overline{G})$ in Theorem 2 of §4.

3 Young diagrams and $\widetilde{Q}$-polynomials

We begin by recalling some basic facts about partitions and their Young diagrams; our main reference is [M]. A *partition* is a sequence

$$\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r) \tag{9}$$

of nonnegative integers in decreasing order. The number of nonzero λ_i's in (9) is called the *length* of λ, denoted $l(\lambda)$; the partitioned number (i.e. the sum of the parts of λ) is the *weight* $|\lambda|$ of λ. We identify a partition with its associated *Young diagram* of boxes; the relation $\lambda \supset \mu$ is defined by the containment of diagrams. If this is the case then the set-theoretic difference $\lambda \smallsetminus \mu$ is the *skew diagram* λ/μ. For any box $x \in \lambda$ the *hook* H_x consists of x together with all boxes directly to the right and below x. The *rim hook* R_x is the skew diagram obtained by projecting H_x along diagonals onto the boundary of λ (an example is shown in Figure 4). The *height* $ht(R_x)$ of R_x is one less than the number of rows it occupies. A skew diagram γ is a *horizontal strip* if it has at most one box in each column. Two boxes in γ are *connected* if they share a vertex or an edge; this defines the connected components of γ.

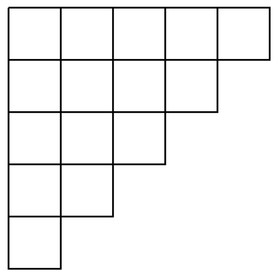

Figure 1: The partition $\rho(5) = (5, 4, 3, 2, 1)$

A partition is *strict* if all its (nonzero) parts are different. We define $\rho(n) = (n, n-1, \ldots, 1)$ and let D_n denote the set of strict partitions λ with $\lambda \subset \rho(n)$. The *shifted diagram* $\mathcal{S}(\lambda)$ of a strict partition λ is obtained from the usual diagram of λ by shifting the i-th row $i-1$ squares to the right, for each $i > 1$ (see Figure 2). For skew diagrams $\mathcal{S}(\lambda/\mu) = \mathcal{S}(\lambda) \smallsetminus \mathcal{S}(\mu)$.

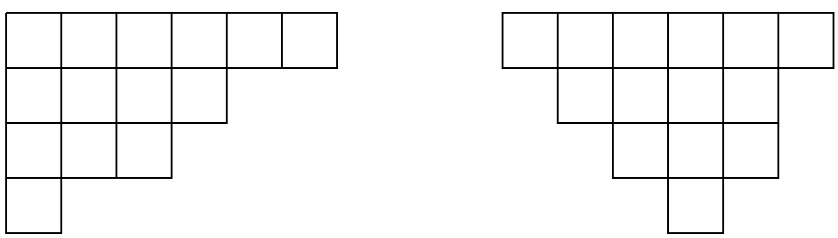

Figure 2: $\lambda = (6, 4, 3, 1)$ and the shifted diagram $\mathcal{S}(\lambda)$

Throughout this paper we use multiindex notation for sets of commuting variables; in particular $X = \{X_1, \ldots, X_n\}$ and $X^2 = \{X_1^2, \ldots, X_n^2\}$. Let $\Lambda_n(X) = \mathbb{Z}[X]^{S_n}$ be the ring of symmetric polynomials in n variables; Λ will denote the ring of symmetric functions in countably many independent variables. We will need a family of symmetric functions modelled on Schur's Q-polynomials (see [Sh]). These $\widetilde{Q}$-polynomials were introduced by Pragacz and Ratajski [PR] in their study of Lagrangian and orthogonal degeneracy loci.

For each i between 1 and n, let $\widetilde{Q}_i = e_i(X)$ be the i-th elementary symmetric function. For i, j nonnegative integers define

$$\widetilde{Q}_{i,j} := \widetilde{Q}_i \widetilde{Q}_j + 2\sum_{k=1}^{j}(-1)^k \widetilde{Q}_{i+k}\widetilde{Q}_{j-k}.$$

If $\lambda = (\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_r \geqslant 0)$ is a partition with r even (by putting $\lambda_r = 0$ if necessary), define

$$\widetilde{Q}_\lambda = \text{Pfaffian}[\widetilde{Q}_{\lambda_i,\lambda_j}]_{1\leqslant i<j\leqslant r}.$$

These polynomials have the following properties ([PR], §4):

(1) If $\lambda_1 > n$, then $\widetilde{Q}_\lambda = 0$.

