Actuarial Mathematics and Life-Table Statistics

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0.1 Preface

This book is the text for an upper-level lecture course (`STAT 470`) at the University of Maryland on actuarial mathematics, in particular on the basics of Life Tables, Survival Models, and Life Insurance Premiums and Reserves. This is a 'topics' course, aiming not so much to prepare the students for specific Actuarial Examinations – since it cuts across the Society of Actuaries' Exams FM, M (Segment MLC) and C – as to present the actuarial material conceptually with reference to ideas from other undergraduate mathematical studies. Such a focus allows undergraduates with solid preparation in calculus (not necessarily mathematics or statistics majors) to explore their possible interests in business and actuarial science. It also allows the majority of such students — who will choose some other avenue, from economics to operations research to statistics, for the exercise of their quantitative talents — to know something concrete and mathematically coherent about the topics and ideas actually useful in Insurance.

The Insurance material on contingent present values and life tables is developed as directly as possible from calculus and common-sense notions, illustrated through word problems. Both the Interest Theory and Probability related to life tables are treated as wonderful concrete applications of the calculus. The lectures require no background beyond a third semester of calculus, but the prerequisite calculus courses must have been solidly understood. It is a truism of pre-actuarial advising that students who have not done really well in and digested the calculus ought not to consider actuarial studies.

It is not assumed that the student has seen a formal introduction to probability. Notions of relative frequency and average are introduced first with reference to the ensemble of a cohort life-table, the underlying formal random experiment being random selection from the cohort life-table population (or, in the context of probabilities and expectations for 'lives aged x', from the subset of l_x members of the population who survive to age x). The calculation of expectations of functions of a time-to-death random variables is rooted on the one hand in the concrete notion of life-table average, which is then approximated by suitable idealized failure densities and integrals. Later, in discussing Binomial random variables and the Law of Large Numbers, the combinatorial and probabilistic interpretation of binomial coefficients are derived from the Binomial Theorem, which the student the is assumed to know as a topic in calculus (Taylor series identification of coefficients of a polynomial.) The general notions of expectation and probability are introduced, but for example the Law of Large Numbers for binomial variables is treated (rigorously) as a topic involving calculus inequalities and summation of finite series. This approach allows introduction of the numerically and conceptually useful large-deviation inequalities for binomial random variables to explain just how unlikely it is for binomial (e.g., life-table) counts to deviate much percentage-wise from expectations when the underlying population of trials is large.

While the basics of actuarial Life Contingencies are treated elsewhere as a problem-solving method using mortality tables presented in a cohort format, some effort is devoted in this book to contrasting the form in which the underlying mortality data are received to the form of the cohort life table used in calculating premiums and reserves. This allows statistics students to connect the basic ideas of life table construction – considered by actuaries a more advanced topic – to the problems of statistical estimation. Accordingly, some material is included on statistics of biomedical studies and on reliability which would not ordinarily find its way into an actuarial course.

The reader is also not assumed to have worked previously with the Theory of Interest. These lectures present Theory of Interest as a mathematical problem-topic, which is rather unlike what is done in typical finance courses. Getting the typical Interest problems — such as the exercises on mortgage refinancing and present values of various payoff schemes — into correct format for numerical answers is often not easy even for good mathematics students. The approach here is to return to the first principles of present-value Equivalence and linear Superposition of payment streams over time. Interest Theory topics are presented here first as a way to learn the skills of applying Equivalence and Superposition principles to real problems, but also as a way of highlighting the relationship between realized payouts under standard Insurance contracts and instances of standard payment streams with random duration. In this approach, insurance reserves are seen as natural generalizations of bond amortization schedules.

While the material in these lectures is presented systematically, it is not separated by chapters into unified topics such as Interest Theory, Probability Theory, Premium Calculation, etc. Instead the introductory material from probability and interest theory are interleaved, and later, various mathematical ideas are introduced as needed to advance the discussion. No book at this level can claim to be fully self-contained, but every attempt has been made to develop the mathematics to fit the actuarial applications as they arise logically.

The coverage of the main body of each chapter is primarily 'theoretical'. At the end of each chapter is an Exercise Set and a short section of Worked Examples to illustrate the kinds of word problems which can be solved by the techniques of the chapter. The Worked Examples sections show how the ideas and formulas work smoothly together, and they highlight the most important and frequently used formulas.

Finally, this book differs from other Actuarial texts in its use of computational tools. Realistic problems on present values of payment streams, on probabilistic survival models related to human lifetimes, and on insurance-contract premiums related to those models, rapidly lead to calculations too difficult to do by hand or by calculator. Actuarial students often do these calculations using `EXCEL` or other spreadsheet programs, but the conceptually based formulas often translate more effectively using mathematical tools in computing platforms like `MATLAB` or the statistical language `R`, especially where building blocks like root-finders or numerical integration routines are needed. In this text, we encourage the students to use the free, open-source `R` platform because of its powerful tools for numerical integration, root-finding, life-table construction, and statistical estimation. Throughout this text, illustrations and Exercise solutions and solutions are given in terms of **R**.

Free web access to the downloadable `R` platform, including manuals, can be found at `http://www.r-project.org/`. There are now many good introductory texts on computing with `R` in statistical applications. One text which combines a general introduction to `R` with the specifics of many statistical data analysis methods, is Venables and Ripley (2002). Some good free tutorial material on `R` can also be found on the web, for example at `http://wiener.math.csi.cuny.edu/Statistics/R/simpleR/`.

Chapter 1

Basics of Probability and the Theory of Interest

This first Chapter supplies some background on elementary Probability Theory and basic Theory of Interest. The reader who has not previously studied these subjects may get an overview here, but will likely want to supplement this Chapter with reading in any of a number of calculus-based introductions to probability and statistics, such as Hogg and Tanis (2005) or Devore (2007), and the basics of the Theory of Interest as covered in the text of Kellison (2008) or Chapter 1 of Gerber (1997).

1.1 Probability, Lifetimes, and Expectation

In the *cohort life-table model*, imagine a number l_0 of individuals born simultaneously and followed until death, resulting in data d_x, l_x for each integer age $x = 0, 1, 2, \ldots,$ where

$$l_x = \text{number of lives aged } x \quad (\text{i.e. alive at birthday } x\ )$$

and

$$d_x \ = \ l_x - l_{x+1} \quad = \text{number dying between ages } \ x,\, x+1$$

Now, allow the age-variable to be denoted by t and to take all real values, not just whole numbers x, and treat $S_0(t)$ as the fraction of individuals in a life table surviving to exact age t. This nonincreasing function $S_0(t)$ would be called the empirical 'survivor' or 'survival' function. Although it takes on only rational values with denominator l_0, it can be approximated by a **survival function** $S(t)$ which is continuous, decreasing, and continuously differentiable (or piecewise continuously differentiable with just a few breakpoints) and takes values exactly $= l_x/l_0$ at integer ages x. Then for all positive real y and t, $S_0(y) - S_0(y+t)$ is the exact and $S(y) - S(y+t)$ the approximated fraction of the initial cohort which fails between time y and $y+t$, and for integers x, k,

$$\frac{S(x) - S(x+k)}{S(x)} = \frac{l_x - l_{x+k}}{l_x}$$

denotes the fraction of those alive at exact age x who fail before $x+k$.

What do probabilities have to do with the cohort life table and survival function ? To answer this, we first introduce probability as simply a relative frequency, using numbers from a cohort life-table like that of the accompanying Illustrative Life Table. In response to a probability question, we supply the fraction of the relevant life-table population, to obtain identities like

Pr(life aged 29 dies between exact ages 35 and 41 or between 52 and 60)

$$= \frac{S(35) - S(41) + S(52) - S(60)}{S(29)} = \Big\{(l_{35} - l_{41}) + (l_{52} - l_{60})\Big\} \Big/ l_{29}$$

where our convention is that a *life aged 29* is one of the cohort known to have survived to the 29th birthday. Note that the event of dying between exact ages 35 and 41 or between 52 and 60 is the union of the nonoverlapping events of the age random variable having value falling in the interval $[35, 41)$ with that of falling in $[52, 60)$.

The idea here is that all of the lifetimes covered by the life table are understood to be governed by an identical "mechanism" of failure, and that any probability question about a single lifetime is really a question concerning the fraction of a specified set of lives, e.g., those alive at age x, whose lifetimes will satisfy the stated property, e.g., who die either between 35 and 41 or between 52 and 60. This "frequentist" notion of probability of an event as the relative frequency with which the event occurs in a large population of (independent) identical units is associated with the phrase "law of large numbers", which will be discussed later. For now, remark only that the life table population should be large for the ideas presented so far to make good sense. See Table 1.1 for an illustration of a cohort life-table with realistic numbers, and for a cohort life table constructed to reflect the best estimates of US male and female mortality rates in 2004, see the Social Security webpage `http://www.ssa.gov/OACT/STATS/table4c6.html`.

The main ideas arising in the discussion so far are really matters of common sense when applied to relative frequency but require formal axioms when used more generally:

- Probabilities are numbers between 0 and 1 assigned to subsets of the entire collection Ω of possible outcomes, with the probability of Ω itself defined equal to 1. In the examples, the subsets which are assigned probabilities include sub-intervals of the interval of possible human lifetimes measured in years, and also disjoint unions of such subintervals. These sets in the real line are viewed as possible *events* summarizing ages at death of newborns in the cohort population. At this point, we regard each set A of ages as determining the subset of the cohort population whose ages at death fall in A.

- The probability $Pr(A \cup B)$ of the union $A \cup B$ of disjoint (i.e., nonoverlapping) sets A and B is necessarily equal to the sum of the separate probabilities $Pr(A)$ and $Pr(B)$.

- When probabilities are requested with reference to a smaller universe of possible outcomes, such as $B =$ *lives aged 29*, rather than all members of a cohort population, the resulting *conditional probabilities* of events A are written $Pr(A \,|\, B)$ and calculated as $Pr(A \cap B)/Pr(B)$, where $A \cap B$ denotes the *intersection* or *overlap* of the two events A, B. *The phrase "lives aged 29" defines an event which in terms of ages at death says simply "age at death is 29 or larger" or, in relation to the cohort population, specifies the subset of the population which survives to exact age 29 (i.e., to the 29th birthday).*

- Two events A, B are defined to be *independent* when $Pr(A \cap B) = Pr(A) \cdot Pr(B)$ or — equivalently, as long as $Pr(B) > 0$ — when the conditional probability $Pr(A|B)$ expressing the probability of A if B were known to have occurred, is the same as the unconditional probability $Pr(A)$.

Table 1.1: Illustrative Life-Table, simulated to resemble realistic US Male life-table up to age 78. For details of simulation, see Section 3.2 below.

Age x	l_x	d_x	x	l_x	d_x
0	100000	2629	40	92315	295
1	97371	141	41	92020	332
2	97230	107	42	91688	408
3	97123	63	43	91280	414
4	97060	63	44	90866	464
5	96997	69	45	90402	532
6	96928	69	46	89870	587
7	96859	52	47	89283	680
8	96807	54	48	88603	702
9	96753	51	49	87901	782
10	96702	33	50	87119	841
11	96669	40	51	86278	885
12	96629	47	52	85393	974
13	96582	61	53	84419	1082
14	96521	86	54	83337	1088
15	96435	105	55	82249	1213
16	96330	83	56	81036	1344
17	96247	125	57	79692	1423
18	96122	133	58	78269	1476
19	95989	149	59	76793	1572
20	95840	154	60	75221	1696
21	95686	138	61	73525	1784
22	95548	163	62	71741	1933
23	95385	168	63	69808	2022
24	95217	166	64	67786	2186
25	95051	151	65	65600	2261
26	94900	149	66	63339	2371
27	94751	166	67	60968	2426
28	94585	157	68	58542	2356
29	94428	133	69	56186	2702
30	94295	160	70	53484	2548
31	94135	149	71	50936	2677
32	93986	152	72	48259	2811
33	93834	160	73	45448	2763
34	93674	199	74	42685	2710
35	93475	187	75	39975	2848
36	93288	212	76	37127	2832
37	93076	228	77	34295	2835
38	92848	272	78	31460	2803
39	92576	261			

Note: see a basic probability textbook, such as Hogg and Tanis (1997) or Devore (2007), for formal definitions and more detailed discussion of the notions of sample space, event, probability, and conditional probability.

The life-table, and the mechanism by which members of the population die, are summarized first through the survivor function $S(t)$ which at integer values of $t = x$ agrees with the ratios l_x/l_0. Note that $S(t)$ has values between 0 and 1, and can be interpreted as the probability for a single individual to survive at least x time units. Since fewer people are alive at larger ages, $S(t)$ is a decreasing function of the continuous age-variable t, and in applications $S(t)$ should be continuous and piecewise continuously differentiable (largely for convenience, and because any analytical expression which would be chosen for $S(t)$ in practice *will* be piecewise smooth). In addition, by definition, $S(0) = 1$. Another way of summarizing the probabilities of survival given by this function is to define the **density** function

$$f(t) \;=\; -\,\frac{dS}{dt}\,(t) = -S'(t) \tag{1.1}$$

as the (absolute) rate of decrease of the function S. Then, by the fundamental theorem of calculus, for any ages $a < b$,

$$Pr(\text{ life aged 0 dies between ages a and b })$$

$$= S(a) - S(b) = \int_a^b (-S'(t))\,dt = \int_a^b f(t)\,dt \tag{1.2}$$

which has the very helpful geometric interpretation that the probability of dying within the interval $[a, b)$ is equal to the area under the density curve $y = f(t)$ over the t-interval $[a, b)$. Note also that the 'probability' rule which assigns the integral $\int_A f(t)\,dt$ to the set A (which may be an interval, a union of intervals, or a still more complicated set) obviously satisfies the first two of the bulleted axioms displayed above, namely that $P(\Omega) = 1$ (where Ω is the *sample space* of all life-table outcomes) and $Pr(A \cup B) = Pr(A) + Pr(B)$ whenever A, B are disjoint or nonoverlapping subsets of Ω.

The **terminal age** ω of a life table is an integer value large enough that $S(\omega)$ is negligibly small, but no value $S(t)$ for $t < \omega$ is zero. For practical purposes, no individual lives to the ω birthday. While ω is finite in real life-tables and in some analytical survival models, most theoretical forms for

$S(t)$ have no finite age ω at which $S(\omega) = 0$, and in those forms $\omega = \infty$ by convention.

In probability theory, the **sample space** Ω is the set of all detailed outcomes of the underlying data-generating experiment. Subsets of the sample space to which probabilities will be assigned are called **events**. In this book, all of the interesting events concern lifetimes, or ages at death. Insurance contract payouts will be expressed as functions of the lifetimes at death of insured lives, and the average or expected values of these payouts will be used to calculate a fair equivalent value of the insurance contract to the insured. The machinery for calculating the average values relates to the concept of *random variable* based on the sample space $\Omega = [0, \infty)$ of lifetimes.

1.1.1 Random Variables and Expectations

Formally, a random variable is a real-valued mapping X defined on a sample space Ω, such that $\{s \in \Omega : X(s) \in (a, b]\}$ is an event with assigned probability whenever $a < b$ are real numbers. The real number $X(s)$ is interpreted as the value which would be observed if the detailed outcome of the underlying random experiment were $s \in \Omega$. The most important feature of a random variable is its **probability distribution**, which is the assignment rule of probabilities to all intervals $(a, b]$ of values for X, denoted for all real numbers $a \le b$ by

$$Pr(a < X \le b) \equiv Pr(\{s \in \Omega : \quad X(s) \in (a, b]\})$$

Remark 1.1 *In datasets derived from actual mortality studies or insurance portfolios, the detailed outcomes can be quite complicated, as discussed in Appendix A. However, in this and succeeding Chapters, we analyze lifetimes based on the* **cohort life table model***, also discussed in Appendix A, which is a simplified model based on the reduced data-structure, in which numbers at risk and numbers of observed failures are tabulated on age intervals of one year.*

Now we are ready to define some terms and motivate the notion of expectation. Think of the age T at which a specified newly born member of the population will die as a **random variable**, which for present purposes means a variable which takes various values t with probabilities governed (at integer ages) by the life table data l_x and the survivor function $S(t)$ or density function $f(t)$ in a formula like the one just given in equation (1.2). Suppose there is a contractual amount Y which must be paid (say, to the heirs of that individual) at the death of the individual at age T, and suppose that the contract provides a specific function $Y = g(T)$ according to which this payment depends on (the whole-number part of) the age T at which death occurs. What is the average value of such a payment over all individuals whose lifetimes are reflected in the life-table ? Since $d_x = l_x - l_{x+1}$ individuals (out of the original l_0) die at ages between x and $x+1$, thereby generating a payment $g(x)$, the total payment to all individuals in the life-table can be written as

$$\sum_x (l_x - l_{x+1}) \, g(x)$$

Thus the average payment, at least under the assumption that $Y = g(T)$ depends only on the largest whole number $[T]$ less than or equal to T, is

$$\begin{aligned} \textstyle\sum_x (l_x - l_{x+1}) \, g(x) \, / \, l_0 \; &= \; \textstyle\sum_x (S(x) - S(x+1)) g(x) \\ &= \; \textstyle\sum_x \int_x^{x+1} f(t) \, g(t) \, dt = \int_0^{\infty} f(t) \, g(t) \, dt \end{aligned} \tag{1.3}$$

This quantity, the total contingent payment over the whole cohort divided by the number in the cohort, is called the **expectation** of the random payment $Y = g(T)$ in this special case, and can be interpreted as the weighted average of all of the different payments $g(x)$ actually received, where the weights are just the relative frequency in the life table with which those payments are received. More generally, if the restriction that $g(t)$ depends only on the integer part $[t]$ of t were dropped , then the expectation of $Y = g(T)$ would be given by the same formula

$$E(Y) = E(g(T)) = \int_0^{\infty} f(t) \, g(t) \, dt \tag{1.4}$$

The foregoing discussion of expectations based on lifetime random variables included an interpretation of the expected value of discrete random variables in terms of weighted averages which holds much more generally. In this chapter, the averages are taken over all lives tabulated in an underlying cohort life table. In Chapter 3, specifically in Section 3.3, averages are taken over large samples of observations of discrete random variables. With the aid of the Law of Large Numbers, the weighted-average interpretation of expectations can be understood as a general mathematical result.

The displayed integral (1.4), like all expectation formulas, can be understood as a weighted average of values $g(T)$ obtained over a population, with weights equal to the probabilities of obtaining those values. Recall from the Riemann-integral construction in Calculus that the integral $\int f(t)g(t)dt$ can be regarded approximately as the sum over very small time-intervals $[t, t+\Delta)$ of the quantities $f(t)g(t)\Delta$, quantities which are interpreted as the base Δ of a rectangle multiplied by its height $f(t)g(t)$, and the rectangle closely matches the area under the graph of the function $f\,g$ over the interval $[t, t+\Delta)$. The term $f(t)g(t)\Delta$ can alternatively be interpreted as the product of the value $g(t)$ — essentially equal to any of the values $g(T)$ which can be realized when T falls within the interval $[t, t+\Delta)$ — multiplied by $f(t)\,\Delta$. The latter quantity is, by the Fundamental Theorem of the Calculus, approximately equal for small Δ to the area under the function f over the interval $[t, t+\Delta)$, and is by definition equal to the probability with which $T \in [t, t+\Delta)$. In summary, $E(Y) = \int_0^\infty g(t)f(t)dt$ is the average of values $g(T)$ obtained for lifetimes T within small intervals $[t, t+\Delta)$ weighted by the probabilities of approximately $f(t)\Delta$ with which those T and $g(T)$ values are obtained. The expectation is a weighted average because the weights $f(t)\Delta$ sum to the integral $\int_0^\infty f(t)dt = 1$.

Remark 1.2 *This way of approximating integrals of continuous integrands by sums corresponding to the integrals of piecewise constant integrands is closely related to the construction of the integral in terms of* **Riemann sums**. *For fuller details, see the definition the Integral via Riemann sums in a calculus book like Ellis and Gulick (2002).*

The same idea and formula in (1.4) can be applied to the restricted population of lives aged x. The resulting quantity is then called the **conditional expected value of** $g(T)$ given that $T \geq x$. The formula will be different in two ways: first, the range of integrationis from x to ∞, because of the resitriction to individuals in the life-table who have survived to exact age x; second, the density $f(t)$ must be replaced by $f(t)/S(x)$, the so-called **conditional density given** $T \geq x$, which is found as follows. From the definition of conditional probability, for $t \geq x$,

$$Pr(t \leq T < t + \Delta \,|\, T \geq x) \;=\; \frac{Pr(\,\{t \leq T < t + \Delta\} \cap \{T \geq x\})}{Pr(T \geq x)}$$

$$= \frac{Pr(t \leq T < t + \Delta)}{Pr(T \geq x)} \;=\; \frac{S(t) - S(t + \Delta)}{S(x)}$$

Thus the density which can be used to calculate conditional probabilities $Pr(a \leq T < b \,|\, T \geq x)$ for $x < a < b$ is

$$\lim_{\Delta \to 0} \frac{1}{\Delta} Pr(t \leq T < t{+}\Delta \,|\, T \geq x) \;=\; \lim_{\Delta \to 0} \frac{S(t) - S(t + \Delta)}{S(x)\,\Delta} \;=\; \frac{-S'(t)}{S(x)} \;=\; \frac{f(t)}{S(x)}$$

In other words, when it is desired to calculate the expectation of a function $Y = g(T)$ of the lifetime variable T only within the conditional or restricted population of individuals with lifetime $\geq x$, then the density $f(t)$ in the expectation formula (1.4) should be replaced by the density which is equal to $f(t)/S(x)$ for all values of t which are $\geq x$, and which is 0 for values $t \in [0, x)$.