(2) $\widetilde{Q}_{i,i} = e_i(X_1^2, \ldots, X_n^2)$.

(3) If $\lambda = (\lambda_1, \ldots, \lambda_r)$ and $\lambda^+ = \lambda \cup (i,i) = (\lambda_1, \ldots, i, i, \ldots, \lambda_r)$ then

$$\widetilde{Q}_{\lambda^+} = \widetilde{Q}_{i,i}\widetilde{Q}_\lambda.$$

(4) The set $\{\widetilde{Q}_\lambda \mid \lambda_1 \leqslant n\}$ is an additive $\mathbb{Z}$-basis of $\Lambda_n(X)$.

(5) The set $\{\widetilde{Q}_\lambda \mid \lambda \in D_n\}$ is a basis for $\Lambda_n(X)$ as a $\Lambda_n(X^2)$-module.

The $\widetilde{Q}$-polynomials can be realized as the duals of certain modified Hall-Littlewood polynomials. More precisely, let $P_\lambda(X;t)$ be the usual Hall-Littlewood polynomials (cf. [M], III.2) and let $Q'_\lambda(X;t)$ be the adjoint basis for the standard scalar product on $\Lambda[t]$; we have $Q'_\lambda(X;t) = Q_\lambda(X/(1-t);t)$ in the sense of λ-rings (see [LLT]). Then ([PR] Prop. 4.9):

$$\widetilde{Q}_\lambda(X) = \omega(Q'_\lambda(X;-1)),$$

where $\omega : \Lambda \to \Lambda$ is the duality involution of [M], I.2.

Since the $\{\widetilde{Q}_\lambda\}$ with $\lambda_1 \leqslant n$ form a basis of Λ_n, there exist integers $e^\nu_{\lambda\mu}$ so that

$$\widetilde{Q}_\lambda \widetilde{Q}_\mu = \sum_\nu e^\nu_{\lambda\mu}\widetilde{Q}_\nu. \tag{10}$$

There are explicit combinatorial rules for generating the coefficients $e^\nu_{\lambda\mu}$, which follow by specializing corresponding formulas for the multiplication of Hall-Littlewood polynomials (see [PR] §4 and [M], III.3.(3.8)). In particular one has the following Pieri type formula for λ *strict* ([PR], Prop. 4.9):

$$\widetilde{Q}_\lambda \widetilde{Q}_k = \sum 2^{m(\lambda,\mu)} \widetilde{Q}_\mu, \tag{11}$$

where the sum is over all partitions $\mu \supset \lambda$ with $|\mu| = |\lambda| + k$ such that μ/λ is a horizontal strip, and $m(\lambda, \mu)$ is the number of connected components of μ/λ not meeting the first column.

For the height calculations in §5 it is useful to have a combinatorial formula for the product $\widetilde{Q}_1^N$. Recall that a *standard tableau* on the Young diagram λ is a numbering of the boxes of λ with the integers $1, 2, \ldots, |\lambda|$ such that the entries are strictly increasing along each row and column. Call a standard tableau T on λ *proper* if in each hook $H_{(i,j)}$ of λ, the number of entries of T less than the $(i, j+1)$ entry is odd (the condition being vacuous if λ has no box in the $(i, j+1)$ position). Let g^λ denote the number of proper standard tableaux on λ.

Proposition 1

$$\widetilde{Q}_1^N = \sum_{|\lambda|=N} 2^{N-l(\lambda)} g^\lambda \widetilde{Q}_\lambda.$$

Proof. This follows from an analysis of the Pieri type formula for the polynomials $Q'_\lambda(X; t)$ given in [M], III.5, Example 7. By specializing $t = -1$ and applying ω we deduce that

$$\widetilde{Q}_\mu(X)\widetilde{Q}_1(X) = \sum_\lambda \psi_{\lambda/\mu}(-1)\widetilde{Q}_\lambda(X) \tag{12}$$

where the sum is over all $\lambda \supset \mu$ with $|\lambda| = |\mu| + 1$ and $\psi_{\lambda/\mu}(t)$ is defined as in [M], III.5.(5.8′). Call a non-empty row of μ *odd* if it contains k boxes and the part k occurs in μ an odd number of times. Then (12) says that

$$\widetilde{Q}_\mu \widetilde{Q}_1 = 2 \sum_\lambda \widetilde{Q}_\lambda + \widetilde{Q}_{\mu \cup 1}, \tag{13}$$

where the sum is over all λ obtained from μ by adding a box in an odd row and $\mu \cup 1 = (\mu_1, \ldots, \mu_{l(\mu)}, 1)$. The equality in the proposition is obtained by repeated application of (13). □

Example 1. Take $n = 2$ and $N = 4$. Clearly $\lambda_1 = (2, 2)$, $\lambda_2 = (2, 1, 1)$ and $\lambda_3 = (1, 1, 1, 1)$ are the only partitions λ with $|\lambda| = 4$ and $\widetilde{Q}_\lambda(X_1, X_2) \neq 0$.

There are 1, 2 and 1 proper standard tableaux on λ_1, λ_2 and λ_3 respectively (Figure 3). This leads to the equation

$$\widetilde{Q}_1(X_1, X_2)^4 = 4\widetilde{Q}_{2,2} + 4\widetilde{Q}_{2,1,1} + \widetilde{Q}_{1,1,1,1} \tag{14}$$

which corresponds to the identity

$$(X_1 + X_2)^4 = 4X_1^2X_2^2 + 4X_1X_2(X_1^2 + X_2^2) + (X_1^2 + X_2^2)^2.$$

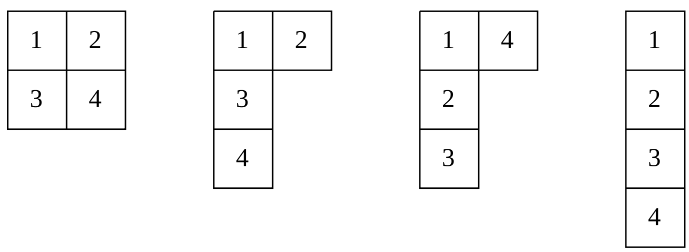

Figure 3: The proper standard tableaux on $(2, 2)$, $(2, 1, 1)$ and $(1, 1, 1, 1)$

4 Arithmetic Schubert calculus

4.1 Classical case

We review here the classical Schubert calculus, which describes the multiplication in $CH(G)$, following [P] §6. To avoid notational confusion we will use $\sigma_\lambda(\mathrm{x})$ in place of $\widetilde{Q}_\lambda(\mathrm{x})$ when referring to polynomials in the Chern roots $\mathrm{x} = \{\mathrm{x}_1, \ldots, \mathrm{x}_n\}$ of the vector bundle S^*, and also when using the other two kinds of root variables discussed in §2.

The abelian group $CH(G)$ is freely generated by the classes $\sigma_\lambda(\mathrm{x}) = \sigma_\lambda(S^*)$, for strict partitions λ contained in the 'triangle' partition $\rho(n)$. $\sigma_\lambda(\mathrm{x})$ is the class of the codimension $|\lambda|$ *Schubert variety* X_λ, defined as follows: if $\{e_1, \ldots, e_n\}$ spans a fixed Lagrangian subspace of E and $F_i = \mathrm{Span}\,\langle e_1, \ldots, e_i\rangle$ then X_λ parametrizes the set

$$\{L \in G(k) \mid \dim(L \cap F_{n+1-\lambda_i}) \geqslant i \quad \text{for } 1 \leqslant i \leqslant l(\lambda)\}$$

over any base field k.

The product formula (10) gives the following multiplication rule in $CH(G)$: for any two partitions $\lambda, \mu \in D_n$,

$$\sigma_\lambda(\mathrm{x})\sigma_\mu(\mathrm{x}) = \sum_{\nu \in D_n} e^\nu_{\lambda\mu} \sigma_\nu(\mathrm{x}); \tag{15}$$

the non-negative integers $e^\nu_{\lambda\mu}$ are the *structure constants* in $CH(G)$. When $\mu = k$ is a single integer then $\sigma_\mu(\mathrm{x}) = \sigma_k(\mathrm{x})$ is the class of a *special Schubert variety*, and (15) specializes to the *Pieri rule* (due to Hiller and Boe [HB]):

$$\sigma_\lambda(\mathrm{x})\sigma_k(\mathrm{x}) = \sum 2^{m(\lambda,\mu)} \sigma_\mu(\mathrm{x}) \tag{16}$$

the sum over all (strict) partitions $\mu \supset \lambda$ with $|\mu| = |\lambda| + k$ such that μ/λ is a horizontal strip, with $m(\lambda, \mu)$ defined as in §3. Note that since $G(\mathbb{C})$ is a hermitian symmetric space, (15) and (16) are valid on the level of harmonic differential forms.