The result of all of this discussion of conditional expected values is the formula, with associated weighted-average interpretation:

$$E(g(T) \,|\, T \geq x) \;=\; \frac{1}{S(x)} \int_x^\infty g(t)\, f(t)\, dt \tag{1.5}$$

1.2 Theory of Interest

1.2.1 Interest Rates and Compounding

Since payments based upon unpredictable occurrences or *contingencies* for insured lives can occur at different times, we study next the Theory of Interest, which is concerned with valuing streams of payments made over time. The general model in the case of constant interest, to which we restrict in the current sub-section, is as follows. Money is deposited in an account like a bank-account and grows according to a schedule determined by both the interest rate and the occasions when interest amounts are *compounded*, that is, deemed to be added to the account. The compounding rules are important because they determine when new interest interest begins to be earned on previously earned interest amounts.

The central concept of compound interest is that, over the fixed time interval of one year, an amount A_0 deposited at the beginning of the interval accumulates to $A_1 = A_0 \cdot (1+i)$ which could be withdrawn at the end of the interval. Since the constant interest rate is quoted as a constant over the period of one year, we have $1+i$ as the *accumulation factor* by which an initial deposit is multiplied to find the balance at the end of one year. By convention, interest rates are generally quoted as *annualized* rates, which means that the interest rate i_h applied to a time-interval $[t, t+h]$ for a period h of less than one year is prorated down to the interval h to give hi_h, which results in an accumulation factor $1 + hi_h$. Thus, for an initial deposit A_0 at time t which is to be retained in a bank account for the time h, so that the accumulated amount is **compounded** (i.e., is calculated by the bank and owned by the depositor) at time $t+h$, the balance which the owner could withdraw at time $t+h$ is $A(0) \cdot (1 + hi_h)$.

If the quoted interest rate i_h is annualized, and if interest earned is to be credited after every successive interval $h = 1/m$, then we say that the interest rate is a **nominal annualized interest rate with m-times-yearly compounding** or simply the *nominal interest rate*, and the standard notation for it is $i^{(m)}$ instead of $i_{1/m}$ as written above.

Banks are not required to calculate interest from the instant (in practical terms, the day) of deposit to the instant (i.e., the day) of withdrawal. In practice, the intervals of *compounding* are generally fractions $h = 1/m$ of a year, usually with $m = 1,\ 2,\ 4,$ or 12. This means that after a deposit of A_0 at time t, the depositor wishing to withdraw the full accumulation or balance at time $t+s$ for $0 < s < h$ owns only the initial amount A_0, because no interest has yet been credited.

The further growth of deposited money over successive time intervals of length $h = 1/m$, if compounded at each additional interval of length $1/m$, is easily understood inductively. With amount A_0 deposited initially at time t, the balance as of time $t+h$ is $A_0(1 + i^{(m)}/m)$ and can be viewed as though it were simultaneously withdrawn and freshly deposited at time $t+h$, after which it would accumulate over the succeeding interval

$[t+h, t+2h]$ by multiplying the deposited amount $A_0(1+i^{(m)}/m)$ by the interval-h accumulation factor $1+i^{(m)}/m$. Thus the balance as of time $t+2h = t+2/m$ is $A_0(1+i^{(m)}/m)^2$. Inductively, for $k \geq 2$, if the balance $A_0(1+i^{(m)}/m)^k$ owned by the depositor at time $t+k/m$ is regarded as instantaneously withdrawn and redeposited as an initial balance for the next interval $[t+k/m, t+(k+1)/m]$, the balance at time $t+(k+1)/m$ is $A_0(1+i^{(m)}/m)^k$ multiplied by the interval-h accumulation factor $1+ih$, or $A_0(1+i^{(m)}/m)^{(k+1)}$.

The overall result of our reasoning about m-times yearly compounded nominal interest is the following:

Proposition 1.1 *The accumulated value of an initial bank deposit of* A_0 *compounded* m *times yearly at nominal interest rate* $i^{(m)}$ *after a time* $k/m+s$, *where* $0 \leq s < 1/m$ *and* $k \geq 0$ *is an integer, is* $(1+i^{(m)}/m)^k \cdot A_0$.

Proposition 1.1 with $k = m$ says that at the annualized nominal interest rate $i^{(m)}$, an initial deposit of A_0 accumulates after exactly one year to a balance of $(1+i^{(m)}/m)^m A_0$. Since the accumulation from the full year of deposit has the effect of multiplying the initial deposit by the factor $(1+i^{(m)}/m)^m$, a factor which would have been $1+i$ at interest rate i compounded yearly. *This proves that the nominal interest rate* $i^{(m)}$ *with m-times-yearly compounding leads to exactly the same accumulation over whole years as a deposit account with the once-yearly compounded "effective" rate*

$$i \equiv i_{\mathit{eff}} = (1+i^{(m)}/m)^m - 1$$

Since any nominal interest rate $i^{(m)}$ has its equivalent effective interest rate $i = i_{\mathit{eff}}$ providing the same yearly accumulations, the nominal interest rates $i^{(m)}$ with different values of m but the same value of i can also be regarded as equivalent. These whole-year-equivalent nominal rates are determined by solving the last equation for $i^{(m)}$ in terms of $i = i_{\mathit{eff}}$:

$$i^{(m)} = m\Big((1+i)^{1/m} - 1\Big) \tag{1.6}$$

For example, with $i = .05$, or 5% effective annual interest, the corresponding nominal rates $i^{(m)}$ for the most common values of m are obtained through the **R** code line:

```
mvec = c(1,2,4,12, 365) ; imvec = mvec*(1.05^(1/mvec) - 1)
```

as

$$i^{(1)} = .05 \ , \quad i^{(2)} = .04939 \ , \quad i^{(4)} = .04909 \ , \quad i^{(12)} = .04889 \ , \quad i^{(365)} = .04879$$

A few simple calculus manipulations allow us to establish the pattern of the displayed $i^{(m)}$ values for all choices of i, m. The right-hand side of equation (1.6) is a function $g(h)$ of $h = 1/m$, where

$$g(h) \ = \ h^{-1} \Big((1+i)^h \ - \ 1 \Big) \ = \ (\exp(h \ln(1+i)) - 1)/h$$

has the form of a *difference quotient* from Calculus. Recall the Taylor series expansion $e^z \ = \ 1 + z + z^2/2 + z^3/3! \cdots$ which is valid for all $z > 0$. Substitute this series with $z = h \ln(1+i)$ into the displayed formula for $g(h)$ to conclude that

$$g(h) \ = \ \sum_{j=1}^{\infty} \frac{(\ln(1+i) \ \cdot \ h)^j}{h \ \cdot \ j!} \ = \ \ln(1+i) \ + \ \sum_{j=1}^{\infty} \frac{1}{j!} \, (\ln(1+i))^j \, h^{j-1}$$

is an increasing function of $h > 0$ and is always greater than its right-hand limit

$$g(0+) \ = \ \lim_{h \searrow 0} \frac{\exp(h \ln(1+i)) \ - \ 1}{h} \ = \ \frac{d}{dh} \Big(e^{h \ \ln(1+i)} \Big)_{h=0} \ = \ \ln(1+i)$$

The information just established concerning the behavior of $i^{(m)} = g(1/m)$ as a function of m for fixed effective interest rate $i = i_{\mathit{eff}}$ is summarized as follows.

Proposition 1.2 *When* $i = i_{\mathit{eff}}$ *is fixed, the nominal annual interest rate* $i^{(m)}$ *for m-times-yearly compounding is a decreasing function of the positive integer* m *and tends as* $m \to \infty$ *to the limiting value, defined as the* **force of interest**,

$$\delta \ = \ \ln(1 + i_{\mathit{eff}}) \ = \ \lim_{m \to \infty} \ i^{(m)}$$

In the displayed $i^{(m)}$ values for $i = .05$, the daily-compounded nominal interest rate was $i^{(365)} = .048973$. The corresponding force of interest, also called the *instaneously* or *continuously compounded nominal interest rate*, is $\delta = \ln(1.05) = .048970$.

The effective interest rate i_{eff} can be expressed through its nominal continuously compounded interest rate δ as $i = e^{\delta}$, and the other nominal rates have similar expressions immediately derived from (1.6):

$$i^{(m)} = m\,(e^{\delta/m} - 1)$$

For all durations t which are rational numbers, i.e., are of the form $t = k/m$ for positive integers k, m, Prop. 1.1 with $s = 0$ says that the accumulation factor for duration $t = k/m$ based on m-times-yearly compounding at effective interest rate i is $(1+i^{(m)}/m)^k = (1+i^{(m)}/m)^{mt} = e^{t\,\delta}$. Since $t = k/m$ is also of the form $kl/(ml)$ for every integer $l \geq 1$, the same reasoning gives $e^{t\delta}$ as the accumulation factor for duration t under the same effective interest rate with ml-times-yearly compounding. Taking the limit as $l \to \infty$, with t fixed and arbitrary $m \geq 1$, says that the accumulation factor over duration t for instantaneous or continuous compounding should be the same. *This is essentially a definition of what accumulation by continuous compounding should mean, but it is the only definition under which continuous compounding is well approximated by compounding arbitrarily (but finitely) many times per day.*

Now it is obvious that the accumulation factor by continuous compounding over a duration $k/m + s$ (for $0 \leq s \leq 1/m$ is nondecreasing in s and must therefore lie within the interval $[e^{\delta k/m}, e^{\delta(k+1)/m}]$. By continuity of the exponential function, there follows:

Proposition 1.3 *The accumulated balance of an initial deposit A_0 under continuous compounding with effective interest rate i, or equivalently with force of interest $\delta = \ln(1+i)$, over a duration $t > 0$ which is not necessarily a rational number, is $\exp(\delta t) \cdot A_0$.*

So far, we have described in Props. 1.1 and 1.3 the mechanism of accumulation under nominal interest rates applying with either m-times-yearly or continuous compounding, and in equation (1.6) and Prop. 1.2 the relations between nominal interest rates, force of interest, and effective interest rates. There is a further term for interest rates which must be disclosed to borrowers under US contracts, namely the *Annual Percentage Rate* or **APR**. Unlike the interest rate terminology discussed up to this point, **APR** is a legal term which refers either ('effective APR') to the effective interest rate or ('nominal APR' or simply 'APR') to the nominal interest, *to which is added in either case* the service fees charged by the lender as a fraction of the beginning-of-year loan balance. The APR disclosure is intended as a consumer protection to the borrower, but may vary across jurisdictions in the way start-up fees (e.g., origination and participation) are required to be reported.

1.2.2 Present Values and Payment Streams

Applications of the theory of interest generally involve comparisons between streams of payments which may be made at different times and may accumulate at different rates of interest. These payments may be deposits into a bank or investment account, or loan repayments, or successive payments designed to accumulate over time at interest to a sufficient reserve fund to meet some future liability.

First, a discrete **payment stream** is a sequence of (positive) deposit amounts α_j made at specified calendar times t_j, $j = 1, 2, \ldots, n$ and which are regarded as accumulating from their times of deposit according to a schedule of interest rates $r(t)$ which remain constant within successive intervals of calendar time t but which may change from one such interval to the next.

Two basic principles govern all problems of valuing such payment streams.

- The **Principle of Equivalence** defines *equivalence at time* τ between two payment streams, one with payments and times $(\alpha_j,\, t_j,\, j = 1, 2, \ldots, n)$ and interest rate function $r(t)$ and the other with payments and times $(\alpha_j^*,\, t_j^*,\, j = 1, 2, \ldots, n^*)$ and interest rate function $r^*(t)$, where $\tau \ge \max_j t_j,\, \max_j t_j^*$. These streams are called *equivalent at* τ if the accumulated values at τ from the two payment streams under their respective interest rate functions are the same.

- The **Principle of Linear Superposition** states that the total accumulated amount resulting at time τ from a payment stream $(\alpha_j,\, t_j,\, j = 1, 2, \ldots, n)$ under interest rate function $r(t)$ is the same as the sum of the accumulated values up to time T of n separate deposit accounts initiated at the respective times t_j with deposits of α_j, all under the same interest rate function $r(t)$.

The two Principles as just stated do not yet tell us how to calculate the accumulated values at τ under interest rate functions $r(t)$ that vary over time. However, we can already see that the first Principle is a definition, while we will see that the second is an essentially obvious restatement of the commutativity of addition together with the fact that the accumulation of discrete payment streams is a well-defined linear function of the payment amounts α_j.

Consider first the case where $r(t) \equiv i$ is constant over the entire time-interval $[\min_j t_j,\, \tau]$. Then Prop. 1.3 gives the contribution of the deposit α_j at time t_j to the accumulated value at τ as $\alpha_j \cdot (1+i)^{\tau - t_j}$. On the other hand, the direct inductive calculation of the accumulated amounts at all times $t_l \geq t_j$ due to the α_j deposit, are also given via Prop. 1.3 as $\alpha_j \cdot (1+i)^{t_l - t_j}$, from which (by continuously compounding at interest rate i from the largest of the times t_l until τ) the final contribution of the α_j deposit to the final accumulation at τ is again seen to be $\alpha_j \cdot (1+i)^{\tau - t_j}$. This argument, with a little more notational effort and an inductive argument over the successively larger deposit times t_l, can be made into a rigorous proof of the Linear Superposition principle in the constant interest-rate environment with continuous compounding. The formula for the continuously compounded accumulated value of the stream at time τ is

$$\sum_{j=1}^{n} \alpha_j \, (1+i)^{\tau - t_j} \;=\; \sum_{j=1}^{n} \alpha_j \, e^{\delta\,(\tau - t_j)} \tag{1.7}$$

If compounding is instead m-times-yearly and all of the time-differences $\tau - t_l$ are integer multiples of $1/m$, then we appeal to Propositions 1.1 and 1.3 to confirm that there is no difference between the accumulated values at effective interest rate i under continuous or m-times-yearly compounding, and formula (1.7) again expresses the accumulated value at τ, which is also equal to

$$\sum_{j=1}^{n} \alpha_j \; (1 + i^{(m)}/m)^{m(\tau - t_j)}$$

The most important application of the principle of equivalence is in finding a deposit amount $\alpha_{\rm PV}$ at a single fixed time t which is equivalent to the payment stream $(\alpha_j,\, t_j,\; j = 1, \ldots, n)$ *at all times* $\tau \geq \max_j t_j$, at the same effective interest rate $i = i_{\it eff}$ with continuous compounding as is used to accumulate $(\alpha_j,\, t_j,\; j = 1, \ldots, n)$. This amount $\alpha_{\rm PV}$ is then called the **present value of the payment stream at time** t . To see why this is possible, consider any fixed $\tau \geq \max(t,\; \max_j t_j)$ and equate the accumulated value (1.7) to the accumulated value $\alpha_{\rm PV}\,(1+i)^{\tau - t}$ of the single deposit at time t using interest rate i, yielding:

$$\sum_{j=1}^{n} \alpha_j \; (1+i)^{\tau - t_j} \; = \; \alpha_{\rm PV}\,(1+i)^{\tau - t} \quad \Longrightarrow \quad \alpha_{\rm PV} \; = \; \sum_{j=1}^{n} \alpha_j \; (1+i)^{t - t_j} \tag{1.8}$$

This equation determining $\alpha_{\rm PV}$ evidently does not depend upon τ. It tells first (with $n = 1$, $t_1 = 0$ in (1.8)) the present value at fixed interest rate i of a payment of 1 exactly t years in the future, $(1+i)^{-t}$. This is the amount which must be put in the bank at time 0 in order to accumulate by the factor $(1+i)^t$ given by Prop. 1.3) to the value 1 at time t. Then, more generally,

> the present value at time 0 under constant interest rate i of a payment stream consisting of payments α_j at future times t_j, $j = 1, \ldots, n$ is equal to the summation $\sum_{j=1}^{n} \alpha_j \; (1+i)^{-t_j}$.

The same phenomenon, that a single deposit $\alpha_{\rm PV}$ at time t_0 can be equivalent at all times τ to a payment stream $(\alpha_j,\, t_j,\; j = 1, \ldots, n)$ turns out to hold more generally whenever the same time-varying interest rate function $r(t)$ is used to accumulate both the single deposit and the payment stream. The proof of this Fact will be left to an Exercise in Section 1.2.4 where accumulation formulas for variable interest rates are discussed. The magnitude $\alpha_{\rm PV}$ is then the general present value of the payment stream at time t_0.

1.2.3 Principal and Interest, and Discount Rates

In this Section, we consider the compounding of interest from the point of view of a borrower of an amount L at time 0, where the interest rate is constant with $i_{eff} = i$. Initially assume continuous compounding for all accumulations. If the borrower plans to make payments α_j, $1 \le j \le n$, at times $0 < t_1 < t_2 < \cdots < t_n$, then by definition

> the **principal** remaining on the loan as of time t is equal to the accumulated value at t of the single deposit L at time 0, minus the accumulated value at t under continuously compounded effective rate i of all payments made at times before t, i.e.,
>
> **Principal at time** $t \;=\; L\,(1+i)^t \;-\; \sum_{j:\, t_j \le t} \alpha_j\,(1+i)^{t-t_j}$.

The principal remaining in the loan just after a payment has been made is the same as the amount the borrower could pay to pay off the loan completely at that instant. In addition, if there are fees or late charges due at the times t_j when payments are made, then those amounts are added to the Principal or Balance owed as of t_j. However, in the present discussion we ignore all such additional fees or charges.

The principal owed on the loan just after time t reflects that as of time t, the lender must be compensated for the amount $(1+i)^t\,L$ to which the original loan amount would accumulate; while the accumulated value of the stream of payments actually made up to time t reduces the debt.

Each payment α_j made can be broken down into the so-called *Interest* and *Principal* portions by the rule:

$$\text{Interest Portion of Paymt at } t_j \;=\; (\text{Principal at } t_{j-1}) \cdot ((1+i)^{t_j - t_{j-1}} - 1)$$

$$\text{Principal Portion of Paymt at } t_j \;=\; \alpha_j \;-\; \text{Interest Portion of Paymt at } t_j$$

The first of these lines is clearly the amount of interest that the principal just after t_{j-1} would have earned at rate i over the time interval $t_j - t_{j-1}$. The amount of the payment at t_j minus the amount of interest at t_j is the amount by which the principal decreases from just after t_{j-1} to just after t_j. This simple Proposition is not quite obvious, but is easily shown by an algebraic rearrangement of terms, given as an Exercise.

Exercise 1.A. Show that the foregoing definitions of Principal and Principal Portions of payments are compatible by deriving the following identity from the definitions. If $\Pi(t)$ denotes the principal owed just after time t, and π_j denotes the principal portion of the payment at t_j, then

$$\Pi(t_{j-1}) \;-\; \pi_j \;=\; \Pi(t_j) \qquad \Box$$

The nominal interest rates $i^{(m)}$ for different periods of compounding were seen in Prop. 1.2 to be related by the formulas

$$(1+i^{(m)}/m)^m \;=\; 1+i \;=\; 1+i_{\mathit{eff}} \quad , \quad i^{(m)} \;=\; m\,\Big\{(1+i)^{1/m} \,-\, 1\Big\} \tag{1.9}$$

Similarly, interest can be said to be governed by the *discount rates* $d^{(m)}$ for various compounding periods, defined by

$$1 \;-\; d^{(m)}/m \;=\; (1+i^{(m)}/m)^{-1}$$

Solving the last equation for $d^{(m)}$ gives

$$d^{(m)} \;=\; i^{(m)}/(1+i^{(m)}/m) \tag{1.10}$$

The idea of discount rates is that if an amount 1 is loaned out at interest, then the amount $d^{(m)}/m$ is the correct amount to be repaid at the *beginning* rather than the end of each fraction $1/m$ of the year, with repayment of the principal of 1 at the end of the year, in order to amount to the same effective interest rate. The reason is that, according to the definition, the amount $1-d^{(m)}/m$ accumulates at nominal interest $i^{(m)}$ to $(1-d^{(m)}/m)\cdot(1+i^{(m)}/m) = 1$ after a time-period of $1/m$.

The quantities $i^{(m)}$ and $d^{(m)}$ are naturally introduced as the interest payments which must be made respectively at the ends and the beginnings of successive time-periods of length $1/m$ in order that the principal owed at each time j/m on an amount 1 borrowed at time 0 will always be 1. To define these terms and justify this assertion, consider first the simplest case, $m=1$. If 1 is to be borrowed at time 0, then the single payment at time 1 which fully compensates the lender, if that lender could alternatively have earned interest rate i, is $(1+i)$, which we view as a payment of 1 *principal* (the face amount of the loan) and i interest. In exactly the same way, if 1 is borrowed at time 0 for a time-period $1/m$, then the repayment at time $1/m$ takes the form of 1 principal and $i^{(m)}/m$ interest. Thus, if 1 was borrowed at time 0, an interest payment of $i^{(m)}/m$ at time $1/m$ leaves an amount 1 still owed, which can be viewed as an amount borrowed on the time-interval $(1/m, 2/m]$. Then a payment of $i^{(m)}/m$ at time $2/m$ still leaves an amount 1 owed at $2/m$, which is deemed borrowed until time $3/m$, and so forth, until the loan of 1 on the final time-interval $((m-1)/m, 1]$ is paid off at time 1 with a final interest payment of $i^{(m)}/m$ together with the principal repayment of 1. The overall result which we have just proved intuitively is:

> 1 at time 0 is equivalent to the stream of m payments of $i^{(m)}/m$ at times $1/m$, $2/m$, $\ldots, 1$ plus the payment of 1 at time 1.