4.2 Schubert calculus in $CH(\overline{G})$

We now turn to an analogous description of the multiplicative structure of $CH(\overline{G})$, which we refer to as 'arithmetic Schubert calculus'. Due to the the power sums in the relations $\mathcal{R}_2$ of §2 we expect to encounter operations on Young diagrams involving rim hooks, as in the $SL(n)$ case (see [T3]). We proceed to give the relevant definitions.

Recall that D_n denotes the set of strict partitions λ with $\lambda \subset \rho(n)$. Let E_n be the set of non-strict partitions λ with $\lambda_1 \leqslant n$ such that exactly one non-zero part r_λ of λ occurs more than once, and further, r_λ occurs at most 3 times. There is a map

$$E_n \longrightarrow D_n \quad : \quad \lambda \longmapsto \overline{\lambda}$$

defined as follows: $\overline{\lambda}$ is obtained from λ by deleting two of the parts r_λ. For example if $\lambda = (6, 4, 4, 4, 2, 1)$ then $\overline{\lambda} = (6, 4, 2, 1)$.

The next definition makes sense in the context of shifted diagrams and follows Macdonald [M], Example III.8.11. Define a *double rim* to be a skew diagram formed by the union of two rim hooks which both end on the main diagonal $\{(i, i) \mid i > 0\}$. A double rim δ can be cut into two non-empty connected pieces: one piece α consisting of the diagonals in δ of length 2 (parallel to the main diagonal), and the other piece being the rim hook

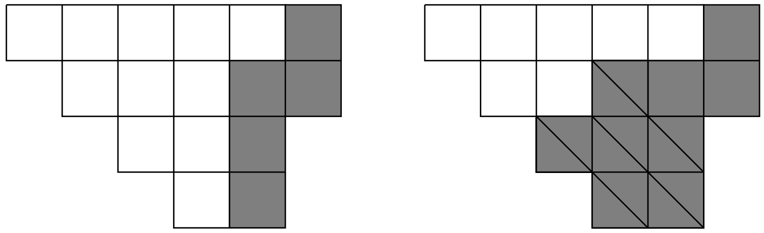

Figure 4: A rim hook and a double rim

$\beta := \delta \setminus \alpha$. In this case we say that the double rim is of type $(\frac{1}{2}|\alpha|, |\beta|)$. Figure 4 shows a single rim hook and a double rim of type $(3, 3)$.
Each double rim $\delta = \alpha \cup \beta$ of type (a, b) has an associated integer $\epsilon(\delta) := (-1)^{a+ht(\beta)}\, 2$. To a single rim hook γ we associate the sign $\epsilon(\gamma) := (-1)^{ht(\gamma)}$.

Suppose that $\lambda \in E_n$ and $\mu \in D_n$ are two Young diagrams with $|\mu| = |\lambda| - 1$. We say that there is a *shifted hook operation* from λ to μ if the shifted skew diagram $\mathcal{S}(\mu/\overline{\lambda})$ is a rim hook or double rim (of weight $2r_\lambda - 1$).

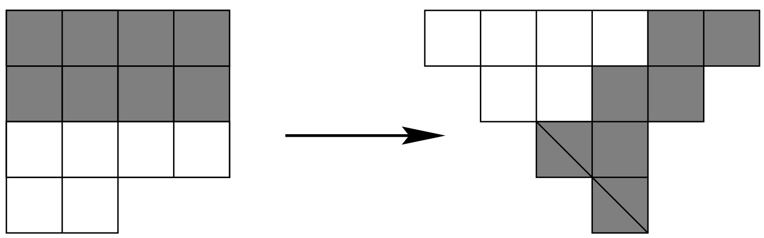

Figure 5: A shifted hook operation from $\lambda = (4, 4, 4, 2)$ to $\mu = (6, 4, 2, 1)$