Similarly, if interest is to be paid at the *beginning* of the period of the loan instead of the end, the interest paid at time 0 for a loan of 1 would be $d = i/(1+i)$, with the only other payment a repayment of principal at time 1. To see that this is correct, note that since interest d is paid at the same instant as receiving the loan of 1 , the net amount actually received is $1 - d = (1+i)^{-1}$, which accumulates in value to $(1-d)(1+i) = 1$ at time 1. Similarly, if interest payments are to be made at the beginnings of each of the intervals $(j/m, (j+1)/m]$ for $j = 0, 1, \ldots, m-1$, with a final principal repayment of 1 at time 1, then the interest payments should be $d^{(m)}/m$. This follows because the amount effectively borrowed (after the immediate interest payment) over each interval $(j/m, (j+1)/m]$ is $(1-d^{(m)}/m)$, which accumulates in value over the interval of length $1/m$ to an amount $(1-d^{(m)}/m)(1+i^{(m)}/m) = 1$. So throughout the year-long life of the loan, the principal owed at (or just before) each time $(j+1)/m$ is exactly 1. The overall result concerning m-period-yearly discount interest is

> 1 at time 0 is equivalent to the stream of m payments of $d^{(m)}/m$ at times 0, $1/m$, $2/m$, $\ldots$, $(m-1)/m$ plus the payment of 1 at time 1.

A useful algebraic exercise to confirm the displayed assertions is:

Exercise 1.B. Verify that the present values at time 0 of the payment streams with m interest payments in the displayed assertions are respectively

$$\sum_{j=1}^{m} \frac{i^{(m)}}{m} (1+i)^{-j/m} + (1+i)^{-1} \qquad \text{and} \qquad \sum_{j=0}^{m-1} \frac{d^{(m)}}{m} (1+i)^{-j/m} + (1+i)^{-1}$$

and that both are equal to 1. These identities are valid for all $i > 0$. □

1.2.4 Variable Interest Rates

Now we formulate the generalization of these ideas to the case of non-constant instantaneously varying, but known or observed, effective interest rate $r(t)$ at time t , corresponding to the instantaneous continuously compounded nominal rate, or *time-varying force of interest*, $\delta(t) = \ln(1+r(t))$. Consider the compounding of interest over successive intervals $[b + kh, b + (k + 1)h]$, where $h = 1/m$ for large m, there is an essentially constant principal amount over each interval of length $1/m$. Since we assume the functions $r(t)$ and therefore $\ln(1+\delta(t))$ are uniformly continuous in t, so that over very short intervals $[b+kh, b+(k+1)h]$ with instantaneous compounding, the interest rate and its associated force of interest are essentially constant, with accumulation factor over the interval given by $e^{h\,\delta(kh)}$. Therefore, if an initial time b and duration $\tau > 0$ are fixed and $[m\tau] = [\tau/h]$ denotes the largest integer $\le m\tau$, we find that the continuous compounding of interest over the time-interval $[b, b + \tau]$ results in an overall accumulation factor of approximately

$$e^{h\delta(b)}\, e^{h\delta(b+h)}\, e^{h\delta(b+2h)} \cdots e^{h\delta(b+([\tau/h]-1)h)} \exp\Big(((\tau - h[\tau/h]) \cdot \delta(b + h[\tau/h])\Big)$$

which has limit as $m \to \infty$ equal to

$$\exp\Big(\lim_{m} \frac{1}{m} \sum_{k=0}^{[m\tau]-1} \delta(b + k/m)\Big) = \exp\Big(\int_0^t \delta(b + s)\, ds\Big)$$

The last step in this chain of equalities relates the concept of continuous compounding to that of the Riemann integral. To specify continuous-time varying interest rates in terms of instantaneous effective rates, we would equate the last displayed formula for the accumulation factor over $[b, b+\tau]$ to

$$\exp\Big(\int_0^t \ln(1+r(b+s))\,ds\Big)$$

Next consider the case of deposits $\alpha_0, \alpha_1, \ldots, \alpha_k, \ldots, \alpha_n$ made at times $0, h, \ldots, kh, \ldots, nh$, where $h = 1/m$ is the given compounding-period, and wherenominal annualized instantaneous interest-rates $\delta(kh)$ (with compounding-period h) apply to the accrual of interest on the interval $[kh, (k+1)h)$. If the accumulated bank balance just after time kh is denoted by B_k , then how can the accumulated bank balance be expressed in terms of α_j and $\delta(jh)$? Clearly

$$B_{k+1} \;=\; B_k \cdot e^{\delta(kh)/m} \,+\, \alpha_{k+1} \quad , \qquad B_0 \;=\; \alpha_0$$

The preceding *difference equation* can be solved in terms of successive summation and product operations acting on the sequences α_j and $\delta(jh)$, as follows. First define a function A_k to denote the accumulated bank balance at time kh for a unit invested at time 0 and earning interest with instantaneous nominal interest rates $\delta(jh)$ applying respectively over the whole compounding-intervals $[jh, (j+1)h)$, $j = 0, \ldots, k-1$. Then by definition, A_k satisfies a *homogeneous equation* analogous to the previous one, which together with its solution is given by

$$A_{k+1} \;=\; A_k \cdot e^{\delta(kh)/m}, \qquad A_0 = 1, \qquad A_k = \prod_{j=0}^{k-1} e^{\delta(jh)/m}$$

We now return to the idea of *equivalent investments* and present value of a payment stream, as discussed in Section 1.2.2. Our object is to determine a single deposit D at time 0 which is *equivalent at time* $\tau = nh$ to a stream of deposits α_j, $j = 0, 1, 2, \ldots, n$, where all amounts accumulate according to the continuously compounded instantaneous effective interest rate $r(t)$ and associated force of interest $\delta(t) = \ln(1 + r(t))$. By approximating the continuous interest rate function $r(t)$ by one which is constant on intervals $[kh, (k+1)h)$, we have just calculated that an amount 1 at time 0 compounds to an accumulated amount A_n at time $\tau = nh$. Therefore, an amount D at time 0 accumulates to $D \cdot A_n$ at time τ, and in particular $D = 1/A_n$ at time 0 accumulates to 1 at time τ. Note, as in (1.8) of Section 1.2.2, this single equivalent deposit D would be the same if the accumulations were valued at any other time $\tau' > nh$. Thus the present value of 1 at time $\tau = nh$ is $1/A_n$. Now define G_k to be the present value of the stream of payments α_j at time jh for $j = 0, 1, \ldots, k$. Since B_k was the accumulated value just after time kh of the same stream of payments, and since the present value at 0 of an amount B_k at time kh is just B_k/A_k, we conclude

$$G_{k+1} \;=\; \frac{B_{k+1}}{A_{k+1}} \;=\; \frac{B_k \, \exp(\delta(km)/m)}{A_k \, \exp(\delta(km)/m)} + \frac{\alpha_{k+1}}{A_{k+1}} \; , \quad k \ge 1 \; , \qquad G_0 = \alpha_0$$

Thus $G_{k+1} - G_k = \alpha_{k+1}/A_{k+1}$, and

$$G_{k+1} \;=\; \alpha_0 + \sum_{i=0}^{k} \frac{\alpha_{i+1}}{A_{i+1}} \;=\; \sum_{j=0}^{k+1} \frac{\alpha_j}{A_j}$$

In summary, we have simultaneously found the solution for the accumulated balance B_k just after time kh and for the present value G_k at time 0 :

$$G_k \;=\; \sum_{i=0}^{k} \frac{\alpha_i}{A_i} \; , \qquad B_k \;=\; A_k \cdot G_k \; , \qquad k = 0, \ldots, n$$

The formulas just developed can be used to give the *internal rate of return* r over the time-interval $[0, \tau]$ of a unit investment which pays amount α_k at times t_k, $k = 0, \ldots, n$, $0 \le t_k \le \tau$. This constant (effective) interest rate r is the one such that

$$\sum_{k=0}^{n} s_k \left(1 + r\right)^{-t_k} \; = 1$$

With respect to the constant interest rate r , the present value of a payment α_k at a time t_k time-units in the future is $\alpha_k \cdot (1 + r)^{-t_k}$. Therefore the stream of payments α_k at times t_k, $(k = 0, 1, \ldots, n)$ becomes equivalent, for the uniquely defined interest rate r, to an immediate (time-0) payment of 1.

Exercise 1.C. As an illustration of the notion of effective interest rate, or internal rate of return, suppose that you are offered an investment option under which an investment of $10,000$ made now is expected to pay 300 yearly for 5 years (beginning 1 year from the date of the investment), and then 800 yearly for the following five years, with the principal of $10,000$ returned to you (if all goes well) exactly 10 years from the date of the investment (at the same time as the last of the 800 payments. If the investment goes as planned, what is the effective interest rate you will be earning on your investment ?

Solution. As in all calculations of effective interest rate, the present value ofthe payment-stream, at the unknown interest rate $r = i_{eff}$, must be balanced with the value (here $10,000$) which is invested. (That is because the indicated payment stream is being regarded as equivalent to bank interest at rate r.) The balance equation in the Example is obviously

$$10,000 \;=\; 300 \sum_{j=1}^{5} (1+r)^{-j} \;+\; 800 \sum_{j=6}^{10} (1+r)^{-j} \;+\; 10,000\,(1+r)^{-10}$$

The right-hand side can be simplified somewhat, in terms of the notation $x = (1+r)^{-5}$, to

$$\frac{300}{1+r}\left(\frac{1-x}{1-(1+r)^{-1}}\right) + \frac{800x}{(1+r)}\left(\frac{1-x}{1-(1+r)^{-1}}\right) + 10000\,x^2$$

$$= \frac{1-x}{r}\,(300+800x) \;+\; 10000x^2 \qquad (1.11)$$

Setting this simplified expression equal to the left-hand side of $10,000$ does not lead to a closed-form solution, since both $x = (1+r)^{-5}$ and r involve the unknown r. Nevertheless, we can solve the equation roughly by 'tabulating' the values of the simplified right-hand side as a function of r ranging in increments of 0.005 from 0.035 through 0.075. (We can guess that the correct answer lies between the minimum and maximum payments expressed as a fraction of the principal.) This tabulation yields:

r	.035	.040	.045	.050	.055	.060	.065	.070	.075
(1.11)	11485	11018	10574	10152	9749	9366	9000	8562	8320

From these values, we can see that the right-hand side is equal to $10,000$ for a value of r falling between 0.05 and 0.055. Interpolating linearly to approximate the answer yields $r = 0.050 + 0.005 * (10000 - 10152)/(9749 - 10152) = 0.05189$, while an accurate equation-solver finds $r = 0.05186$. For example, the root-finding function in **R** is called *uniroot*, and the **R** code for computing the effective interest rate in this Example is:

```
       Rsolv = function(r) { x = (1+r)^(-5)
                  (1-x)*(300+800*x)/r + 10000*x^2 - 10000 }
       uniroot(Rsolv, c(.035,.075))$root
[1] 0.05185676
```

1.2.5 Continuous-time Payment Streams

There is a completely analogous development for continuous-time deposit streams with continuous compounding. Suppose $D(t)$ to be the **rate** per unit time at which savings deposits are made, so that if we take m to go to ∞ in the previous discussion, we have $D(t) = \lim_{m\to\infty} m\alpha_{[mt]}$, where $[\cdot]$ again denotes greatest-integer. Taking $\delta(t)$ to be the time-varying nominal interest rate with continuous compounding, and $B(t)$ to be the accumulated balance as of time t (analogous to the quantity $B_{[mt]} = B_k$ from before, when $t = k/m$), we replace the previous difference-equation by

$$B(t+h) \;=\; B(t)\,(1 + h\,\delta(t)) \;+\; h\,D(t) \;+\; o(h)$$

where $o(h)$ denotes a remainder such that $o(h)/h \to 0$ as $h \to 0$. Subtracting $B(t)$ from both sides of the last equation, dividing by h, and letting h decrease to 0, yields a *differential equation* at times $t > 0$:

$$B'(t) \;=\; B(t)\,\delta(t) + D(t)\;, \qquad A(0) = \alpha_0 \tag{1.12}$$

The method of solution of (1.12), which is the standard one from differential equations theory of multiplying through by an *integrating factor*, again has a natural interpretation in terms of present values. The integrating factor $1/A(t) = \exp(-\int_0^t \delta(s)\,ds)$ is the present value at time 0 of a payment of 1 at time t, and the quantity $B(t)/A(t) \;=\; G(t)$ is then the present value of the deposit stream of α_0 at time 0 followed by continuous deposits at rate $D(t)$. The ratio-rule of differentiation yields

$$G'(t) \;=\; \frac{B'(t)}{A(t)} - \frac{B(t)\,A'(t)}{A^2(t)} \;=\; \frac{B'(t)\;-\;B(t)\,\delta(t)}{A(t)} \;=\; \frac{D(t)}{A(t)}$$

where the substitution $A'(t)/A(t) \equiv \delta(t)$ has been made in the third expression. Since $G(0) = B(0) = \alpha_0$, the solution to the differential equation (1.12) becomes

$$G(t) = \alpha_0 + \int_0^t \frac{D(s)}{A(s)}\, ds \; , \qquad B(t) = A(t)\, G(t)$$

Finally, the formula can be specialized to the case of a constant unit-rate payment stream ($D(x) = 1$, $\delta(x) = \delta = \ln(1+i)$, $0 \le x \le T$) with no initial deposit (i.e., $\alpha_0 = 0$). By the preceding formulas, $A(t) = \exp(t \ln(1+i)) = (1+i)^t$, and the present value of such a payment stream is

$$\int_0^T 1 \cdot \exp(-t \ln(1+i))\, dt = \frac{1}{\delta}\Big(1 - (1+i)^{-T}\Big)$$

Recall that the *force of interest* $\delta = \ln(1+i)$ is the limiting value obtained from the nominal interest rate $i^{(m)}$ using the difference-quotient representation:

$$\lim_{m\to\infty} i^{(m)} \;=\; \lim_{m\to\infty} \frac{\exp((1/m)\ln(1+i)) \,-\, 1}{1/m} \;=\; \ln(1+i)$$

The present value of a payment at time T in the future is then

$$\Big(1 \,+\, \frac{i^{(m)}}{m}\Big)^{-mT} \;=\; (1+i)^{-T} \;=\; \exp(-\delta\, T)$$

1.3 *Exercise Set 1*

The first homework set covers the basic definitions in two areas: (i) probability as it relates to events defined from *cohort life-tables*, including the theoretical machinery of population and conditional survival, distribution, and density functions and the definition of expectation; (ii) the theory of interest and present values, with special reference to the idea of income streams of equal value at a fixed rate of interest.

(1). For how long a time should \$100 be left to accumulate at 5% interest so that it will amount to twice the accumulated value (over the same time period) of another \$100 deposited at 3% ?

(2). Use a calculator or computer to answer the following numerically:

(a) Suppose you sell for \$6,000 the right to receive for 10 years the amount of \$1,000 per year payable quarterly (starting at the end of the first quarter). What effective rate of interest makes this a fair sale price ? (You will have to solve numerically or graphically, or interpolate a tabulation, to find it.)

(b) \$100 deposited 20 years ago has grown at interest to \$235. The interest was compounded twice a year. What were the nominal and effective interest rates ?

(c) How much should be set aside (the same amount each year) at the beginning of each year for 10 years to amount to \$1000 at the end of the 10th year at the interest rate of part (b) ?

In the following problems, $S(t)$ denotes the probability for a newborn in a designated population to survive to exact age t . If a *cohort life table* is under discussion, then the probability distribution relates to a randomly chosen member of the newborn cohort.

(3). Assume that a population's survival probability function is given by $S(t) = 0.1(100-t)^{1/2}$, for $0 \le t \le 100$.

(a) Find the probability that a life aged 0 will die between exact ages 19 and 36.

(b) Find the probability that a life aged 36 will die before exact age 51.

(4). For members of the poulation in Problem **(3)**,

(a) Find the expected age at death of a newborn (life aged 0).

(b) Find the expected age at death of a life aged 20.

(5). Use the Illustrative Life-table (Table 1.1) to calculate the following probabilities. (In each case, assume that the indicated span of years runs from birthday to birthday.) Find the probability

(a) that a life aged 26 will live at least 30 more years;

(b) that a life aged 22 will die between ages 45 and 55;

(c) that a life aged 25 will die either before age 50 or after age 70.

(6). In a special population, you are given the following facts:

(i) The probability that two independent lives, respectively aged 25 and 45, *both* survive 20 years is 0.7.

(ii) The probability that a life aged 25 will survive 10 years is 0.9.

Then find the probability that a life aged 35 will survive to age 65.

(7). Suppose that you borrowed $1000 at 6% effective rate, to be repaid in 5 years in a lump sum, and that after holding the money idle for 1 year you invested the money and earned 8% effective for theremaining four years. What is the effective interest rate you earned (ignoring interest costs) over 5 years on the $1000 which you borrowed ? Taking interest costs into account, what is the present value of your profit over the 5 years of the loan ? Also re-do the problem if instead of repaying all principal and interest at the end of 5 years, you must make a payment of accrued interest at the end of 3 years, with the additional interest and principal due in a single lump-sum at the end of 5 years.

(8). Find the total present value at 5% APR of payments of $1 at the end of 1, 3, 5, 7, and 9 years and payments of $2 at the end of 2, 4, 6, 8, and 10 years.

(9). Find the present value at time 0 at a 6% effective interest rate of a series payments of 100 at times $1, 2, 3$ and of 300 at times $6, 7, 8$.

(10). Find the present value at time 0 of payments of 100 at ten successive times $1, 2, \ldots, 10$ if the instanteous effective interest rate applying at all times t in the time interval $[0, 10]$ is $r(t) = .07 - (.002)t$.

(11). Find the internal rate of return (i.e., the equivaent constant effective interest rate) over the time interval $[0, 7]$ of an investment which pays bank interest of 4% at times in $[0, 5]$ if you make deposits of 1000 at each of the times $t = 0, 2, 4,$ if the interest rate earned on the time interval $[5, 7]$ is 6%, and if the total balance is withdrawn at time 7.

(12). (i) Find the payment amount K such that a loan of $10,000$ at a 7% effective annual interest rate is repaid in exactly three payments consisting of an amount K at times 1 and 3 years and of $2K$ at 5 years.

(ii) After finding K in part (i), decompose each of the three loan repayment amounts K, K, $2K$ at respective times, 1, 3, 5 into their principal and interest portions.

1.4 Worked Examples

Example 1. How many years does it take for money to triple in value at interest rate i ?

The equation to solve is $3 = (1+i)^t$, so the answer is $\ln(3)/\ln(1+i)$, with numerical answer given by

$$t \;=\; \begin{cases} 22.52 & \text{for } \; i = 0.05 \\ 16.24 & \text{for } \; i = 0.07 \\ 11.53 & \text{for } \; i = 0.10 \end{cases}$$

Example 2. Suppose that a sum of $1000 is borrowed for 5 years at 5%, with interest deducted immediately in a lump sum from the amount borrowed, and principal due in a lump sum at the end of the 5 years. Suppose further that the amount received is invested and earns 7%. What is the value of the net profit at the end of the 5 years ? What is its present value (at 5%) as of time 0 ?

First, the amount received by the borrower at time 0 is $1000\,(1-d)^5 = 1000/(1.05)^5 = 783.53$, where $d = .05/1.05$, since the amount received should compound to precisely the principal of 1000 at 5% interest in 5 years. Next, the compounded value of 783.53 for 5 years at 7% is $783.53\,(1.07)^5 = 1098.94$, so the net profit at the end of 5 years, after paying off the principal of 1000, is 98.94. The present value of the profit ought to be calculated with respect to the 'going rate of interest', which in this problem is presumably the rate of 5% at which the money is borrowed, so is $98.94/(1.05)^5 = 77.52$.

Example 3. For the following small cohort life-table (first 3 columns) with 5 age-categories, find the probabilities for all values of $[T]$, both unconditionally and conditionally for lives aged 2, and find the expectation of both $[T]$ and $(1.05)^{-[T]-1}$.

The basic information in the table is the first column l_x of numbers surviving. Then $d_x = l_x - l_{x+1}$ for $x = 0, 1, \ldots, 4$. The random variable T is the life-length for a randomly selected individual from the age=0 cohort, and therefore $Pr([T] = x) = Pr(x \le T < x+1) = d_x/l_0$. The conditional probabilities given survivorship to age-category 2 are simply the ratios with numerator d_x for $x \ge 2$, and with denominator $l_2 = 65$.

x	l_x	d_x	$Pr([T]=x)$	$Pr([T]=x \mid T \ge 2)$	1.05^{-x-1}
0	100	20	0.20	0	0.95238
1	80	15	0.15	0	0.90703
2	65	10	0.10	0.15385	0.86384
3	55	15	0.15	0.23077	0.82770
4	40	40	0.40	0.61538	0.78353
5	0	0	0	0	0.74622

In terms of the columns of this table, we evaluate from the definitions and formula (1.3)

$$E([T]) \;=\; 0 \cdot (0.20) + 1 \cdot (0.15) + 2 \cdot (0.10) + 3 \cdot (0.15) + 4 \cdot (0.40) \;=\; 2.4$$

$$E([T] \,|\, T \ge 2) \;=\; 2 \cdot (0.15385) + 3 \cdot (0.23077) + 4 \cdot (0.61538) \;=\; 3.4615$$

$$E(1.05^{-[T]-1}) \;=\; 0.95238 \cdot 0.20 \;+\; 0.90703 \cdot 0.15 \;+\; 0.86384 \cdot 0.10 +$$
$$+\, 0.8277 \cdot 0.15 \;+\; 0.78353 \cdot 0.40 \;=\; 0.8497$$

The expectation of $[T]$ is interpreted as the average per person in the cohort life-table of the number of completed whole years before death. The quantity $(1.05)^{-[T]-1}$ can be interpreted as the present value at birth of a payment of 1 to be made at the end of the year of death, and the final expectation calculated above is the average of that present-value over all the individuals in the cohort life-table, if the going rate of interest is 5%.

Example 4. Suppose that the death-rates $q_x = d_x/l_x$ *for integer ages* x *in a cohort life-table follow the functional form*

$$q_x \;=\; \begin{cases} 4 \cdot 10^{-4} & \textit{for } 5 \le x < 30 \\ 8 \cdot 10^{-4} & \textit{for } 30 \le x \le 55 \end{cases}$$

between the ages x *of 5 and 55 inclusive. Find analytical expressions for* $S(x)$, l_x, d_x *at these ages if* $l_0 = 10^5$, $S(5) = .96$.

The key formula expressing survival probabilities in terms of death-rates q_x is:

$$\frac{S(x+1)}{S(x)} = \frac{l_{x+1}}{l_x} = 1 - q_x$$

or

$$l_x = l_0 \cdot S(x) = (1-q_0)(1-q_1)\cdots(1-q_{x-1})$$

So it follows that for $x = 5, \ldots, 30$,

$$\frac{S(x)}{S(5)} = (1 - .0004)^{x-5}\ , \qquad l_x = 96000 \cdot (0.9996)^{x-5}$$

so that $S(30) = .940446$, and for $x = 31, \ldots, 55$,

$$S(x) = S(30) \cdot (.9992)^{x-30} = .940446\ (.9992)^{x-30}$$

The death-counts d_x are expressed most simply through the preceding expressions together with the formula $d_x = q_x\, l_x$.

1.5 Useful Formulas from Chapter 1

$$S(x) \;=\; \frac{l_x}{l_0} \quad , \qquad d_x = l_x - l_{x+1}$$

p. 1

$$P(x \le T < x+k) \;=\; S(x) - S(x+k) \;=\; \frac{l_x - l_{x+k}}{l_x}$$

p. 2

$$f(t) \;=\; -S'(t) \quad , \qquad S(y) - S(y+t) \;=\; \int_y^{y+t} f(s)\, ds$$

p. 5

$$E\left(g(T) \,\middle|\, T \ge x \right) \;=\; \frac{1}{S(x)} \int_x^{\infty} g(t)\, f(t)\, dt$$

p. 9

$$1 + i \;=\; 1 + i_{\mathit{eff}} \;=\; \left(1 + \frac{i^{(m)}}{m}\right)^m \;=\; \left(1 - \frac{d^{(m)}}{m}\right)^{-m} \;=\; e^{\delta}$$

p. 18

Chapter 2

Theory of Interest and Force of Mortality

The parallel development of Interest and Probability Theory topics continues in this Chapter. For application in Insurance, we are preparing to value uncertain payment streams in which times of payment may also be uncertain. The interest theory allows us to express the present values of deterministic or *certain* payment streams compactly, while the probability material prepares us to find and interpret average or expected values of present values expressed as functions of random lifetime variables.

This installment of the course covers: (a) further formulas and topics in the pure (i.e., non-probabilistic) theory of interest, and (b) more discussion of lifetime random variables, in particular of *force of mortality* or hazard-rates, and theoretical families of life distributions.

2.1 More on Theory of Interest

In this Section, we define notations and find compact formulas for present values of some standard payment streams. To this end, newly defined payment streams are systematically expressed in terms of previously considered ones. There are two primary methods of manipulating one payment-stream to give another for the convenient calculation of present values:

- First, if one payment-stream can be obtained from a second one precisely by delaying all payments by the same amount t of time, then the present value of the first one is v^t multiplied by the present value of the second.

- Second, if a payment-stream A can be obtained as the superposition of payment streams B and C, i.e., can be obtained by paying the sum of the timed payment amounts defining the streams B and C, then the present value of stream A is the sum of the present values of B and C.

The following subsection contains several useful applications of these methods. For another simple illustration, see Worked Example 2 at the end of the Chapter.

2.1.1 Annuities & Actuarial Notation

The general present value formulas above will now be specialized to the case of constant (instantaneous) interest rate $r(t) \equiv i \equiv e^\delta$ at all times $t \geq 0$, and some very particular streams of payments s_j at times t_j, related to periodic premium and annuity payments. The effective interest rate is always denoted by $i = i_{\mathit{eff}}$, and as before the m-times-per-year equivalent nominal interest rate is denoted by $i^{(m)}$. Also, from now on the standard and convenient notation

$$v \;\equiv\; 1/(1+i) \;=\; 1/\left(1+\frac{i^{(m)}}{m}\right)^m$$

will be used for the present value of a payment of 1 one year later.

(i) If $s_0 = 0$ and $s_1 = \cdots = s_{nm} = 1/m$ in the discrete setting, where $m \geq 1$ denotes the number of payments per year, and $t_j = j/m$, then the payment-stream is called an **immediate annuity**, and its present value G_n is given the notation $a^{(m)}_{\overline{n|}}$ and is equal, by the geometric-series summation formula, to

$$m^{-1}\sum_{j=1}^{nm}\left(1+\frac{i^{(m)}}{m}\right)^{-j} = \left(1+\frac{i^{(m)}}{m}\right)^{-1}\frac{1-(1+i^{(m)}/m)^{-nm}}{m(1-(1+i^{(m)}/m)^{-1})}$$

which shows that

$$a^{(m)}_{\overline{n}|} = \frac{1 - ((1+i^{(m)}/m)^{-m})^n}{m\,(1+i^{(m)}/m - 1)} \;=\; \frac{1-v^n}{i^{(m)}} \tag{2.1}$$

All of these immediate annuity values, for fixed v, n but varying m, are roughly comparable because all involve a total payment of 1 per year. Formula (2.1) shows that all of the values $a^{(m)}_{\overline{n}|}$ differ only through the factors $i^{(m)}$, which differ by only a few percent for varying m and fixed i, as shown in Table 2.1. Recall from formula (1.9) that $i^{(m)} = m\{(1+i)^{1/m} - 1\}$.

If, instead of the payment stream defining the immediate annuity, $s_0 = 1/m$ but $s_{nm} = 0$, then nm deposits of $1/m$ are made at an arithmetic progression of times from $1/m$ to n inclusive, and the present value notation changes to $\ddot{a}^{(m)}_{\overline{n}|}$. The payment stream is then called an **annuity-due**, and the present valueis given by any of the equivalent formulas

$$\ddot{a}^{(m)}_{\overline{n}|} \;=\; (1 + \frac{i^{(m)}}{m})\, a^{(m)}_{\overline{n}|} \;=\; \frac{1-v^n}{m} \;+\; a^{(m)}_{\overline{n}|} \;=\; \frac{1}{m} + a^{(m)}_{\overline{n-1/m}|} \tag{2.2}$$

The first of these formulas recognizes the annuity-due payment-stream as identical to the annuity-immediate payment-stream shifted earlier by the time $1/m$ and therefore worth more by the accumulation-factor $(1+i)^{1/m} = 1 + i^{(m)}/m$. The third expression in (2.2) represents the annuity-due stream as being equal to the annuity-immediate stream with the payment of $1/m$ at $t = 0$ added and the payment of $1/m$ at $t = n$ removed. The final expression says that if the time-0 payment is removed from the annuity-due, the remaining stream coincides with the annuity-immediate stream consisting of $nm - 1$ (instead of nm) payments of $1/m$.

In the limit as $m \to \infty$ for fixed n, the notation $\overline{a}_{\overline{n}|}$ denotes the **continuous annuity**, that is, the present value of an annuity paid instantaneously at constant unit rate, with the limiting nominal interest-rate which was shown in the previous chapter to be $\lim_m i^{(m)} = i^{(\infty)} = \delta$. The limiting behavior of the nominal interest rate can be seen rapidly from the formula

$$i^{(m)} \;=\; m\Big((1+i)^{1/m} \,-\, 1\Big) \;=\; \delta \,\cdot\, \frac{\exp(\delta/m) - 1}{\delta/m}$$

since $(e^z - 1)/z$ converges to 1 as $z \to 0$. Then by (2.1) and (2.2),

$$\overline{a}_{\overline{n}|} \;=\; \lim_{m\to\infty} \ddot{a}^{(m)}_{\overline{n}|} \;=\; \lim_{m\to\infty} a^{(m)}_{\overline{n}|} \;=\; \frac{1-v^n}{\delta} \tag{2.3}$$

Table 2.1: Values of nominal interest rates $i^{(m)}$ (upper number) and $d^{(m)}$ (lower number), for various choices of effective annual interest rate i and number m of compounding periods per year.

$i =$	.02	.03	.05	.07	.10	.15
$m = 2$	.0199	.0298	.0494	.0688	.0976	.145
	.0197	.0293	.0482	.0665	.0931	.135
3	.0199	.0297	.0492	.0684	.0968	.143
	.0197	.0294	.0484	.0669	.0938	.137
4	.0199	.0297	.0491	.0682	.0965	.142
	.0198	.0294	.0485	.0671	.0942	.137
6	.0198	.0296	.0490	.0680	.0961	.141
	.0198	.0295	.0486	.0673	.0946	.138
12	.0198	.0296	.0489	.0678	.0957	.141
	.0198	.0295	.0487	.0675	.0949	.139

Remark 2.1 *The definition and formulas for the immediate annuity* $a^{(m)}_{\overline{n}|}$ *and the annuity-due* $\ddot{a}^{(m)}_{\overline{n}|}$ *remain valid if* nm *but not necessarily* n *itself is an integer. In the limit as* $m \to \infty$, *the continuous annuity definition* $\overline{a}_{\overline{n}|}$ *and formula remain valid with any positive real number* n. □

A handy formula for annuity-due present values follows easily by recalling that

$$1 - \frac{d^{(m)}}{m} = \left(1 + \frac{i^{(m)}}{m}\right)^{-1} \qquad \text{implies} \qquad d^{(m)} = \frac{i^{(m)}}{1 + i^{(m)}/m}$$

Then, by (2.2) and (2.1),

$$\ddot{a}^{(m)}_{\overline{n}|} = (1 - v^n) \cdot \frac{1 + i^{(m)}/m}{i^{(m)}} = \frac{1 - v^n}{d^{(m)}} \tag{2.4}$$

In case m is 1, the superscript $^{(m)}$ is omitted from all of the annuity notations. In the limit where $n \to \infty$, the notations become $a^{(m)}_{\overline{\infty}|}$ and $\ddot{a}^{(m)}_{\overline{\infty}|}$, and the annuities are called **perpetuities** (respectively immediate and due) with present-value formulas obtained from (2.1) and (2.4) as:

$$a^{(m)}_{\overline{\infty}|} = \frac{1}{i^{(m)}} \quad , \qquad \ddot{a}^{(m)}_{\overline{\infty}|} = \frac{1}{d^{(m)}} \tag{2.5}$$

We now build some more general annuity-related present values out of the standard functions $a^{(m)}_{\overline{n}|}$ and $\ddot{a}^{(m)}_{\overline{n}|}$.

(ii). Consider first the case of the **increasing perpetual annuity-due**, denoted $(I^{(m)}\ddot{a})^{(m)}_{\overline{\infty}|}$, which is defined as the present value of a stream of payments $(k+1)/m^2$ at times k/m, for $k = 0, 1, \ldots$ forever. Clearly the present value is

$$(I^{(m)}\ddot{a})^{(m)}_{\overline{\infty}|} \;=\; \sum_{k=0}^{\infty} m^{-2}\,(k+1)\,\Big(1+\frac{i^{(m)}}{m}\Big)^{-k}$$

Here are two methods to sum this series, the first purely mathematical, the second based on actuarial intuition. First, without worrying about the strict justification for differentiating an infinite series term-by-term,

$$\sum_{k=0}^{\infty} (k+1)\,x^k \;=\; \frac{d}{dx}\sum_{k=0}^{\infty} x^{k+1} \;=\; \frac{d}{dx}\,\frac{x}{1-x} \;=\; (1-x)^{-2}$$

for $0 < x < 1$, where the geometric-series formula has been used to sum the second expression. Therefore, with $x = (1+i^{(m)}/m)^{-1}$ and $1 - x = (i^{(m)}/m)/(1+i^{(m)}/m)$,

$$(I^{(m)}\ddot{a})^{(m)}_{\overline{\infty}|} = m^{-2}\Big(\frac{i^{(m)}/m}{1+i^{(m)}/m}\Big)^{-2} \;=\; \Big(\frac{1}{d^{(m)}}\Big)^2 \;=\; \Big(\ddot{a}^{(m)}_{\overline{\infty}|}\Big)^2$$

and (2.5) has been used in the last step. Another way to reach the same result is to recognize the increasing perpetual annuity-due as $1/m$ multiplied by the superposition of perpetuities-due $\ddot{a}^{(m)}_{\overline{\infty}|}$ paid at times $0, 1/m, 2/m, \ldots$, and therefore its present value must be $\ddot{a}^{(m)}_{\overline{\infty}|} \cdot \ddot{a}^{(m)}_{\overline{\infty}|}$. As an aid in recognizing this equivalence, consider each annuity-due $\ddot{a}^{(m)}_{\overline{\infty}|}$ paid at a time j/m as being equivalent to a stream of payments $1/m$ at time j/m, $1/m$ at $(j+1)/m$, etc. Putting together all of these payment streams gives a total of $(k+1)/m$ paid at time k/m, of which $1/m$ comes from the annuity-due starting at time 0, $1/m$ from the annuity-due starting at time $1/m$, up to the payment of $1/m$ from the annuity-due starting at time k/m.

(iii). The **increasing perpetual annuity-immediate** $(I^{(m)}a)^{(m)}_{\overline{\infty}|}$ — the same payment stream as in the increasing annuity-due, but deferred by a time $1/m$ — is related to the perpetual annuity-due in the obvious way

$$(I^{(m)}a)^{(m)}_{\overline{\infty}|} \;=\; v^{1/m}\,(I^{(m)}\ddot{a})^{(m)}_{\overline{\infty}|} \;=\; (I^{(m)}\ddot{a})^{(m)}_{\overline{\infty}|}\Big/(1+i^{(m)}/m) \;=\; \frac{1}{i^{(m)}\,d^{(m)}}$$

(iv). Now consider the **increasing annuity-due of finite duration** n years. This is the present value $(I^{(m)}\ddot{a})^{(m)}_{\overline{n}|}$ of the payment-stream of $(k+1)/m^2$ at time k/m, for $k = 0, \ldots, nm-1$. Evidently, this payment-stream is equivalent to $(I^{(m)}\ddot{a})^{(m)}_{\overline{\infty}|}$ minus the sum of n multiplied by an annuity-due $\ddot{a}^{(m)}_{\overline{\infty}|}$ starting at time n together with an increasing annuity-due $(I^{(m)}\ddot{a})^{(m)}_{\overline{\infty}|}$ starting at time n. (To see this clearly, equate the payments $0 = (k+1)/m^2 - n \cdot \frac{1}{m} - (k - nm + 1)/m^2$ received at times k/m for $k \geq nm$.) Thus

$$\begin{aligned}
(I^{(m)}\ddot{a})^{(m)}_{\overline{n}|} &= (I^{(m)}\ddot{a})^{(m)}_{\overline{\infty}|} \Big(1 - v^n\Big) - n\ddot{a}^{(m)}_{\overline{\infty}|} v^n \\
&= \ddot{a}^{(m)}_{\overline{\infty}|} \left(\ddot{a}^{(m)}_{\overline{\infty}|} - v^n \Big[\ddot{a}^{(m)}_{\overline{\infty}|} + n\Big]\right) \\
&= \ddot{a}^{(m)}_{\overline{\infty}|} \left(\ddot{a}^{(m)}_{\overline{n}|} - n\, v^n\right)
\end{aligned}$$

where, in the last line, recall that $v = (1+i)^{-1} = (1 + i^{(m)}/m)^{-m}$ and that $\ddot{a}^{(m)}_{\overline{n}|} = \ddot{a}^{(m)}_{\overline{\infty}|}(1 - v^n)$. The latter identity is easy to justify either by the formulas (2.4) and (2.5) or by regarding the annuity-due payment stream as a superposition of the payment-stream up to time $n - 1/m$ and the payment-stream starting at time n. As an exercise, fill in details of a second, intuitive verification, analogous to the second verification in pargraph (ii) above.

(v). The **decreasing annuity** $(D^{(m)}\,\ddot{a})^{(m)}_{\overline{n}|}$ is defined as (the present value of) a stream of payments starting with n/m at time 0 and decreasing by $1/m^2$ after every time-period of $1/m$, with no further payments at or after time n. The easiest way to obtain the present value is through the identity

$$(I^{(m)}\ddot{a})^{(m)}_{\overline{n}|} \; + \; (D^{(m)}\ddot{a})^{(m)}_{\overline{n}|} \; = \; (n + \frac{1}{m}\,)\; \ddot{a}^{(m)}_{\overline{n}|} \tag{2.6}$$

Again, as usual, the method of proving this is to observe that in the payment-stream whose present value is given on the left-hand side, the payment amount at each of the times j/m, for $j = 0, 1, \ldots, nm-1$, is

$$\frac{j+1}{m^2} \; + \; (\frac{n}{m} \; - \; \frac{j}{m^2}) \; = \; \frac{1}{m}\,(n \; + \; \frac{1}{m})$$

2.1.2 Loan Repayment: Mortgage, Bond, Sinking Fund

Perhaps the most common application of interest theory is the calculation of the payment amounts needed to repay a loan according to a few standard repayment plans. In ordinary consumer purchases or long-term fixerd-rate loans on the purchase of a house, the usual repayment plan is a series of *level* or equal payments made m times yearly (usually with $m = 12$, and usually with first payment at time 0) for a total duration of n years, so that at the last payment (the nm'th payment, at time $n - 1/m$ if the first payment was made at time 0) the loan has been completely paid off. We refer to this kind of repayment schedule with level payments as a **mortgage loan**. [1]

Recall that the present value of a payment stream of amount c per year, with c/m paid at times $1/m, 2/m, \ldots, n-1/m, n/m$, is $c\, a_{\overline{n|}}^{(m)}$. Thus, if an amount L has been borrowed for a term of n years, to be repaid by equal installments at the end of every period $1/m$, at fixed nominal interest rate $i^{(m)}$, then the level payment c/m or **installment payment** amount is obtained by equating $L = c\, a_{\overline{n|}}^{(m)}$

$$\text{Mortgage Payment} \quad = \quad L/(m\, a_{\overline{n|}}^{(m)}) \quad = \quad \frac{L\, i^{(m)}}{m\,(1 - v^n)} \tag{2.7}$$

where $v = 1/(1+i) = (1 + i^{(m)}/m)^{-m}$.

A second kind of plan to repay a loan is to pay equal amounts to cover only the interest amounts accrued every $1/m$ year on the principal for a duration of n years, with first payment made at time $1/m$ and last at time n, with the principal (the original amount borrowed) also repaid in a lump sum at time n. This arrangement is used by corporations or government agencies which issue **bonds**: the borrowing agency receives the loan amount from investors at time 0, called the **face amount** of the bond, and regularly issues interest payments after every period of $1/m$ year (usually with $m = 2$ or 4), and finally repays or **redeems** the face or principal amount of the bond at the end of the n year **term** of the bond. (The regular interest payments used to be called *coupon* payments because of small paper coupons attached to the paper bond document, and which the investor would regularly redeem at a bank, at scheduled times, for the interest payment amount.) The reader should look

[1] In legal and historical terms, 'mortgage' refers to the way in which the promise to repay is secured by the house or other property purchased with the amount borrowed.

back to Section 1.2.3 where loan repayment amounts were formally broken down into *interest* and *principal* portions, in order to confirm the sense of this bond repayment plan. Since each payment by the borrower at times k/m for $k = 1, 2, \ldots, nm$ (apart from the final lump sum principal repayment) consists of interest only (equal to the original face amount multiplied by $(1+i)^{1/m} - 1 = i^{(m)}/m$), each of the intermediate payments contains 0 principal portion, and the result of Exercise 1.A in Sec. 1.2.3 shows that the original face amount or principal is also the principal or balance owed on the loan just after each interest payment, up to the final redemption when the principal is paid.

Now a corporation or governmental agency which borrows money from investors by issuing a bond, will often obligate itself through a formal legal arrangement to devote a certain category of income to a so-called **sinking fund**, an investment account maintained by a trustee. Apart from the bond interest payments at a contractual effective interst rate i made directly to investors (the lenders), the borrowing agency will also pay regular amounts at intervals of $1/m'$ year to the sinking fund trustee for the same term of n years as the bond, with the intention that the sinking fund will accumulate at its own possibly different investment interest rate i' to the principal or face amount of the issued bond at time n, at which time the principal is repaid directly to the bond investors. At (or just after) an intermediate times k/m, the amount built up in the sinking fund is referred to as a **reserve** toward the ultimate redemption of the principal of the bond. We will see, later in this book, that *insurance reserves* generalize these deterministic reserves to Insurances, where the future payouts are *not* determinstic but rather contingent on the mortality experience of the portfolio of insured lives.