It is clear that there is at most one such operation from λ to μ; it determines an integer $\epsilon_{\lambda\mu} \in \{\pm 1, \pm 2\}$ defined by

$$\epsilon_{\lambda\mu} = (-1)^{r_\lambda - 1}\epsilon(\mathcal{S}(\mu/\overline{\lambda}))$$

and a rational number ψ_λ^μ by

$$\psi_\lambda^\mu = \epsilon_{\lambda\mu} 2^{l(\lambda) - l(\mu) - 1} \mathcal{H}_{2r_\lambda - 1}.$$

If there is no shifted hook operation from λ to μ then set $\psi_\lambda^\mu = 0$. Figure 5 shows a shifted hook operation involving a double rim of type $(1, 5)$ with $\epsilon_{\lambda\mu} = 2$ and $\psi_\lambda^\mu = \mathcal{H}_7$.

Next we define the *arithmetic structure constants* $\widetilde{e}^{\nu}_{\lambda\mu}$: for any $\nu \in E_n$ and λ, μ strict such that $|\nu| = |\lambda| + |\mu| - 1$ let

$$\widetilde{e}^{\nu}_{\lambda\mu} = \sum_{\rho \in E_n} \psi^{\nu}_{\rho} e^{\rho}_{\lambda\mu} \tag{17}$$

where the $e^{\rho}_{\lambda\mu}$ are defined by (10). Note that only partitions ρ such that there is a shifted hook operation from ρ to ν contribute to the sum (17). We can now state our main result:

Theorem 2 (a) *Let p be an integer between* 0 *and* $\binom{n+1}{2} + 1$*. Each element* $z \in CH^p(\overline{G})$ *has a unique expression*

$$z = \sum_{\substack{\lambda \in D_n \\ |\lambda| = p}} c_\lambda \sigma_\lambda(\widehat{x}) + \sum_{\substack{\lambda \in D_n \\ |\lambda| = p-1}} \gamma_\lambda \sigma_\lambda(x),$$

where $c_\lambda \in \mathbb{Z}$ and $\gamma_\lambda \in \mathbb{R}$.

(b) *For λ and μ in D_n we have the multiplication rules*

$$\sigma_\lambda(\widehat{x}) \cdot \sigma_\mu(\widehat{x}) = \sum_{\substack{\nu \in D_n \\ |\nu| = |\lambda| + |\mu|}} e^{\nu}_{\lambda\mu} \sigma_\nu(\widehat{x}) + \sum_{\substack{\nu \in D_n \\ |\nu| = |\lambda| + |\mu| - 1}} \widetilde{e}^{\nu}_{\lambda\mu} \sigma_\nu(x),$$

$$\sigma_\lambda(\widehat{x}) \cdot \sigma_\mu(x) = \sum_{\substack{\nu \in D_n \\ |\nu| = |\lambda| + |\mu|}} e^{\nu}_{\lambda\mu} \sigma_\nu(x),$$

$$\sigma_\lambda(x) \cdot \sigma_\mu(x) = 0.$$

Proof. The morphism $\epsilon : CH(G) \to CH(\overline{G})$ defined by $\epsilon(\sigma_\lambda(\mathrm{x})) = \sigma_\lambda(\widehat{x})$ for each $\lambda \in D_n$ splits the exact sequence (5). We thus have an isomorphism of abelian groups

$$CH(\overline{G}) \cong CH(G) \oplus \mathrm{Harm}(G_{\mathbb{R}})$$

and the statement (a) follows.

The second and third equalities in (b) follow immediately from the definition of multiplication in $CH(\overline{G})$ and the algebra isomorphism (3). For instance we have

$$\sigma_\lambda(\widehat{x}) \cdot \sigma_\mu(x) = \sigma_\lambda(x)\sigma_\mu(x) = \sum_{\substack{\nu \in D_n \\ |\nu| = |\lambda| + |\mu|}} e^{\nu}_{\lambda\mu} \sigma_\nu(x)$$

because the last equality holds in $CH(G)$.

To prove the first equality, note that properties (2) and (3) of $\widetilde{Q}$-polynomials from §3 imply that for $\lambda \in E_n$,

$$\sigma_\lambda(\widehat{x}) = \sigma_{\overline{\lambda}}(\widehat{x}) \cdot e_{r_\lambda}(\widehat{x}^2) = (-1)^{r_\lambda - 1}\mathcal{H}_{2r_\lambda - 1} p_{2r_\lambda - 1}(x)\sigma_{\overline{\lambda}}(x), \tag{18}$$

where we have used relation $\mathcal{R}_2$ of §2. If a partition λ with $\lambda_1 \leqslant n$ is not in $D_n \cup E_n$ then $\sigma_\lambda(\widehat{x}) = 0$. Indeed, $\widetilde{Q}_\lambda$ for such λ has at least 2 non-trivial factors of the form $e_j(X^2)$, which correspond to differential form terms in the arithmetic setting. But all such products vanish in $CH(\overline{G})$.

We now need a rule for multiplying a $\widetilde{Q}_\mu$-polynomial by an odd power sum in the polynomial ring $\mathbb{Z}[X]$ modulo the ideal generated by the $\widetilde{Q}_\lambda$ for non-strict λ. The calculus of $\widetilde{Q}$-polynomials in this quotient coincides with that in the ring of Schur's Q-polynomials modulo the ideal generated by the Q_λ with λ not contained in $\rho(n)$. This follows because both rings are naturally isomorphic to $CH(G)$ ([P] §6, [PR]).

Note that under the above isomorphism a power sum p_r is mapped to $2p_r$; this follows by considering the image of Newton's identity

$$p_k - e_1 p_{k-1} + e_2 p_{k-2} - \cdots + (-1)^k k e_k = 0$$

in both rings. We can now use the analysis in [M], Example III.8.11 to obtain the required multiplication rule. The reader is warned that there is a missing factor of 2 in formula (8) of loc. cit. (in the double rim case). Using the correct version of the formula and the previous remarks gives, for $\mu \in D_n$ and r odd,

$$p_r(x)\sigma_\mu(x) = \sum_\nu \epsilon(\mathcal{S}(\nu/\mu)) 2^{l(\mu)-l(\nu)+1}\sigma_\nu(x), \tag{19}$$

the sum over all strict $\nu \supset \mu$ with $|\nu| = |\mu| + r$ such that $\mathcal{S}(\nu/\mu)$ is a rim hook or a double rim. Now combine (18) with (19) to get

Proposition 2 *For partitions $\lambda \in E_n$ we have*

$$\sigma_\lambda(\widehat{x}) = \sum_\nu \psi_\lambda^\nu \sigma_\nu(x), \tag{20}$$

the sum over all $\nu \in D_n$ that can be obtained from λ by a shifted hook operation. If $\lambda \notin D_n \cup E_n$ then $\sigma_\lambda(\widehat{x}) = 0$.

The proof is completed by writing the identity

$$\sigma_\lambda(\widehat{x})\cdot\sigma_\mu(\widehat{x}) = \sum_{\substack{\nu\in D_n\\ |\nu|=|\lambda|+|\mu|}} e^\nu_{\lambda\mu}\sigma_\nu(\widehat{x}) + \sum_{\substack{\rho\in E_n\\ |\rho|=|\lambda|+|\mu|}} e^\rho_{\lambda\mu}\sigma_\rho(\widehat{x}),$$

using (20) to replace the classes in the second sum, and comparing with (17). □

Using the Pieri type formula (11) we obtain the following special case of Theorem 2:

Corollary 1 *(Arithmetic Pieri rule): Let $C(\lambda,k)$ be the set of partitions $\mu\supset\lambda$ with $|\mu|=|\lambda|+k$ such that μ/λ is a horizontal strip. Then for $\lambda\in D_n$ we have*

$$\sigma_\lambda(\widehat{x})\cdot\sigma_k(\widehat{x}) = \sum_\mu 2^{m(\lambda,\mu)}\sigma_\mu(\widehat{x}) + \sum_\nu\left(\sum_\rho \psi^\nu_\rho\right)2^{m(\lambda,\rho)}\sigma_\nu(x).$$

where the first (classical) sum is over $\mu\in D_n\cap C(\lambda,k)$ and the second sum is over ν and ρ with $\rho\in E_n\cap C(\lambda,k)$.