Remark 2.2 *Note that, if a borrowed amount* L *is repaid by regular interest payments and a sinking fund, and if the number of payments per year into the fund is* $m' = m$ *and the effective interest rate for the sinking fund is* $i' = i$, *then evidently the sum of the regular* m*-times-yearly interest and sinking fund payment at each time* k/m *is precisely the same as the mortgage payment (2.7). For* $m' \neq m$ *or* $i' \neq i$, *a separate calculation is needed, leading to several exercises at the end of the Chapter. The following Exercise with sketched solution gives an example.*

Exercise 2.A. A small city issues a bond for ten million dollars for ten years at 4% nominal quarterly interest ($m = 4$) and creates a sinking fund into which it will make annual deposits ($m' = 1$), from tax receipts, on which its financial advisors claim it can safely earn an effective annual rate of $i' = .055$. Find the amount of the level annual sinking fund deposit.

Solution of Exercise 2.A. Since the interest payments are made at the contractual interest rate i (corresponding to $i^{(4)} = .04$), the sinking fund must appreciate to the loan amount of $L = 10^7$ at time $t = 10$. Therefore the sinking fund present value at time 10 (equal to its present value at $t = 0$ accumulated by the factor $(1 + i')^{10}$ must be equated to $L = 10^7$. So the annual sinking fund deposit D is found through the equality

$$L = 10^7 = (1 + i')^{10} \cdot D \, a'_{\overline{10}|} = D \, \frac{1 - (1.055)^{-10}}{.055} \, 1.055^{10}$$

where a' indicates that the annuity is calculated at interest rate i'.

2.1.3 Loan Amortization & Mortgage Refinancing

We analyze next the breakdown between principal and interest in repaying a mortgage loan by level payments (2.7). Of the payment made at time $(k+1)/m$, how much can be attributed to interest and how much to principal ? Consider the present value at time 0 of the debt for a unit ($L = 1$) loan amount less the accumulated amounts paid through time k/m :

$$1 - \left(m \, a^{(m)}_{\overline{k/m}|}\right) / \left(m \, a^{(m)}_{\overline{n}|}\right) = 1 - \frac{1 - v^{k/m}}{1 - v^n} = \frac{v^{k/m} - v^n}{1 - v^n}$$

The remaining debt, per unit of loan amount, valued just after time k/m, is denoted from now on by $B_{n,\,k/m}$. It is greater than the displayed present value at 0 by a factor $(1 + i)^{k/m}$, so is equal to

$$B_{n,\,k/m} = (1 + i)^{k/m} \, \frac{v^{k/m} - v^n}{1 - v^n} = \frac{1 - v^{n-k/m}}{1 - v^n} \tag{2.8}$$

The amount of interest for a loan amount of 1 after time $1/m$ is $(1 + i)^{1/m} - 1 = i^{(m)}/m$. Therefore the interest included in the payment at

$(k+1)/m$ is $i^{(m)}/m$ multiplied by the value $B_{n,\,k/m}$ of outstanding debt just after k/m. Thus the next total payment of $i^{(m)}/(m(1-v^n))$ consists of the two parts

$$\text{Amount of interest} \;=\; m^{-1}\, i^{(m)}\, (1 - v^{n-k/m})/(1-v^n)$$

$$\text{Amount of principal} \;=\; m^{-1} i^{(m)} v^{n-k/m}/(1-v^n)$$

By definition, the principal included in each payment is the amount of the payment minus the interest included in it. These formulas show in particular that the amount of principal repaid in each successive payment increases geometrically in the payment number, which at first seems surprising. Note as a check on the displayed formulas that the outstanding balance $B_{n,(k+1)/m}$ immediately after time $(k+1)/m$ is re-computed as $B_{n,\,k/m}$ minus the interest paid at $(k+1)/m$, or

$$\frac{1-v^{n-k/m}}{1-v^n} \;-\; \frac{i^{(m)}}{m}\,\frac{v^{n-k/m}}{1-v^n} \;=\; \frac{1-v^{n-k/m}(1+i^{(m)}/m)}{1-v^n}$$

$$= \frac{1-v^{n-(k+1)/m}}{1-v^n} \;=\; \Big(1 - a^{(m)}_{\overline{(k+1)/m}|} \Big/ a^{(m)}_{\overline{n}|}\Big)\; v^{-(k+1)/m} \tag{2.9}$$

as was derived above by considering the accumulated value of amounts paid. The general definition of the principal repaid in each payment is the excess of the payment over the interest since the past payment on the total balance due immediately following that previous payment.

2.1.4 Illustration on Mortgage Refinancing

Suppose that a 30–year, nominal-rate 8%, \$100,000 mortgage payable monthly is to be refinanced at the end of 8 years for an additional 15 years (instead of the 22 which would otherwise have been remaining to pay it off) at 6%, with a refinancing closing-cost amount of \$1500 and *2 points.* (The points are each 1% of the refinanced balance including closing costs, and costs plus points are then extra amounts added to the initial balance of the refinanced mortgage.) Suppose that the **new** pattern of payments is to be valued at each of the nominal interest rates 6%, 7%, or 8%, due to uncertainty about what the interest rate will be in the future, and that these valuations will be taken into account in deciding whether to take out the new loan.

The monthly payment amount of the initial loan in this example was $\$100,000(.08/12)/(1-(1+.08/12)^{-360}) = \$733.76$, and the present value as of time 0 (the beginning of the old loan) of the payments made through the end of the 8^{th} year is $(\$733.76)\cdot(12a^{(12)}_{\overline{8}|}) = \$51,904.69$. Thus the present value, *as of the end of 8 years*, of the payments still to be made under the *old* mortgage, is $\$(100,000-51,904.69)(1+.08/12)^{96} = \$91,018.31$. Thus, if the loan were to be refinanced, the new refinanced loan amount would be $\$91,018.31 + 1,500.00 = \$92,518.31$. If *2 points* must be paid in order to lock in the rate of 6% for the refinanced 15-year loan, then this amount is $(.02)92518.31 = \$1850.37$. The new principal balance of the refinanced loan is $92518.31 + 1850.37 = \$94,368.68$, and this is the present value at a nominal rate of 6% of the future loan payments, no matter what the term of the refinanced loan is. The new monthly payment (for a 15-year duration) of the refinanced loan is $\$94,368.68(.06/12)/(1-(1+.06/12)^{-180}) = \$796.34$.

For purposes of comparison, what is the present value at the current going rate of 6% (nominal) of the continuing stream of payments under the old loan ? That is a 22-year stream of monthly payments of $\$733.76$, as calculated above, so the present value at 6% is $\$733.76\cdot(12a^{(12)}_{\overline{22}|}) = \$107,420.21$. Thus, if the new rate of 6% were really to be the correct one for the next 22 years, and each loan would be paid to the end of its term, then it would be a financial disaster *not* to refinance. Next, suppose instead that right after re-financing, the economic rate of interest would be a nominal 7% for the next 22 years. In that case *both* streams of payments would have to be re-valued — the one before refinancing, continuing another 22 years into the future, and the one after refinancing, continuing 15 years into the future. The respective present values (as of the end of the 8^{th} year) at nominal rate of 7% of these two streams are:

Old loan: $733.76\,(12a^{(12)}_{\overline{22}|}) = \$98,700.06$

New loan: $796.34\,(12a^{(12)}_{\overline{15}|}) = \$88,597.57$

Even with these different assumptions, and despite closing-costs and points, it is well worth re-financing.

Exercise 2.B. Suppose that you can forecast that you will in fact sell your house in precisely 5 more years after the time when you are re-financing. At the time of sale, you would pay off the cash principal balance, whatever it is. Calculate and compare the present values (at each of 6%, 7%, and 8% nominal interest rates) of your payment streams to the bank, (a) if you continue the old loan without refinancing, and (b) if you re-finance to get a 15-year 6% loan including closing costs and points, as described above.

2.1.5 Computational illustration in R

All of the calculations described above are very easy to program in any language from Fortran to Mathematica, and also on a programmable calculator; but they are also very handily organized within a spreadsheet, which seems to be the way that MBA's, bank-officials, and actuaries will learn to do them from now on.

In this section, an **R** function (*cf.* Venables & Ripley 2002) is provided to do some comparative refinancing calculations. Concerning the syntax of **R**, the only explanation necessary at this point is that * denotes multiplication, and ^ denotes exponentiation.

The function *RefExmp* given below calculates mortgage payments, balances for purposes of refinancing both before and after application of administrative costs and points, and the present value under any interest rate (not necessarily the ones at which either the original or refinanced loans are taken out) of the stream of repayments to the bank up to and including the lump-sum payoff which would be made, for example, at the time of selling the house on which the mortgage loan was negotiated. The output of the function is a list which, in each numerical example below, is displayed in 'unlisted' form, horizontally as a vector. Lines beginning with the symbol # are comment-lines.

The outputs of the function are as follows. *Oldpayment* is the monthly payment on the original loan of face-amount *Loan* at nominal interest $i^{(12)} =$ *OldInt* for a term of *OldTerm* years. *NewBal* is the balance $B_{n,\,k/m}$ of formula (2.8) for $n =$ *OldTerm*, $m = 12$, and k/m $=$ *RefTim*, and the refinanced loan amount is a multiple $1+$ *Points* of *NewBal*, which is equal to *RefBal* + *Costs*. The new loan, at nominal interest rate *NewInt*, has

```
 R FUNCTION CALCULATING REFINANCE PAYMENTS & VALUES

RefExmp
function(Loan, OldTerm, RefTim, NewTerm, Costs, Points,
       PayoffTim, OldInt, NewInt, ValInt)
{
# Function calculates present value of future payment stream
#      underrefinanced loan.
# Loan = original loan amount;
# OldTerm = term of initial loan in years;
# RefTim = time in years after which to refinance;
# NewTerm = term of refinanced loan;
# Costs = fixed closing costs for refinancing;
# Points = fraction of new balance as additional costs;
# PayoffTim (no bigger than NewTerm) = time (from refinancing-
#      time at which new loan balance is to be paid off in
#      cash (eg at house sale);
# The three interest rates OldInt, NewInt, ValInt are
#      nominal 12-times-per-year, and monthly payments
#      are calculated.
   vold = (1 + OldInt/12)^(-12)
   Oldpaymt = ((Loan * OldInt)/12)/(1 - vold^OldTerm)
   NewBal = (Loan * (1 - vold^(OldTerm - RefTim)))/
       (1 - vold^OldTerm)
   RefBal = (NewBal + Costs) * (1 + Points)
   vnew = (1 + NewInt/12)^(-12)
   Newpaymt = ((RefBal * NewInt)/12)/(1 - vnew^NewTerm)
   vval = (1 + ValInt/12)^(-12)
   Value = (Newpaymt * 12 * (1 - vval^PayoffTim))/ValInt +
        (RefBal *  vval^PayoffTim * (1 - vnew^(NewTerm -
         PayoffTim)))/(1 -  vnew^NewTerm)
   list(Oldpaymt = Oldpaymt, NewBal = NewBal,
       RefBal = RefBal, Newpaymt = Newpaymt, Value = Value)
}
```

monthly payments *Newpaymt* for a term of *NewTerm* years. The loan is to be paid off *PayoffTim* years after *RefTim* when the new loan commences, and the final output of the function is the present value at the start of the refinanced loan with nominal interest rate *ValInt* of the stream of payments made under the refinanced loan up to and including the lump sum payoff.

We begin our numerical illustration by reproducing the quantities calculated in the previous subsection:

```
> unlist(RefExmp(100000, 30, 8, 15, 1500, 0.02, 15,
         0.08, 0.06, 0.06))
 Oldpaymt NewBal RefBal Newpaymt Value
   733.76  91018  94368   796.33 94368
```

Note that, since the payments under the new (refinanced) loan are here valued at the same interest rate as the loan itself, the present value Value of all payments made under the loan must be equal to the the refinanced loan amount *RefBal*.

The comparisons of the previous Section between the original and refinanced loans, at (nominal) interest rates of 6, 7, and 8 %, are all recapitulated easily using this function. To use it, for example, in valuing the old loan at 7%, the arguments must reflect a 'refinance' with no costs or points for a period of 22 years at nominal rate 6%, as follows:

```
> unlist(RefExmp(100000,30,8,22,0,0,22,0.08,0.08,0.07))
 Oldpaymt  NewBal RefBal Newpaymt  Value
   733.76  91018  91018   733.76   98701
```

(The small discrepancies between the values found here and in the previous subsection are due to the rounding used there to express payment amounts to the nearest cent.)

We consider next a numerical example showing break-even point for refinancing by balancing costs versus time needed to amortize them.

Suppose that you have a 30-year mortage for \$100,000 at nominal rate $i^{(12)} = 9\%$, *with level monthly payments, and that after 7 years of payments you refinance to obtain a new 30-year mortgage at 7% nominal interest (* $=$

$i^{(m)}$ for $m = 12$), with closing costs of \$1500 and 4 points (i.e., 4% of the total refinanced amount including closing costs added to the initial balance), also with level monthly payments. Figuring present values using the new interest rate of 7%, what is the time K (to the nearest month) such that if both loans — the old and the new — were to be paid off in exactly K years after the time (the 7-year mark for the first loan) when you would have refinanced, then the remaining payment-streams for both loans from the time when you refinance are equivalent (i.e., have the same present value from that time) ?

We first calculate the present value of payments under the new loan. As remarked above in the previous example, since the same interest rate is being used to value the payments as is used in figuring the refinanced loan, the valuation of the new loan *does not depend upon the time K to payoff.* (It is figured here as though the payoff time K were 10 years.)

```
> unlist(RefExmp(1.e5, 30,7,30, 1500,.04, 10, 0.09,0.07,0.07))
 Oldpaymt NewBal RefBal Newpaymt Value
   804.62  93640  98946   658.29 98946
```

Next we compute the value of payments under the old loan, at 7% nominal rate, also at payoff time $K = 10$. For comparison, the value under the old loan for payoff time 0 (i.e., for cash payoff at the time when refinancing would have occurred) coincides with the New Balance amount of \$93640.

```
> unlist(RefExmp(1.e5, 30,7,23, 0,0, 10, 0.09,0.09,0.07))
 Oldpaymt NewBal RefBal Newpaymt  Value
   804.62  93640  93640   804.62 106042
```

The values found in the same way when the payoff time K is successively replaced by 4, 3, 3.167, 3.25 are 99979, 98946, 98593, 98951. Thus, the payoff-time K at which there is essentially no difference in present value at nominal 7% between the old loan or the refinanced loan with costs and points (which was found to have Value 98946), is 3 years and 3 months after refinancing.

2.2 Force of Mortality & Analytical Models

Up to now, the function $S(t)$ called the *survivor* or *survival* function has been defined to be equal to the life-table ratio l_x/l_0 at all integer ages $t = x$, and to be piecewise continuously differentiable for all positive real values of t. Intuitively, for all positive real y and t, $S(y) - S(y+t)$ is the fraction of the initial life-table cohort which dies between ages y and $y+t$, and $(S(y) - S(y+t))/S(y)$ represents the fraction of those alive at exact age y who fail before $y+t$. An equivalent representation is $S(y) = \int_y^\infty f(t)\, dt$, where $f(t) \equiv -S'(t)$ is called the *failure density.* If T denotes the random variable which is the age at death for a newly born individual governed by the same causes of failure as the life-table cohort, then $Pr(T \geq y) = S(y)$, and according to the Fundamental Theorem of Calculus,

$$\lim_{\epsilon \to 0+} \frac{P(y \leq T \leq y+\epsilon)}{\epsilon} \;=\; \lim_{\epsilon \to 0+} \frac{1}{\epsilon} \int_y^{y+\epsilon} f(t)\, dt \;=\; f(y)$$

as long as the failure density is a continuous function.

Two further useful actuarial notations, often used to specify the theoretical lifetime distribution, are:

$${}_tp_y \;=\; P\big(T \geq y+t \,|\, T \geq y\big) \;=\; S(y+t)/S(y)$$

and

$${}_tq_y \;=\; 1 - {}_tp_y \;=\; P\big(T \leq y+t \,|\, T \geq y\big) \;=\; (S(y) - S(y+t))/S(y)$$

The quantity ${}_tq_y$ is referred to as the *age-specific death rate* for periods of length t. In the most usual case where $t = 1$ and $y = x$ is an integer, the notation ${}_1q_x$ is replaced by q_x, and ${}_1p_x$ is replaced by p_x. The rate q_x would be estimated from the cohort life table as the ratio d_x/l_x of those who die between ages x and $x+1$ as a fraction of those who reached age x. The way in which this quantity varies with x is one of the most important topics of study in actuarial science. For example, one important way in which numerical analysis enters actuarial science is that one wishes to interpolate the values ${}_1q_y$ smoothly as a function of y. The topic called "Graduation Theory" among actuaries is the mathematical methodology of Interpolation and Spline-smoothing applied to the raw function $q_x = d_x/l_x$.

To give some idea what a realistic set of death-rates looks like, Figure 2.1 pictures the age-specific 1-year death-rates q_x for the simulated life-table given as Table 1.1 on page 4. Additional granularity in the death-rates can be seen in Figure 2.2, where the logarithms of death-rates are plotted. After a very high death-rate during the first year of life (26.3 deaths per thousand live births), there is a year-by-year decline in death-rates roughly from 1.45 per thousand in the second year to 0.34 per thousand in the eleventh year. (But there were small increases in rate from ages 4 to 7 and from 8 to 9, which are as likely due to statistical irregularity as to real increases in risk.) Between ages 11 and 40, there is an erratic but roughly linear increase of death-rates per thousand from 0.4 to 3.0. However, at ages beyond 40 there is a rapid increase in death-rates as a function of age. As can be seen from Figure 2.2, the values q_x seem to increase roughly as a power c^x where $c \in [1.08, 1.10]$. (Compare this behavior with the Gompertz-Makeham Example (v) below.) This exponential behavior of the age-specific death-rate for large ages suggests that the death-rates pictured could reasonably be extrapolated to older ages using the formula

$$q_x \;\approx\; q_{78} \cdot (1.0885)^{x-78} \quad , \qquad x \geq 79 \tag{2.10}$$

where the number 1.0885 was found as $log(q_{78}/q_{39})/(78-39)$.

Now consider the behavior of ${}_\epsilon q_x$ as ϵ gets small. It is clear that ${}_\epsilon q_x$ must also get small, roughly proportionately to ϵ, since the probability of dying between ages x and $x+\epsilon$ is approximately $\epsilon\, f(x)$ when ϵ gets small.

<u>Definition</u>: The limiting death-rate ${}_\epsilon q_x/\epsilon$ per unit time as $\epsilon \searrow 0$ is called by actuaries the **force of mortality** $\mu(x)$. In reliability theory or biostatistics, the same function is called the *failure intensity, failure rate*, or *hazard intensity.*

The reasoning above shows that for small ϵ,

$$\frac{{}_\epsilon q_y}{\epsilon} \;=\; \frac{1}{\epsilon S(y)} \int_y^{y+\epsilon} f(t)\, dt \;\longrightarrow\; \frac{f(y)}{S(y)}\,, \qquad \epsilon \searrow 0$$

Thus

$$\mu(y) \;=\; \frac{f(y)}{S(y)} \;=\; \frac{-S'(y)}{S(y)} \;=\; -\frac{d}{dy}\ln(S(y))$$

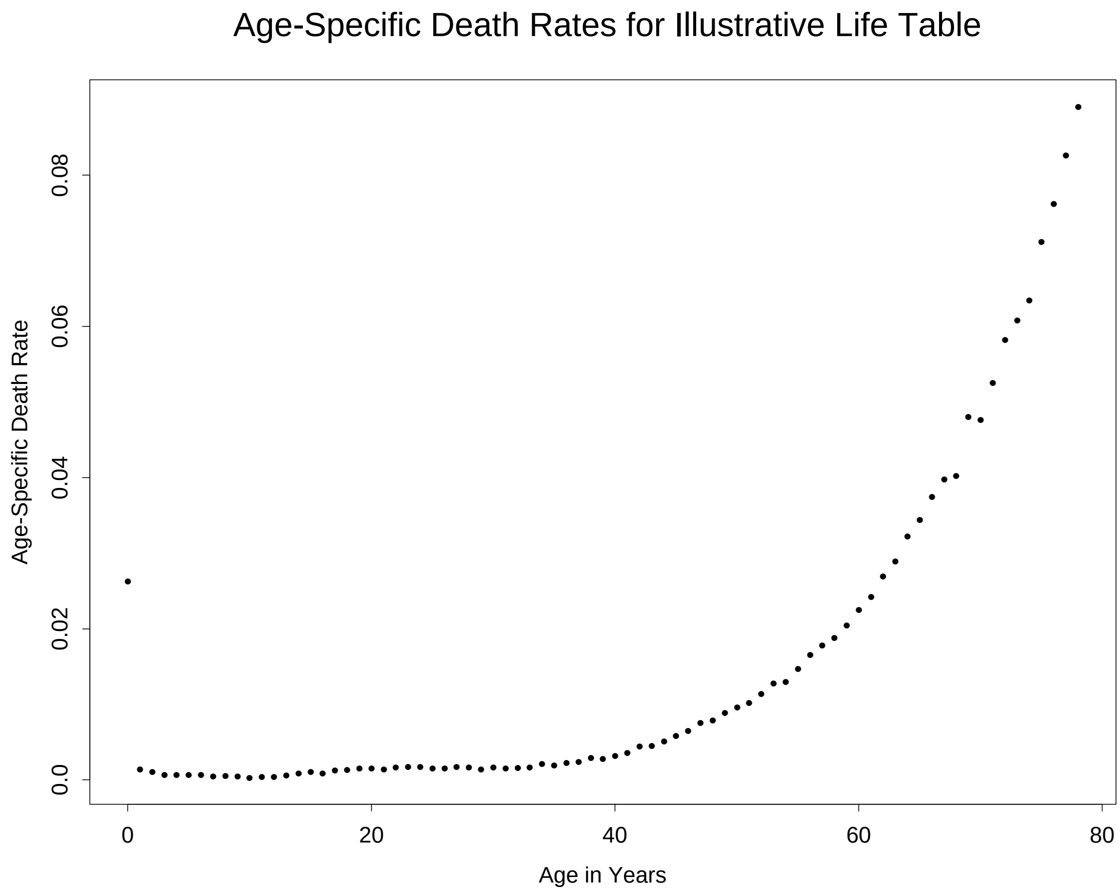

Figure 2.1: Plot of age-specific death-rates q_x versus x, for the simulated illustrative life table given in Table 1.1, page 4.