5 Height calculation

The Lagrangian Grassmannian G has a natural embedding in projective space given by the very ample line bundle $\mathcal{O}(1):=\det S^*$. The metric on S induces a metric on $\mathcal{O}(1)$ which is the restriction of the Fubini-Study metric under the composition

$$LG(n,2n)\hookrightarrow G(n,2n)\overset{i}{\hookrightarrow}\mathbb{P}^{\binom{2n}{n}-1}$$

where i is the Plücker embedding of the usual $SL(n)$-Grassmannian $G(n,2n)$ in projective space (compare [LaSe] §4). This metric coincides with the one induced from the Plücker (i.e. the minimal) embedding of $LG(n,2n)$ itself in projective space.

In geometry the *degree* of $G(k)$ (for any field k) with respect to $\mathcal{O}(1)$ is given by

$$\deg(G(k)) = 2^{n(n-1)/2}g^{\rho(n)} \tag{21}$$

where the partition $\rho(n)$ and $g^{\rho(n)}$ were defined in §3; this follows from Proposition 1. The Faltings height [F] of G under its Plücker embedding (which equals its height with respect to $\overline{\mathcal{O}}(1)$) is an arithmetic analogue of the geometric degree. In this section we will use the results of §4 to compute this number; our formula will be an 'arithmetic perturbation' of (21).

The *height* of G with respect to $\overline{\mathcal{O}}(1)$ is the number

$$ht_{\overline{\mathcal{O}}(1)}(G) = \widehat{\deg}(\widehat{c}_1(\overline{\mathcal{O}}(1))^d \,|\, G) = \widehat{\deg}(\sigma_1^d(\widehat{x})). \tag{22}$$

Here the arithmetic degree map $\widehat{\deg}$ is defined as in [BoGS] and $d = \binom{n+1}{2}+1$ is the absolute dimension of G. In $CH(\overline{G})$ we have

$$\sigma_1^d(\widehat{x}) = r_d \sigma_{\rho(n)}(x) = r_d \sigma_{\rho(n)}(\overline{S}^*)$$

for some rational number r_d; the height (22) is then given by

$$ht_{\overline{\mathcal{O}}(1)}(G) = \frac{1}{2}\int_{G(\mathbb{C})} r_d \sigma_{\rho(n)}(\overline{S}^*) = \frac{r_d}{2}$$

as $\sigma_{\rho(n)}(\overline{S}^*)$ is dual to the class of a point in $G(\mathbb{C})$.

A single rim hook β which ends on the main diagonal of a shifted diagram will be referred to as a double rim of type $(0, |\beta|)$. Define the following set of diagrams:

$$\mathcal{E}(n) = \{\lambda \in E_n \ : \ |\lambda| = d\} = \{[a,b]_n \mid 0 \leqslant a + 2b < n\}$$

where $[a,b]_n$ denotes the unique diagram $\lambda \in E_n$ of weight d such that $\mathcal{S}(\rho(n)/\overline{\lambda})$ is a double rim of type $(a, 2b+1)$. There are exactly $\frac{1}{4}(n^2+2n+[n]_2)$ diagrams in $\mathcal{E}(n)$, where $[n]_2 = 0$ or 1 depending on whether n is even or odd. For instance one has

$$\begin{aligned}\mathcal{E}(3) &= \{[0,0]_3, [0,1]_3, [1,0]_3, [2,0]_3\} \\ &= \{(3,2,1,1), (2,2,2,1), (3,2,2), (3,3,1)\}.\end{aligned}$$

These correspond to the diagrams in Figure 6.

Theorem 3 *The height of the Lagrangian Grassmannian G with respect to $\overline{\mathcal{O}}(1)$ is*

$$ht_{\overline{\mathcal{O}}(1)}(G) = 2^{n(n-1)/2} \sum_{0 \leqslant a+2b<n} (-1)^b 2^{-\delta_{a0}} \mathcal{H}_{2a+2b+1} g^{[a,b]_n}$$

where δ_{ij} is the Kronecker delta.