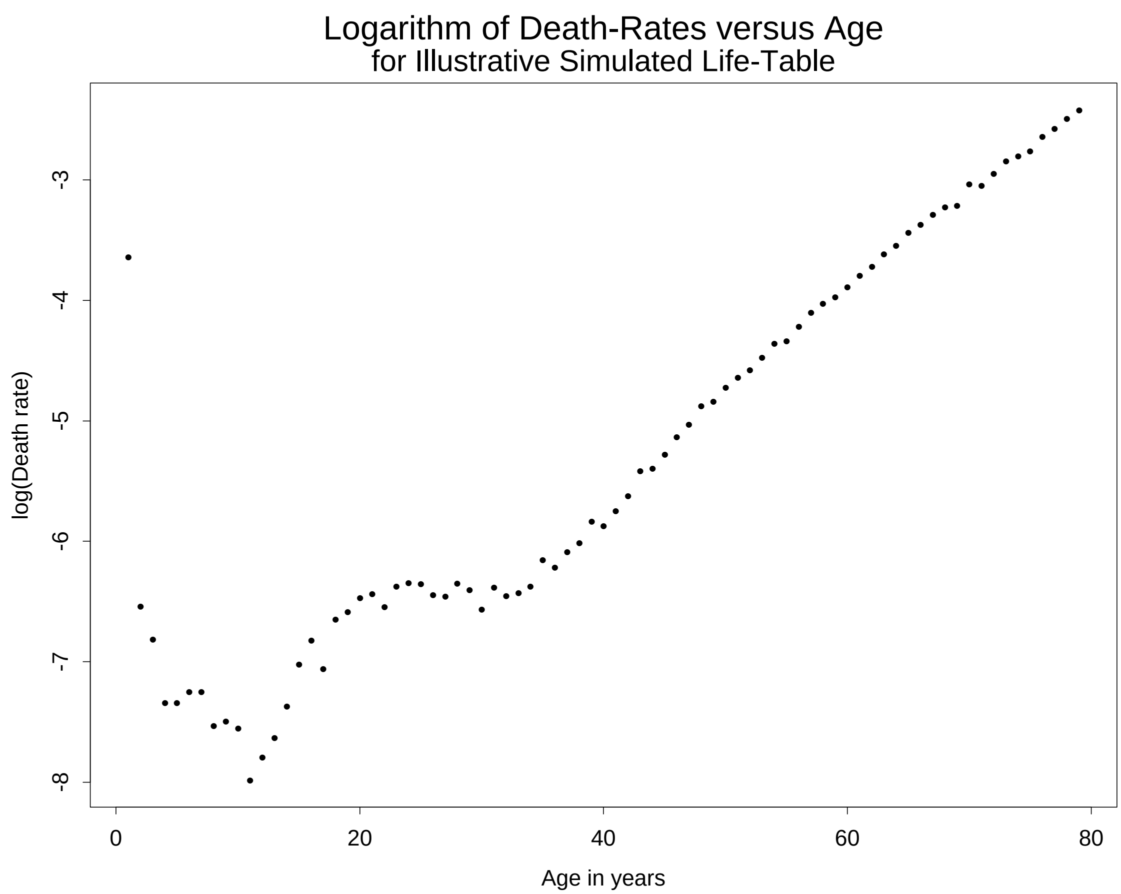

Figure 2.2: Plot of logarithm $\log(q_x)$ of age-specific death-rates as a function of age x, for the simulated illustrative life table given in Table 1.1, page 4. The rates whose logarithms are plotted here are the same ones shown in Figure 2.1.

where the chain rule for differentiation was used in the last step. Replacing y by t and integrating both sides of the last equation between 0 and y, we find

$$\int_0^y \mu(t)\,dt \;=\; \Big(-\ln(S(t))\Big)_0^y \;=\; -\ln(S(y))$$

since $S(0) = 1$. Similarly,

$$\int_y^{y+t} \mu(s)\,ds \;=\; \ln S(y) - \ln S(y+t)$$

Now exponentiate to obtain the useful formulas

$$S(y) = \exp\Big\{ -\int_0^y \mu(t)\,dt \Big\} \;, \qquad {}_tp_y = \frac{S(y+t)}{S(y)} = \exp\Big\{ -\int_y^{y+t} \mu(s)\,ds \Big\}$$

Examples:

(i) If $S(t) = (\omega - t)/\omega$ for $0 \le t \le \omega$ (the *uniform failure distribution* on $[0, \omega]$), then $\mu(t) = (\omega - t)^{-1}$. Note that this hazard function increases to ∞ as t increases to ω.

(ii) If $S(t) = e^{-\mu t}$ for $t \ge 0$ (the *exponential failure distribution* on $[0, \infty)$), then $\mu(t) = \mu$ is constant.

(iii) If $S(t) = \exp(-\lambda t^\gamma)$ for $t \ge 0$, then mortality follows the *Weibull life distribution model* with *shape parameter* $\gamma > 0$ and *scale parameter* λ. The force of mortality takes the form

$$\mu(t) \;=\; \lambda\,\gamma\,t^{\gamma-1}$$

This model is very popular in engineering reliability. It has the flexibility that by choice of the shape parameter γ one can have

(a) $(\gamma > 1)$ failure rate increasing as a function of x

(b) $(\gamma = 1)$ constant failure rate (exponential model), or

(c) $(0 < \gamma < 1)$ decreasing failure rate.

But what one cannot have, in the examples considered so far, is a force-of-mortality function which decreases on part of the time-axis and increases elsewhere.

(iv) Two other models for positive random variables which are popular in various statistical applications are the **Gamma**, with

$$S(y) \;=\; \int_y^\infty \beta^\alpha \, t^{\alpha-1} \, e^{-\beta t} \, dt \,/ \int_0^\infty z^{\alpha-1} \, e^{-z} \, dz \;, \qquad \alpha, \, \beta > 0$$

and the **Lognormal**, with

$$S(y) \;=\; 1 - \Phi\Big(\frac{\ln y - m}{\sigma} \Big) \quad , \qquad m \;\; \text{real}, \;\; \sigma > 0$$

where

$$\Phi(z) \equiv \frac{1}{2} + \int_0^z e^{-u^2/2} \, \frac{du}{\sqrt{2\pi}}$$

is called the *standard normal distribution function*. In the Gamma case, the expected lifetime is α/β, while in the Lognormal, the expectation is $\exp(m + \sigma^2/2)$. Neither of these last two examples has a convenient or interpretable force-of-mortality function.

Increasing force of mortality intuitively corresponds to aging, where the causes of death operate with greater intensity or effect at greater ages. Constant force of mortality, which is easily seen from the formula $S(y) = \exp(-\int_0^y \mu(t)\, dt)$ to be equivalent to exponential failure distribution, would occur if mortality arose only from pure accidents unrelated to age. Decreasing force of mortality, which really does occur in certain situations, reflects what engineers call "burn-in", where after a period of initial failures due to loose connections and factory defects the nondefective devices emerge and exhibit high reliability for a while. The decreasing force of mortality reflects the fact that the devices known to have functioned properly for a short while are known to be correctly assembled and are therefore highly likely to have a standard length of operating lifetime. In human life tables, infant mortality corresponds to burn-in: risks of death for babies decrease markedly after the one-year period within which the most severe congenital defects and diseases of infancy manifest themselves. Of course, human life tables also exhibit an aging effect at high ages, since the high-mortality diseases like heart disease and cancer strike with greatest effect at higher ages. Between infancy and late middle age, at least in western countries, hazard rates are relatively flat. This pattern of initial decrease, flat middle, and final increase of the force-of-mortality, seen clearly in Figure 2.1, is called a *bathtub shape* and requires new survival models.

As shown above, the failure models in common statistical and reliability usage *either* have increasing force of mortality functions *or* decreasing force of mortality, but not both. Actuaries have developed an analytical model which is somewhat more realistic than the preceding examples for human mortalty at ages beyond childhood. While the standard form of this model does not accommodate a bathtub shape for death-rates, a simple modification of it does.

Example (v). (*Gompertz-Makeham* forms of the force of mortality). Suppose that $\mu(y)$ is defined directly to have the form $A + B\, c^y$. (The Bc^y term was proposed by Gompertz, the additive constant A by Makeham. Thus the *Gompertz* force-of-mortality model is the special case with $A = 0$, or $\mu(y) = Bc^y$.) By choice of the parameter c as being respectively greater than or less than 1, one can arrange that the force-of-mortality curve either be increasing or decreasing. Roughly realistic values of c for human mortality will be only slightly greater than 1: if the Gompertz (non-constant) term in force-of-mortality were for example to quintuple in 20 years, then $c \approx 5^{1/20} = 1.084$, which may be a reasonable value except for very advanced ages. (Compare the comments made in connection with Figures 2.1 and 2.2: for middle and higher ages in the simulated illustrative life table of Table 1.1, which corresponds roughly to US male mortality of around 1960, the figure of c was found to be roughly 1.09.) Note that in any case the Gompertz-Makeham force of mortality is strictly convex (i.e., has strictly positive second derivative) when $B > 0$ and $c \neq 1$. The Gompertz-Makeham family could be enriched still further, with further benefits of realism, by adding a linear term Dy. If $D < -B\ln(c)$, with $0 < A < B$, $c > 1$, then it is easy to check that

$$\mu(y) \;=\; A + B\, c^y + Dy$$

has a bathtub shape, initially decreasing and later increasing.

Figures 2.3 and 2.4 display the shapes of force-of-mortality functions (iii)-(v) for various parameter combinations chosen in such a way that the expected lifetime is 75 years. This restriction has the effect of reducing the number of free parameters in each family of examples by 1. One can see from these pictures that the Gamma and Weibull families contain many very similar shapes for force-of-mortality curves, but that the lognormal and Makeham families are quite different.

Figure 2.5 shows survival curves from several analytical models plotted on the same axes as the 1959 US male life-table data from which Table 1.1 was simulated. The previous discussion about bathtub-shaped force of mortality functions should have made it clear that none of the analytical models presented could give a good fit at all ages, but the Figure indicates the rather good fit which can be achieved to realistic life-table data at ages 40 and above. The models fitted all assumed that $S(40) = 0.925$ and that for lives aged 40, $T - 40$ followed the indicated analytical form. Parameters for all models were determined from the requirements of median age 72 at death (equal by definition to the value t_m for which $S(t_m) = 0.5$) and probability 0.04 of surviving to age 90. Thus, all four plotted survival curves have been designed to pass through the three points $(40, 0.925)$, $(72,\ 0.5)$, $(90,\ 0.04)$. Of the four fitted curves, clearly the Gompertz agrees most closely with the plotted points for 1959 US male mortality. The Gompertz curve has parameters $B = 0.00346$, $c = 1.0918$, the latter of which is close to the value 1.0885 used in formula (2.10) to extrapolate the 1959 life-table death-rates to older ages.

2.2.1 Comparison of Forces of Mortality

What does it mean to say that one lifetime, with associated survival function $S_1(t)$, has hazard (i.e. force of mortality) $\mu_1(t)$ which is a constant multiple κ at all ages of the force of mortality $\mu_2(t)$ for a second lifetime with survival function $S_2(t)$? It means that the cumulative hazard functions are *proportional*, i.e.,

$$-\ln S_1(t) \;=\; \int_0^t \mu_1(x)dx \;=\; \int_0^t \kappa\,\mu_2(x)dx \;=\; \kappa(-\ln S_2(t))$$

and therefore that

$$S_1(t) \;=\; (S_2(t))^\kappa \quad , \qquad \text{all} \quad t \ge 0$$

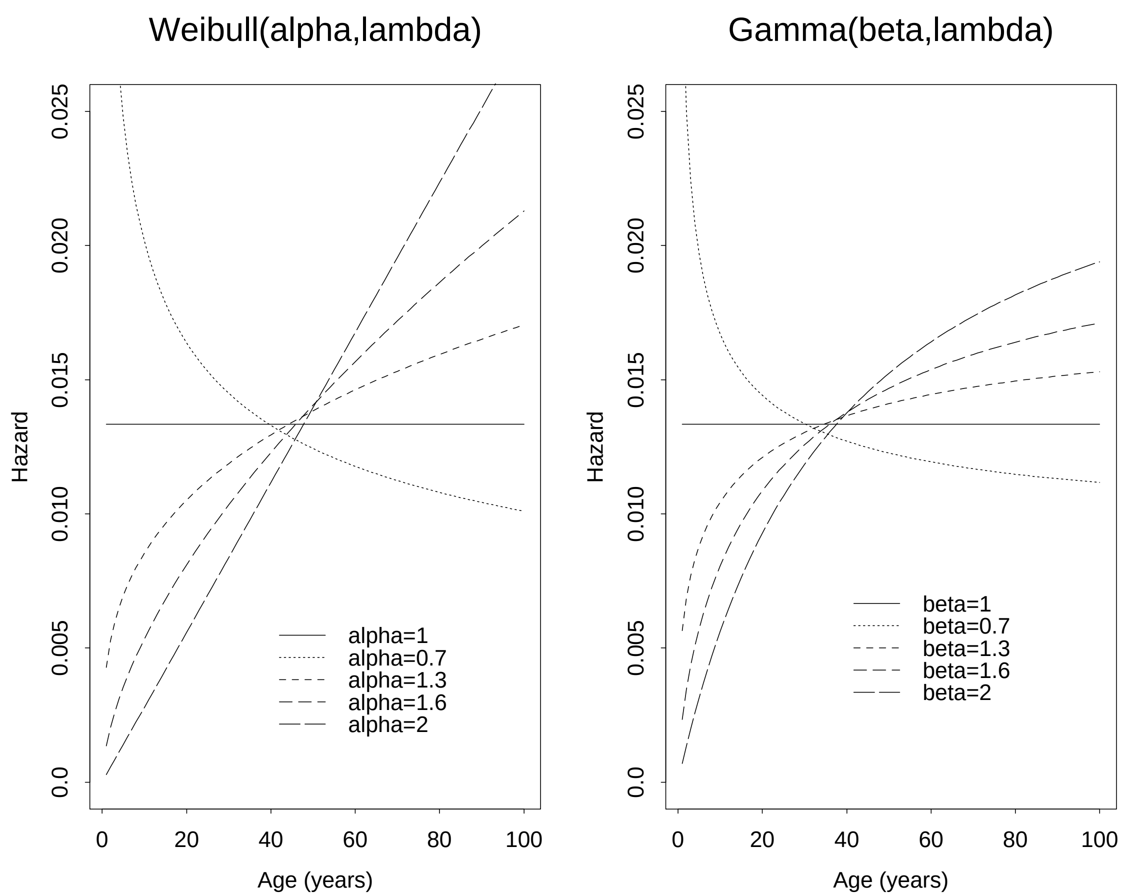

Figure 2.3: Force of Mortality Functions for Weibull and Gamma Probability Densities. In each case, the parameters are fixed in such a way that the expected survival time is 75 years.

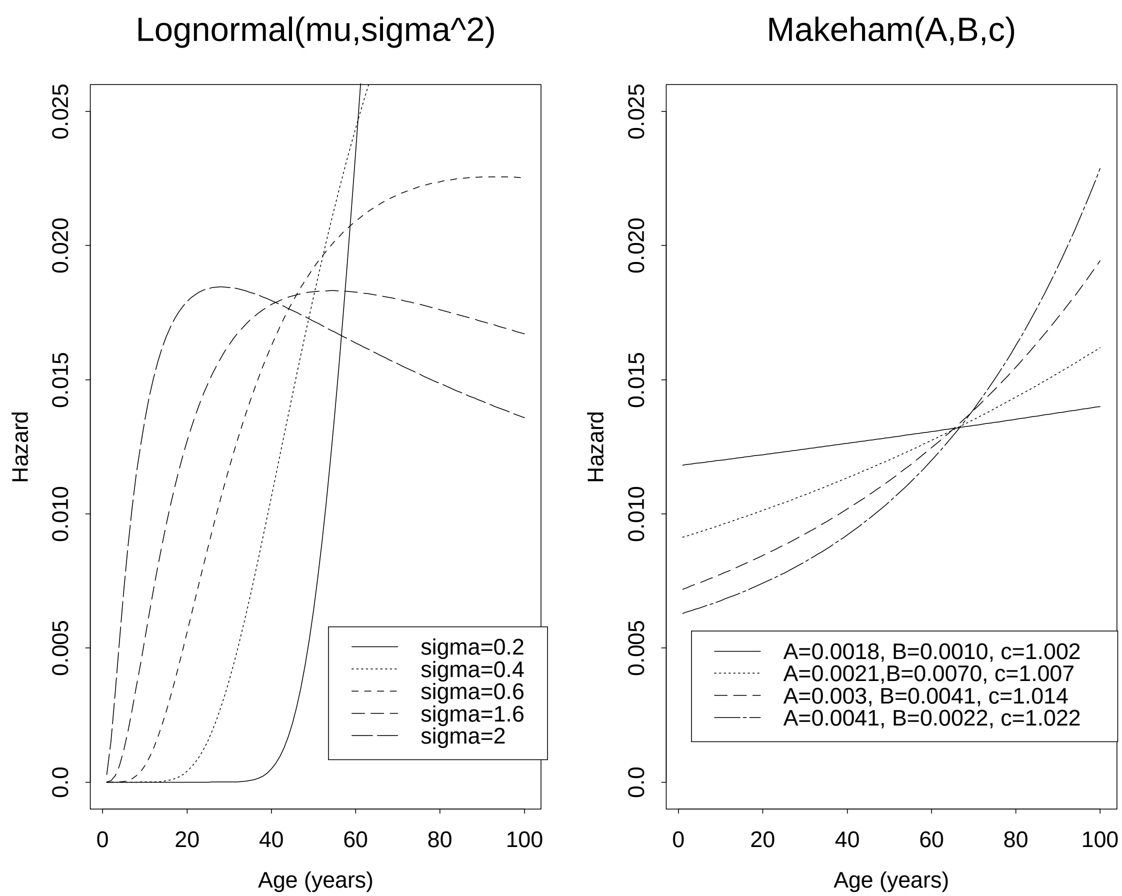

Figure 2.4: Force of Mortality Functions for Lognormal and Makeham Densities. In each case, the parameters are fixed in such a way that the expected survival time is 75 years.

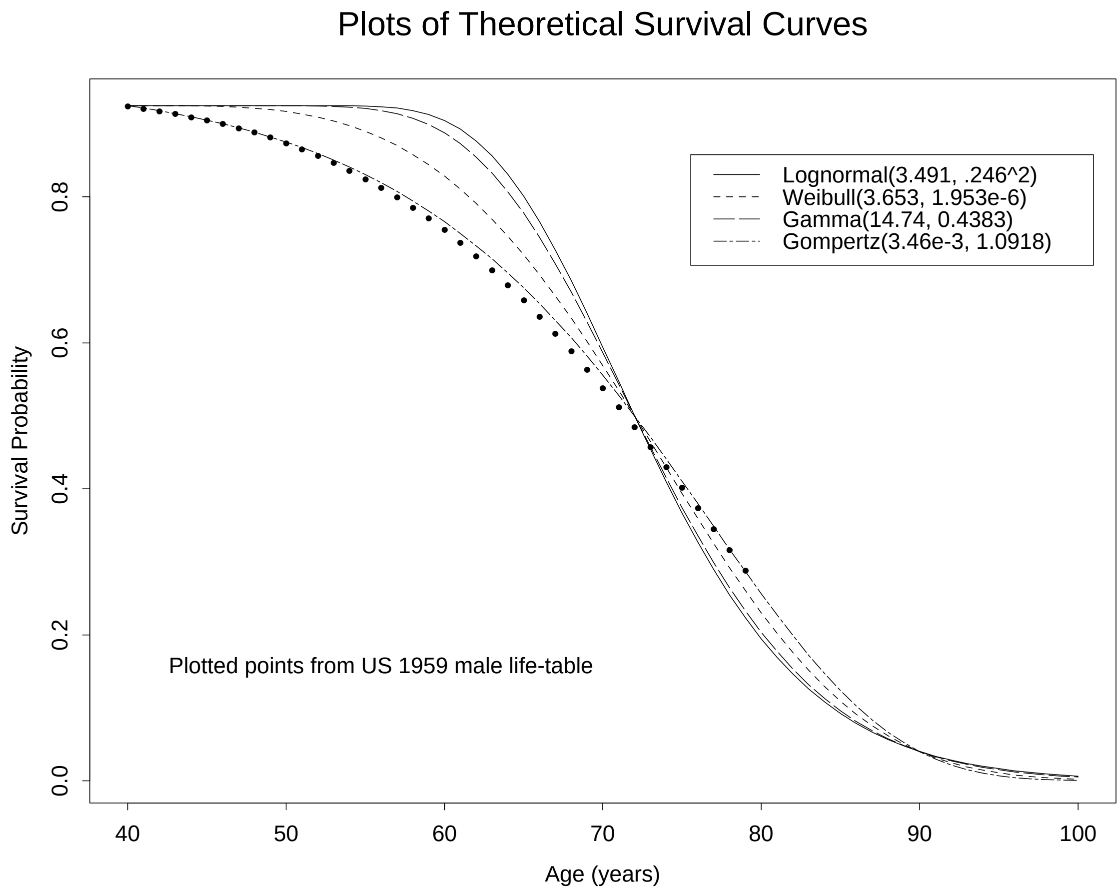

Figure 2.5: Theoretical survival curves, for ages 40 and above, plotted as lines for comparison with 1959 US male life-table survival probabilities plotted as points. The four analytical survival curves — Lognormal, Weibull, Gamma, and Gompertz — are taken as models for age-at-death minus 40, so if $S_{\rm theor}(t)$ denotes the theoretical survival curve with indicated parameters, the plotted curve is $(t,\ 0.925 \cdot S_{\rm theor}(t-40))$. The parameters of each analytical model were determined so that the plotted probabilities would be $0.925,\ 0.5,\ 0.04$ respectively at $t = 40,\ 72,\ 90$.

This remark is of especial interest in biostatistics and epidemiology when the factor κ is allowed to depend (e.g., by a regression model $\ln(\kappa) = \beta \cdot Z$) on other measured variables (*covariates*) Z. This model is called the *(Cox) Proportional-Hazards model* and is treated at length in books on survival data analysis (Cox and Oakes 1984, Kalbfleisch and Prentice 1980) or biostatistics (Lee 1992).

Example. Consider a setting in which there are four subpopulations of the general population, categorized by the four combinations of values of two binary covariates $Z_1, Z_2 = 0, 1$. Suppose that these four combinations have respective conditional probabilities for lives aged x (or relative frequencies in the general population aged x)

$$P_x(Z_1 = Z_2 = 0) = 0.15 \quad , \quad P_x(Z_1 = 0, \, Z_2 = 1) = 0.2$$
$$P_x(Z_1 = 1, \, Z_2 = 0) = 0.3 \quad , \quad P_x(Z_1 = Z_2 = 1) = 0.35$$

and that for a life aged x and all $t > 0$,

$$Pr(T \geq x + t \,|\, T \geq x, \; Z_1 = z_1, \; Z_2 = z_2) \;=\; \exp(-2.5 \, e^{0.7 z_1 - .8 z_2} \, t^2 / 20000)$$

It can be seen from the conditional survival function just displayed that the forces of mortality at ages greater than x are

$$\mu(x + t) \;=\; (2.5 \, e^{0.7 z_1 - .8 z_2}) \, t / 10000$$

so that the force of mortality at all ages is multiplied by $e^{0.7} = 2.0138$ for individuals with $Z_1 = 1$ versus those with $Z_1 = 0$, and is multiplied by $e^{-0.8} = 0.4493$ for those with $Z_2 = 1$ versus those with $Z_2 = 0$. The effect on age-specific death-rates is approximately the same. Direct calculation shows for example that the ratio of age-specific death rate at age x+20 for individuals in the group with $(Z_1 = 1, Z_2 = 0)$ versus those in the group with $(Z_1 = 0, Z_2 = 0)$ is not precisely $e^{0.7} = 2.014$, but rather

$$\frac{1 - \exp(-2.5 e^{0.7}((21^2 - 20^2)/20000)}{1 - \exp(-2.5((21^2 - 20^2)/20000)} \;=\; 2.0085$$

Various calculations, related to the fractions of the surviving population at various ages in each of the four population subgroups, can be performed easily . For example, to find

$$Pr(Z_1 = 0, \; Z_2 = 1 \,|\, T \geq x + 30)$$

we proceed in several steps (which correspond to an application of Bayes' rule, *viz.* Hogg and Tanis 1997, sec. 2.5, or Devore 2007):

$$Pr(T \geq x{+}30,\, Z_1 = 0\, Z_2 = 1 |\, T \geq x) \;=\; 0.2\, \exp(-2.5 e^{-0.8} \frac{30^2}{20000}) \;=\; 0.1901$$

and similarly

$$Pr(T \geq x + 30 \,|\, T \geq x) \;=\; 0.15\, \exp(-2.5(30^2/20000)) \;+\; 0.1901 \;+$$

$$+\; 0.3\, \exp(-2.5 * e^{0.7} \frac{30^2}{20000}) \;+\; 0.35\, \exp(-2.5 e^{0.7-0.8} \frac{30^2}{20000}) \;=\; 0.8795$$

Thus, by definition of conditional probabilities (restricted to the cohort of lives aged x), taking ratios of the last two displayed quantities yields

$$Pr(Z_1 = 0,\, Z_2 = 1 \,|\, T \geq x + 30) \;=\; \frac{0.1901}{0.8795} \;=\; 0.2162$$

□

In biostatistics and epidemiology, the measured variables $\underline{Z} = (Z_1, \ldots, Z_p)$ recorded for each individual in a survival study might be: indicator of a specific disease or diagnostic condition (e.g., diabetes, high blood pressure, specific electrocardiogram anomaly), quantitative measurement of a risk-factor (dietary cholesterol, percent caloric intake from fat, relative weight-to-height index, or exposure to a toxic chemical), or indicator of type of treatment or intervention. In these fields, the objective of such detailed models of covariate effects on survival can be: to correct for incidental individual differences in assessing the effectiveness of a treatment; to create a prognostic index for use in diagnosis and choice of treatment; or to ascertain the possible risks and benefits for health and survival from various sorts of life-style interventions. The multiplicative effects of various risk-factors on age-specific death rates are often highlighted in the news media.

In an insurance setting, categorical variables for risky life-styles, occupations, or exposures might be used in *risk-rating*, i.e., in individualizing insurance premiums. While risk-rating is used routinely in casualty and property insurance underwriting, for example by increasing premiums in response to recent claims or by taking location into account, it can be politically sensitive in a life-insurance and pension context. In particular, while differences in mortality by gender and to some extgent by family health history can be used in calculating insurance and annuity premiums, as can certain life-style factors like smoking, it is currently illegal to use racial differences and differences based on genetic testing in this way.

All life insurers must be conscious of the extent to which their policyholders as a group differ from the general population with respect to mortality. Insurers can collect special mortality tables on special groups, such as employee groups or voluntary organizations, and regression-type models like the Cox proportional-hazards model may be useful in quantifying group mortality differences when the special-group mortality tables are not based upon large enough cohorts for long enough times to be fully reliable. See Chapter 6, Section 4, for discussion about the modification of insurance premiums for select groups.

2.3 *Exercise Set 2*

(1). The sum of the present value of $\$1$ paid at the end of n years and $\$1$ paid at the end of $2n$ years is $\$1$. Find $(1+r)^{2n}$, where $r =$ annual interest rate, compounded annually.

(2). Suppose that an individual aged 20 has random lifetime Z with continuous density function

$$f_(t) \;=\; \frac{1}{360}\left(1 + \frac{t}{10}\right)\;, \qquad \text{for} \qquad 20 \le t \le 80$$

and 0 for other values of t. (*The random variable Z in this problem is a particular type of age-at-death variable* T *conditioned on being* ≥ 20.)

(a) If this individual has a contract with your company that you must pay his heirs $10^6 \cdot (1.4 - Z/50)$ dollars at the exact date of his death if this occurs between ages 20 and 70, then what is the expected payment ?

(b) If the value of the death-payment described in (a) should properly be discounted by the factor $\exp(-0.08\cdot(Z-20))$, i.e. by the nominal interest rate of $e^{0.08}-1$ per year) to calculate the present value of the payment, then what is the expected present value of the payment under the insurance contract ?

(3). Suppose that a continuous random variable T has hazard rate function (= force of mortality)

$$h(t) \;=\; 10^{-3} \cdot \Big[7.0 - 0.5t + 2e^{t/20} \Big] \;, \qquad t > 0$$

This **is** a legitimate hazard rate of Gompertz-Makeham type since its minimum, which occurs at $t = 20\ln(5)$, is $(17 - 10\ln(5)) \cdot 10^{-4} = 9.1 \cdot 10^{-5} > 0$.

(a) Construct a cohort life-table with $h(t)$ as "force of mortality", based on integer ages up to 70 and cohort-size (= *radix*) $l_0 = 10^5$. (Give selected numerical entries, preferably calculated by means of a little computer program. If you do the arithmetic using hand-calculators and/or tables, give only the values for ages which are multiples of 10.)

(b) Find the probability that the random variable T exceeds 30, given that it exceeds 3. **Hint:** find a closed-form formula for $S(t) = P(T \geq t)$.

(4). Do the Mortgage-Refinancing exercise given as Exercise 2.B in the Illustration on mortgage refinancing at the end of Section 2.1.

(5). (a) The mortality pattern of a certain population may be described as follows: out of every 98 lives born together, one dies annually until there are no survivors. Find a simple function that can be used as $S(x)$ for this population, and find the probability that a life aged 30 will survive to attain age 35.

(b) Suppose that for x between ages 12 and 40 in a certain population, 10% of the lives aged x die before reaching age $x{+}1$. Find a simple function that can be used as $S(x)$ for this population, and find the probability that a life aged 30 will survive to attain age 35.

(6). Suppose that a survival distribution (i.e., survival function based on a cohort life table) has the property that ${}_1p_x = \gamma \cdot (\gamma^2)^x$ for some fixed γ between 0 and 1, for every real ≥ 0. What does this imply about S(x) ? (*Give as much information about S as you can.*)

(7). If the instantaneous interest rate is $r(t) = 0.01 \cdot t$ for $0 \leq t \leq 3$, then find the equivalent single *effective rate of interest* for money invested at interest throughout the interval $0 \leq t \leq 3$.

(8). Find the accumulated value of \$100 at the end of 15 years if the nominal interest rate compounded quarterly (i.e., $i^{(4)}$) is 8% for the first 5 years, if the effective rate of discount is 7% for the second 5 year interval (i.e. the interval ranging from time 5 to 10), and if the nominal rate of discount compounded semiannually ($m = 2$) is 6% for the third 5 year interval.

(9). Suppose that you borrow \$1000 for 3 years at 6% effective rate, to be repaid in level payments every six months (twice yearly).

(a) Find the level payment amount P.

(b) What is the present value of the payments you will make if you skip the 2nd and 4th payments ? (You may express your answer in terms of P.)

(10). A survival function has the form $S(t) = \max(0, \frac{c-t}{c+t})$. If a mortality table is derived from this survivalfunction with a radix l_0 of 100,000 at age 0, and if $l_{35} = 44,000$:

(i) What is the terminal age of the table ?

(ii) What is the probability of surviving from birth to age 60 ?

(iii) What is the probability of a person at exact age 10 dying between exact ages 30 and 45 ?

(11). A separate life table has been constructed for each calendar year of birth, Y, beginning with $Y = 1950$. The mortality functions for the various tables are denoted by the appropriate superscript ^{Y}. For each Y and for all ages t

$$\mu_Y(t) \;=\; A \cdot k(Y) \,+\, B\,c^t \quad , \qquad p_t^{Y+1} \;=\; (1+r)\,p_t^Y$$

where k is a function of Y alone and A, B, r are constants (with $r > 0$). If $k(1950) = 1$, then derive a general expression for $k(Y)$.

(12). A standard mortality table follows Makeham's Law with force of mortality

$$\mu(t) \;=\; A \,+\, B\,c^t \qquad \text{at all ages } t$$

A separate, higher-risk mortality table also follows Makeham's Law with force of mortality

$$\mu^*(t) \;=\; A^* \,+\, B^*\, c^t \qquad \text{at all ages } t$$

with the same constant c. If for all starting ages the probability of surviving 6 years according to the higher-risk table is equal to the probability of surviving 9 years according to the standard table, then express each of A^* and B^* in terms of A, B, c.

(13). A homeowner borrows $100,000 at effective annual rate 7% from a bank, agreeing to repay by 30 equal yearly payments beginning one year from the time of the loan.

(a) How much is each payment ?

(b) Suppose that after paying the first 3 yearly payments, the homeowner misses the next two (i.e. pays nothing on the 4^{th} and 5^{th} anniversaries of the loan). Find the outstanding balance at the 6^{th} anniversary of the loan, figured at 7%). This is the amount which, if paid as a lump sum at time 6, has present value together with the amounts already paid of $100,000 at time 0.

(14). A deposit of 300 is made into a fund at time $t = 0$. The fund pays interest for the first three years at a nominal monthly rate $d^{(12)}$ of discount. From $t = 3$ to $t = 7$, interest is credited according to the force of interest $\delta_t = 1/(3t + 3)$. As of time $t = 7$, the accumulated value of the fund is 574. Calculate $d^{(12)}$.

(15). Calculate the price at which you would sell a $10,000 30-year coupon bond with nominal 6% semi-annual coupon ($n = 30,\ m - 2,\ i^{(m)} = 0.06$), 15 years after issue, if for the next 15 years, the effective interest rate for valuation is $i_{\mathit{eff}} = 0.07$.

(16). A 6% 'zero-coupon' 30-year bond was issued exactly 15 years ago for a face amount of $10,000. This bond contractually entitles the bearer to receive 30 years after the issue date the amount accumulated at $i = i_{\mathit{eff}} = 0.06$ on the face amount. Calculate the fair price at which you would sell this zero-coupon bond, if for the next 15 years, the effective interest rate will be $i'_{\mathit{eff}} = 0.07$.

(17). Suppose that the borrower of a \$100,000 30-year loan with half-yearly payments ($m = 2$) start in six months from the time of borrowing, and with nominal interest rate $i^{(2)} = .04$, has made all payments except for two that he skipped, the 23'rd and 56'th payments. What lump-sum payment did the borrower have to make at the end of 30 years, in addition to his final payment, in order to pay off the loan completely ?

(18). In Problem (17), the missed payments did not result in any additional fees or charges, only in continuing accrued interest on the amounts of the missed payments. Suppose that the missed payments in (17) actually result in late fees of \$200 each of which is added to the balance at the time(s) of missed payments. Now answer the same question as in (17) about the amount of the final lump-sum payment required.

(19). One of the curves plotted in the first part of Figure 2.3 is the lognormal(m, σ^2) hazard intensity where $\sigma = 1.3$ is fixed and where m was determined from it by the requirement that the expectation of survival time was 75 years. Now find the value m associated with $\sigma = 1.3$ if the **median** survival time is fixed at 72 years, and find the force of mortality for this lognormal at 65 years.

(20). A small city issues a bond for twenty million dollars for ten years at 5% nominal half-yearly interest ($m = 2$) and creates a sinking fund into which it will make twice-yearly deposits ($m' = 2$), from its tax revenue.

(a) Find the amount of the level payment the city must make into the sinking fund if the interest it earns on that fund is 5%, and find the **reserve**, or accumulated balance, in the sinking fund after 6 years.

(b) Answer the same questions if the city knows it can earn 6% on the money it deposits into its sinking fund.

2.4 Worked Examples

Example 1. How large must a half-yearly payment be in order that the stream of payments starting immediately be equivalent (in present value terms) at 6% interest to a lump-sum payment of \$5000, if the payment-stream is to last (a) 10 years, (b) 20 years, or (c) forever ?

If the payment size is P, then the balance equation is

$$5000 \;=\; 2\,P \cdot \ddot{a}^{(2)}_{\overline{n}|} \;=\; 2\,P\,(1-1.06^{-n})/d^{(2)}$$

Since $d^{(2)} = 2(1-1/\sqrt{1.06}) = 2 \cdot 0.02871$, the result is

$$P \;=\; (5000 \cdot 0.02871)/(1-1.06^{-n}) \;=\; 143.57/(1-1.06^{-n})$$

So the answer to part (c), in which $n = \infty$, is \$143.57. For parts (a) and (b), respectively with $n = 10$ and 20, the answers are \$325.11, \$208.62.

Example 2. Assume m *is divisible by* 2. *Express in two different ways the present value of the perpetuity of payments* $1/m$ *at times* $1/m, 3/m, 5/m, \ldots$, *and use either one to give a simple formula.*

This example illustrates the general methods enunciated at the beginning of Section 2.1. Observe first of all that the specified payment-stream is exactly the same as a stream of payments of $1/m$ at times $0, 2/m, 4/m, \ldots$ forever, deferred by a time $1/m$. Since this payment-stream starting at 0 is exactly one-half that of the stream whose present value is $\ddot{a}^{(m/2)}_{\overline{\infty}|}$, a first present value expression is

$$v^{1/m}\,(1/2)\,\ddot{a}^{(m/2)}_{\overline{\infty}|}$$

A second way of looking at the payment-stream at odd multiples of $1/m$ is as the perpetuity-due payment stream ($1/m$ at times k/m for all $k \ge 0$) **minus** the payment-stream discussed above of amounts $1/m$ at times $2k/m$ for all nonnegative integers k. Thus the present value has the second expression

$$\ddot{a}^{(m)}_{\overline{\infty}|} \;-\; (1/2)\,\ddot{a}^{(m/2)}_{\overline{\infty}|}$$

Equating the two expressions allows us to conclude that

$$(1/2)\,\ddot{a}^{(m/2)}_{\overline{\infty}|} \;=\; \ddot{a}^{(m)}_{\overline{\infty}|} \Big/ (1 + v^{1/m})$$

Substituting this into the first of the displayed present-value expressions, and using the simple expression $1/d^{(m)}$ for the present value of the perpetuity-due, shows that that the present value requested in the Example is

$$\frac{1}{d^{(m)}} \cdot \frac{v^{1/m}}{1 + v^{1/m}} \;=\; \frac{1}{d^{(m)}\,(v^{-1/m} + 1)} \;=\; \frac{1}{d^{(m)}\,(2 + i^{(m)}/m)}$$

and this answer is valid whether or not m is even.

Example 3. Suppose that you are negotiating a car-loan of $10,000. *Would you rather have an interest rate of* 4% *for 4 years,* 3% *for 3 years,* 2% *for 2 years, or a cash discount of* $500 *? Show how the answer depends upon the interest rate with respect to which you calculate present values, and give numerical answers for present values calculated at 6% and 8%. Assume that all loans have monthly payments paid at the beginning of the month (e.g., the 4 year loan has 48 monthly payments paid at time 0 and at the ends of 47 succeeding months).*

The monthly payments for an n-year loan at interest-rate i is $10000/(12\,\ddot{a}^{(12)}_{\overline{n}|}) = (10000/12)\,d^{(12)}/(1-(1+i)^{-n})$. Therefore, the present value at interest-rate r of the n-year monthly payment-stream is

$$10000 \cdot \frac{1-(1+i)^{-1/12}}{1-(1+r)^{-1/12}} \cdot \frac{1-(1+r)^{-n}}{1-(1+i)^{-n}}$$

Using interest-rate $r = 0.06$, the present values are calculated as follows:

For 4-year 4% loan: $9645.77

For 3-year 3% loan: $9599.02

For 2-year 2% loan: $9642.89

so that the most attractive option is the cash discount (which would make the present value of the debt owed to be $9500). Next, using interest-rate $r = 0.08$, the present values of the various options are:

For 4-year 4% loan: $9314.72

For 3-year 3% loan: $9349.73

For 2-year 2% loan: $9475.68

so that the most attractive option in this case is the 4-year loan. (The cash discount is now the least attractive option.)

Example 4. Suppose that the force of mortality $\mu(y)$ *is specified for exact ages* y *ranging from* 5 *to* 55 *as*

$$\mu(y) \;=\; 10^{-4} \cdot (20 - 0.5|30-y|)$$

Then find analytical expressions for the survival probabilities $S(y)$ *for exact* *ages* y *in the same range, and for the (one-year) death-rates* q_x *for integer ages* $x = 5, \ldots, 54$, *assuming that* $S(5) = 0.97$.