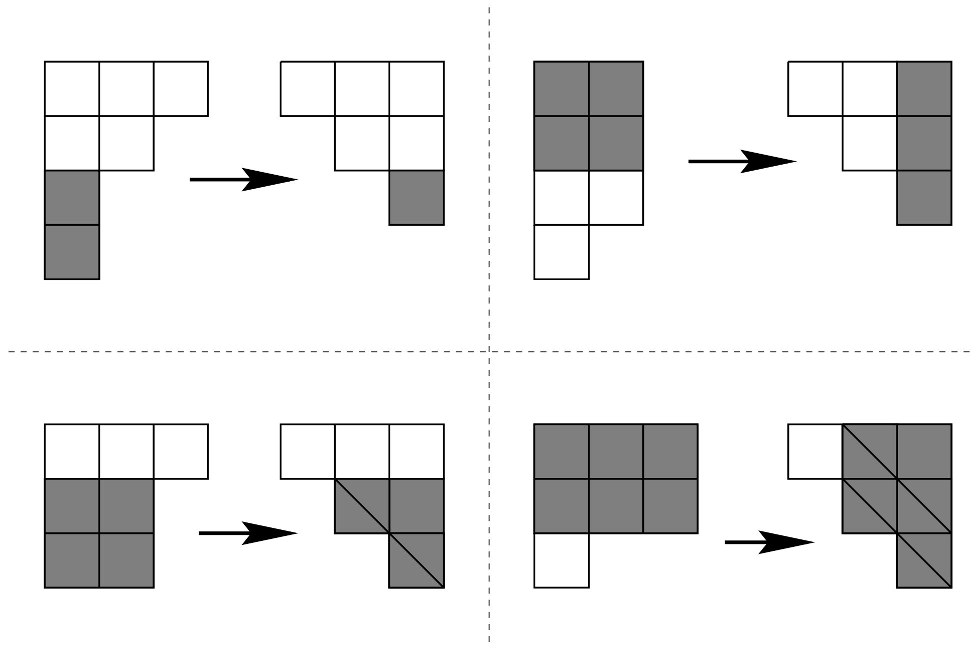

Figure 6: The four diagrams in $\mathcal{E}(3)$ and their operations to $\rho(3)$

Proof. Use Propositions 1 and 2 to obtain

$$\sigma_1(\widehat{x})^d = \sum_{\lambda\in\mathcal{E}(n)} 2^{d-l(\lambda)} g^\lambda \sigma_\lambda(\widehat{x}) = \sum_{\lambda\in\mathcal{E}(n)} 2^{d-l(\lambda)} g^\lambda \psi_\lambda^{\rho(n)} \sigma_{\rho(n)}(x).$$

For $\lambda = [a,b]_n$ we have $r_\lambda = a+b+1$ and $\epsilon(\mathcal{S}(\rho(n)/\overline{\lambda})) = (-1)^a 2^{1-\delta_{a0}}$, so

$$\psi_\lambda^{\rho(n)} = (-1)^b 2^{l(\lambda)-n-\delta_{a0}} \mathcal{H}_{2a+2b+1}.$$

Therefore

$$ht_{\overline{\mathcal{O}}(1)}(G) = \frac{1}{2} \sum_{0\leqslant a+2b<n} (-1)^b 2^{d-n-\delta_{a0}} \mathcal{H}_{2a+2b+1} g^{[a,b]_n}$$

and the result follows. □

Note that for $n>1$, $ht_{\overline{\mathcal{O}}(1)}(G)$ is a number in $\sum_{k=1}^n \frac{1}{2k-1}\mathbb{Z}$; the presence of only odd denominators is in harmony with the fact that the odd power sums form a $\mathbb{Q}$-basis of the ring of Q-polynomials (cf. [M], III.8).

Example 2. When $n = 1$, $G = \mathbb{P}^1$ is projective space and the formula gives $ht_{\overline{\mathcal{O}}(1)}(G) = \frac{1}{2}$. For $n = 2$ we have

$$\mathcal{E}(2) = \{[0,0]_2, [1,0]_2\} = \{(2,1,1), (2,2)\}$$

and their g numbers $g^{(2,1,1)} = 2$ and $g^{(2,2)} = 1$ were calculated in Example 1. Theorem 3 now gives

$$ht_{\overline{\mathcal{O}}(1)}(LG(2,4)) = 2(1 + \mathcal{H}_3) = \frac{17}{3}.$$

Finally one can check that for the diagrams in Figure 6,

$$g^{[0,0]_3} = 8, \qquad g^{[0,1]_3} = 1, \qquad g^{[1,0]_3} = 3, \qquad g^{[2,0]_3} = 4,$$

which leads to

$$ht_{\overline{\mathcal{O}}(1)}(LG(3,6)) = 32 + 20\mathcal{H}_3 + 32\mathcal{H}_5.$$

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University of Pennsylvania, Philadelphia, PA 19104, USA

Email: harryt@math.upenn.edu