The key formulas connecting force of mortality and survival function are here applied separately on the age-intervals $[5, 30]$ and $[30, 55]$, as follows. First for $5 \le y \le 30$,

$$S(y) = S(5) \exp(-\int_5^y \mu(z)\, dz) = 0.97 \exp\Big(-10^{-4}(5(y-5)+0.25(y^2-25))\Big)$$

so that $S(30) = 0.97\, e^{-0.034375} = 0.93722$, and for $30 \le y \le 55$

$$S(y) = S(30) \exp\Big(- 10^{-4} \int_{30}^y (20 + 0.5(30 - z))\, dz\Big)$$

$$= 0.9372 \exp\Big(- .002(y - 30) + 2.5 \cdot 10^{-5} (y - 30)^2\Big)$$

The death-rates q_x therefore have two different analytical forms: first, in the case $x = 5, \ldots, 29$,

$$q_x = S(x + 1)/S(x) = \exp\Big(- 5 \cdot 10^{-5}\, (10.5 + x)\Big)$$

and second, in the case $x = 30, \ldots, 54$,

$$q_x = \exp\Big(- .002 + 2.5 \cdot 10^{-5} (2(x - 30) + 1)\Big)$$

2.5 Useful Formulas from Chapter 2

$$v \;=\; 1/(1+i)$$

p. 34

$$a^{(m)}_{\overline{n}|} \;=\; \frac{1-v^n}{i^{(m)}} \quad , \qquad \ddot{a}^{(m)}_{\overline{n}|} \;=\; \frac{1-v^n}{d^{(m)}}$$

pp. 35–35

$$a^{(m)}_{\overline{n}|} \;=\; v^{1/m}\,\ddot{a}^{(m)}_{\overline{n}|}$$

p. 35

$$\ddot{a}^{(\infty)}_{\overline{n}|} \;=\; a^{(\infty)}_{\overline{n}|} \;=\; \overline{a}_n \;=\; \frac{1-v^n}{\delta}$$

p. 35

$$a^{(m)}_{\overline{\infty}|} \;=\; \frac{1}{i^{(m)}} \quad , \qquad \ddot{a}^{(m)}_{\overline{\infty}|} \;=\; \frac{1}{d^{(m)}}$$

p. 36

$$(I^{(m)}\ddot{a})^{(m)}_{\overline{n}|} \;=\; \ddot{a}^{(m)}_{\overline{\infty}|}\left(\ddot{a}^{(m)}_{\overline{n}|} \;-\; n\,v^n\right)$$

p. 38

$$(D^{(m)}\ddot{a})^{(m)}_{\overline{n}|} \;=\; (n+\frac{1}{m})\,\ddot{a}^{(m)}_{\overline{n}|} \;-\; (I^{(m)}\ddot{a})^{(m)}_{\overline{n}|}$$

p. 38

n-yr m'thly Mortgage Paymt : $\dfrac{Loan\ Amt}{m\,\ddot{a}^{(m)}_{\overline{n}|}}$

p. 39

$$\text{n-yr Mortgage Bal. amt } \frac{k}{m} + \ : \qquad B_{n,k/m} \;=\; \frac{1 - v^{n-k/m}}{1 - v^n}$$

p. 42

$$_t p_y \;=\; \frac{S(y+t)}{S(y)} \;=\; \exp\left(- \int_0^t \mu(y+s)\, ds \right)$$

p. 48

$$_t q_y \;=\; 1 \;-\; _t p_y$$

p. 48

$$q_x \;=\; _1 q_x \;=\; \frac{d_x}{l_x} \quad , \qquad p_x \;=\; _1 p_x \;=\; 1 \;-\; q_x$$

p. 48

$$\mu(y+t) \;=\; \frac{f(y+t)}{S(y+t)} \;=\; -\frac{\partial}{\partial t} \ln S(y+t)$$

p. 49

$$S(y) \;=\; \exp(- \int_0^y \mu(t)\, dt)$$

p. 52

$$\text{Unif. Failure Dist.:} \qquad S(t) \;=\; \frac{\omega - t}{\omega} \;\; , \quad f(t) \;=\; \frac{1}{\omega} \;\; , \quad 0 \le t \le \omega$$

p. 52

$$\text{Expon. Dist.:} \quad S(t) \;=\; e^{-\mu t} \; , \quad f(t) \;=\; \mu e^{-\mu t} \; , \quad \mu(t) \;=\; \mu \; , \quad t > 0$$

p. 52

$$\text{Weibull. Dist.:} \quad S(t) \;=\; e^{-\lambda t^\gamma} \;\;, \quad \mu(t) \;=\; \lambda\gamma t^{\gamma-1} \;\;, \quad t > 0$$

p. 52

$$\text{Makeham:} \qquad \mu(t) \;=\; A + Bc^t \qquad , \quad t \ge 0$$

$$\text{Gompertz:} \qquad \mu(t) \;=\; Bc^t \qquad , \quad t \ge 0$$

$$S(t) \;=\; \exp\left(-At - \frac{B}{\ln c}\,(c^t - 1)\right)$$

p. 54

Chapter 3

More Probability Theory for Life Tables

This Chapter introduces several key ideas in Probability Theory which are essential for an understanding of the book's core actuarial topics in Chapter 4 and 5. The first of these ideas is that survival from one year to the next can be regarded for each member of a population as a coin-toss experiment, with survival probability p_x for a life aged x, independently of all other members of the population. This point of view also provides a convenient vehicle for conducting computer simulations of population survival experience for large or small life-table populations. Since the life-table summarizes outcomes on a large number of coin-toss experiments, we study next through limit theorems (law of large numbers and central limit theorem) the high degree of predictability of these outcomes at the population level. This predictability will be used in later chapters to justify consideration of expected present values of contractual payouts to describe an insurer's liability, so we prepare the ground by presenting background theory and rules of manipulation for expectations of discrete-valued random variables. Finally, we complete our probability background with further material on interpreting, approximating and calculating with probabilities and expectations using theoretical models of survival between successive years of age.

3.1 Binomial Variables and Limit Theorems

This Section develops basic machinery for the theory of random variables which count numbers of successes in large numbers of independent biased coin-tosses. The motivation is that in large life-table populations, the number l_{x+t} who survive t time-units after age x can be regarded as the number of successes or heads in a large number l_x of independent coin-toss trials corresponding to the further survival of each of the l_x lives aged x, which for each such life has probability ${}_tp_x$. The one preliminary mathematical result that the student is assumed to know is the **Binomial Theorem** stating that (for positive integers N and arbitrary real numbers x, y, z),

$$(1+x)^N = \sum_{k=0}^{N} \binom{N}{k} x^k \, , \qquad (y+z)^N = \sum_{k=0}^{N} \binom{N}{k} y^k \, z^{N-k}$$

Recall that the first of these assertions follows by equating the k^{th} derivatives of both sides at $x = 0$, where $k = 0, \ldots, N$. The second assertion follows immediately, in the nontrivial case when $z \neq 0$, by applying the first assertion with $x = y/z$ and multiplying both sides by z^N. This Theorem also has a direct combinatorial consequence. Consider the two-variable polynomial

$$(y+z)^N \;=\; (y+z)\cdot(y+z)\cdots(y+z) \qquad\qquad N \;\; \textit{factors}$$

expanded by making all of the different choices of y or z from each of the N factors $(y+z)$, multiplying each combination of choices out to get a monomial $y^j \, z^{N-j}$, and adding all of the monomials together. Each combined choice of y or z from the N factors $(y+z)$ can be represented as a sequence $(a_1, \ldots, a_n) \in \{0, 1\}^N$, where $a_i = 1$ would mean that y is chosen and $a_i = 0$ would mean that z is chosen in the i^{th} factor. Now this combinatorial fact is immediately deduced from the Binomial Theorem: since the coefficient $\binom{N}{k}$ is the total number of monomial terms $y^k \, z^{N-k}$ which are collected when $(y+z)^N$ is expanded as described, and since these monomial terms arise only from the combinations $(a_1, \ldots, a_N)$ of $\{y, z\}$ choices in which precisely k of the values a_j are 1's and the rest are 0's,

> The number of symbol-sequences $(a_1, \ldots, a_N) \in \{0, 1\}^N$ such that $\sum_{j=1}^{N} a_j = k$ is given, for each $k = 0, 1, \ldots, N$, by

$\binom{N}{k} = N(N-1)\cdots(N-k+1)/k!$. This number $\binom{N}{k}$, spoken as 'N choose k', therefore counts all of the ways of choosing k element subsets (the positions j from 1 to N where 1's occur) out of N objects.

The random experiment of interest in this Section consists of a number N of independent tosses of a coin, with probability p of coming up heads each time. Such coin-tossing experiments — independently replicated two-outcome experiments with probability p of one of the outcomes, designated 'success' — are called *Bernoulli*(p) trials. The space of possible heads-and-tails configurations, or *sample space* for this experiment, consists of the strings of N zeroes and ones, with each string $\underline{\mathbf{a}} = (a_1, \ldots, a_N) \in \{0,1\}^N$ being assigned probability $p^a\,(1-p)^{N-a}$, where $a \equiv \sum_{j=1}^N a_j$. Because of the finite additivity axiom of probabilities (saying that $\Pr(A \cup B) = \Pr(A) + \Pr(B)$ for disjoint events A, B), the rule by which probabilities are assigned to sets or *events* A of more than one string $\underline{\mathbf{a}} \in \{0,1\}^N$ is to add the probabilities of all individual strings $\underline{\mathbf{a}} \in A$. We are particularly interested in the event (denoted $[X = k]$) that precisely k of the coin-tosses are heads, i.e., in the subset $[X = k] \subset \{0,1\}^N$ consisting of all strings $\underline{\mathbf{a}}$ such that $\sum_{j=1}^N a_j = k$. Since each such string has the same probability $p^k\,(1-p)^{N-k}$, and since, according to the discussion following the Binomial Theorem above, there are $\binom{N}{k}$ such strings, the probability which is necessarily assigned to the event of k successes is

$$\Pr(\text{ k successes in N Bernoulli(p) trials }) = P(X = k) = \binom{N}{k}\,p^k\,(1-p)^{N-k}$$

By virtue of this result, the random variable X equal to the number of successes in N Bernoulli(p) trials, is said to have the **Binomial distribution** with **probability mass function** $p_X(k) = \binom{N}{k}\,p^k\,(1-p)^{N-k}$.

With the notion of Bernoulli trials and the binomial distribution in hand, we now begin to regard the ideal probabilities $S(x+t)/S(x)$ as true but unobservable probabilities ${}_tp_x = p$ with which each of the l_x lives aged x will survive to age $x + t$. Since the mechanisms which cause those lives to survive or die can ordinarily be assumed to be acting independently in a probabilistic sense, we can regard the number l_{x+t} of lives surviving to the (possibly fractional) age $x+t$ as a Binomial random variable with parameters $N = l_x$, $p = {}_tp_x$. From this point of view, *if derived from an actual cohort* *dataset of size equal to the radix*, the observed life-table counts l_x would be treated as *random data* which reflect but do not define the underlying probabilities ${}_xp_0 = S(x)$ of survival to age x. However, common sense and experience suggest that, when l_0 is large, and therefore the other life-counts l_x for moderate values x are also large, the observed ratios l_{x+t}/l_x should reliably be very close to the 'true' probability ${}_tp_x$. In other words, the ratio l_{x+t}/l_x is a *statistical estimator* of the unknown constant ${}_tp_x$. The good property, called *consistency*, of this estimator to be close with very large probability (based upon large life-table size) to the heads-probability it estimates, is established in the famous **Law of Large Numbers**. We state and prove the result here only in the setting of binomial random variables, sketching in Section 3.3 how it implies a more general result for finite-valued discrete random variables. A more precise quantitative inequality concerning binomial probabilities, a *Large Deviation Inequality* which is important in its own right but more difficult, is stated and proved in the Appendix to this Chapter, Section 3.9.

Theorem 3.1 (Coin-toss Law of Large Numbers) *Suppose that X is a Binomial(N,p) random variable, denoting the number of successes in N Bernoulli(p) trials.* **Law of Large Numbers**. *For arbitrarily small fixed $\delta,\, \epsilon > 0$, not depending upon N, the number N of Bernoulli trials can be chosen so large that*

$$\Pr\Big(|\, X/N - p \,| \;\geq\; \delta \Big) \;\leq\; \epsilon$$

Proof. Since the event $[\; |X/N - p| \geq \delta \;] = [\; |X - Np| \geq N\delta \;]$ is the union of the disjoint events $[X = k]$ for $|k - Np| \geq N\delta$, which in turn consist of all outcome-strings $(a_1, \ldots, a_N) \in \{0,1\}^N$ for which $\sum_{j=1}^N a_j = k$ with $|k - Np| \geq N\delta$, the subset of the binomial probability mass function values $p_X(k)$ with $|k - Np| \geq N\delta$ are summed to provide

$$\Pr(|X/N - p| \geq \delta) = \sum_{k:\, |k-Np| \geq N\delta} \Pr(X = k) = \sum_{k:\, |k-Np| \geq N\delta} \binom{N}{k} p^k \,(1-p)^{N-k}$$

This summation is term-by-term less than or equal to

$$\sum_{k:\, |k-Np| \geq N\delta} \binom{N}{k} p^k \,(1-p)^{N-k} \,\frac{(k - Np)^2}{(N\delta)^2} \;\leq\; \sum_{k=0}^{N} \binom{N}{k} p^k \,(1-p)^{N-k} \,\frac{(k - Np)^2}{(N\delta)^2} \tag{3.1}$$

where we have made the second some larger by including more nonnegative terms in it. However, direct summation shows

$$\sum_{k=0}^{N} k \binom{N}{k} p^k (1-p)^{N-k} = \sum_{k=1}^{N} kp \frac{N \cdot (N-1)!}{k(k-1)!(N-k)!} p^{k-1} (1-p)^{N-k}$$

which after replacing $k-1$ by l, becomes with the aid of the Binomial Theorem

$$= Np \sum_{l=0}^{N-1} \binom{N-1}{l} p^l (1-p)^{N-1-l} = Np$$

and similarly (now with $j = k-2$)

$$\sum_{k=0}^{N} k(k-1) \binom{N}{k} p^k (1-p)^{N-k} = \sum_{k=2}^{N} p^2 \frac{N(N-1)(N-2)!}{(k-2)!(N-k)!} p^{k-2} (1-p)^{N-k}$$

$$= N(N-1)p^2 \sum_{j=0}^{N-2} \binom{N-2}{j} p^j (1-p)^{N-2-j} = N(N-1)\, p^2$$

Putting together the last calculations, and simplifying algebraically, it is easy to check via the equality $(k-Np)^2 = k(k-1) - (2Np-1)k + (Np)^2$, that

$$\sum_{k=0}^{N} (k-Np)^2 \binom{N}{k} p^k (1-p)^{N-k} = N\, p(1-p)$$

Substituting this final relation into (3.1) now shows that

$$\Pr(|X/N - p| \geq \delta) \leq \frac{Np(1-p)}{(N\delta)^2} = \frac{p(1-p)}{N\delta^2}$$

The assertion of the Theorem now follows by taking $N \geq (p(1-p)/(\epsilon\delta^2)$. □

3.1.1 Probability Bounds & Approximations

Theorem 3.1 provides only a very crude upper bound to the probability with which $|X/N - p| \geq \delta$. A much more accurate upper bound in given in Theorem 3.2 of the Appendix to the Chapter (Sec. 3.9). To see why more accurate bounds are needed, consider the case where $N = 1000$, $p = 0.1$, and $\delta = 0.03$. The exact Binomial$(1000, 0.1)$ probability of (number of successes in N Bernoulli(p) trials falling in) $[0, N(p-\delta)] \cup [N(p+\delta), N] = [0, 70] \cup [130, 1000]$ is 0.001916, while the upper bound established in the proof of Theorem 3.1 is $(.1)(.9)/(1000(.03)^2) = 0.1$, more than fifty times larger ! On the other hand, the upper bound provided by the inequalities of Theorem 3.2, as in formula (3.30), is 0.0198.

Much closer approximations to the exact probabilities for Binomial(N, p) random variables to fall in intervals around Np are obtained from the *Normal distribution approximation*

$$\Pr(a \le X \le b) \;\approx\; \Phi\left(\frac{b - Np}{\sqrt{Np(1-p)}}\right) - \Phi\left(\frac{a - Np}{\sqrt{Np(1-p)}}\right) \qquad (3.2)$$

where Φ is the *standard normal distribution function* given explicitly in integral form in formula (3.29) below. This approximation is the **DeMoivre-Laplace Central Limit Theorem** (Feller vol. 1, 1957, pp. 168-73), which says precisely that the difference between the left- and right-hand sides of (3.2) converges to 0 when p remains fixed, as $n \to \infty$. Moreover, the refined form of the DeMoivre-Laplace Theorem given in the Feller (1957, p. 172) reference says that each of the *ratios* of probabilities

$$\Pr(X < a) \Big/ \Phi\Big(\frac{a - Np}{\sqrt{Np(1-p)}}\Big) \quad , \qquad \Pr(X > b) \Big/ \Big[1 - \Phi\Big(\frac{b - Np}{\sqrt{Np(1-p)}}\Big)\Big]$$

converges to 1 if the 'deviation' ratios $(b - Np)/\sqrt{Np(1-p)}$ and $(a - Np)/\sqrt{Np(1-p)}$ are of smaller order than $N^{-1/6}$ when N gets large. This result suggests the approximation

$$\text{Normal approx.} \;=\; \Phi\left(\frac{N\delta}{\sqrt{Np(1-p)}}\right) - \Phi\left(\frac{-N\delta}{\sqrt{Np(1-p)}}\right) \qquad (3.3)$$

for the true binomial probability $\Pr(|X/N - p| \le \delta)$. In the example discussed above, with $N = 1000, p = 0.1, \delta = 0.03$, where the exact Binomial(N, p) probability of $|X/N - p| \le \delta$ was $1 - .00192 = .99808$, the normal approximation (3.3) is 0.99843.

To give a feeling for the probabilities with which observed life-table ratios reflect the true underlying survival-rates, we have collected in Table 3.1

Table 3.1: Probabilities (in col. 6) with which various Binomial(l_x, ${}_kp_x$) random variables lie within a factor $1 \pm \epsilon$ of their expectations, together with lower bounds (in Col. 7) for these probabilities derived from the large-deviation inequalities (3.30)-(3.31). The final column contains the normal approximations based on (3.3) to the exact probabilities in column 6.

Cohort $n = l_x$	Age x	Time k	Prob. $p = {}_kp_x$	Toler. ϵ	Pr. within factor $1 \pm \epsilon$	Lower bound	Normal approx.
10000	40	3	0.99	.003	.9969	.9760	.9972
10000	40	5	0.98	.004	.9952	.9600	.9949
10000	40	10	0.94	.008	.9985	.9866	.9985
1000	40	10	0.94	.020	.9863	.9120	.9877
10000	70	5	0.75	.020	.9995	.9950	.9995
1000	70	5	0.75	.050	.9938	.9531	.9938
10000	70	10	0.50	.030	.9973	.9778	.9973
1000	70	10	0.50	.080	.9886	.9188	.9886

various exact binomial probabilities and their counterparts from the approximation of (3.3) and the inequality (3.30) of Section 3.9. The illustration concerns cohorts of lives aged x of various sizes l_x, together with 'theoretical' probabilities ${}_kp_x$ with which these lives will survive for a period of $k = 1,\ 5,$ or 10 years. The probability experiment determining the size of the surviving cohort l_{x+k} is modelled as the tossing of l_x independent coins with common heads-probability ${}_kp_x$: then the surviving cohort-size l_{x+k} is viewed as the *Binomial*(l_x, ${}_kp_x$) random variable equal to the number of heads in those coin-tosses. In Table 3.1 are given various combinations of $x,\ l_x,\ k,\ {}_kp_x$ which might realistically arise in an insurance-company life-table, together, with the true and estimated (from Theorem 3.2 and from (3.3)) probabilities with which the ratios l_{x+k}/l_x agree with ${}_kp_x$ to within a fraction ϵ of the latter. Columns 6 and 7 in the Table show how likely the life-table ratios are to be close to the 'theoretical' values, but also show that the lower bounds, while also often close to 1, are still noticeably smaller than the actual values.

Although the deviation-ratios in estimating life-table probabilities are often close to or larger than $N^{-1/6}$, not smaller as they should be for applicability of (3.3), the normal approximations in the final column of Table 3.1

below are sensationally close to the correct binomial probabilities in column 6. A still more refined theorem which justifies this is given by Feller (1972, section XVI.7 leading up to formula 7.28, p. 553).

3.2 Simulation of Discrete Lifetimes

We began by regarding life-table ratios l_x/l_0 in large cohort life-tables as *defining* integer-age survival probabilities $S(x) = {}_xp_0$. We said that if the life-table was representative of a larger population of prospective insureds, then we could imagine a newly presented life aged x as being randomly chosen from the life-table cohort itself. We motivated the conditional probability ratios in this way, and similarly expectations of functions of life-table death-times were averages over the entire cohort. Although we found the calculus-based formulas for life-table conditional probabilities and expectations to be useful, at that stage they were only ideal approximations of the more detailed but still exact life-table ratios and sums. At the next stage of sophistication, we began to describe the (conditional) probabilities ${}_tp_x \equiv S(x+t)/S(x)$ based upon a smooth survival function $S(x)$ as a true but unknown survival distribution, hypothesized to be of one of a number of possible theoretical forms, governing each member of the life-table cohort and of further prospective insureds. Finally, the life-table can also be viewed as in Appendix A as an idealized set of *data*, with each ratio l_{x+t}/l_x equal to the relative frequency of success among a set of l_x imagined $Bernoulli({}_tp_x)$ trials which Nature performs upon the cohort of lives aged x . With the mathematical justification of the Law of Large Numbers, we come full circle: these relative frequencies are random variables, but they are not very random. That is, if the size l_x of the cohort of surviving lives aged x is large, the later fractions l_{x+t}/l_x of survivors at $x+t$ to those at x are extremely likely to lie within a very small tolerance of ${}_tp_x$. The Law of Large Numbers applies equally when the age-x survivors have been sampled by some more complicated method than simply watching a cohort from birth. Thus, in the realistic data-collection scenarios discussed in Appendix A, where the sizes l_x of lives under observation at age x are large but the probabilities p_x are unknown, the life-table ratios l_{x+1}/l_x are highly accurate *statistical estimators* of the life-table probabilities .

Table 3.2: Illustrative Real and Simulated Life-Table Data

Age x	l_x in 1959-61 Life-Table	Simulated l_x^*
9	96801	96753
19	96051	95989
29	94542	94428
39	92705	92576
49	88178	87901
59	77083	76793
69	56384	56186
79	28814	28657

To make this discussion more concrete, we illustrate the difference between the entries in a life-table and the entries one would observe as data in a randomly generated life-table of the same size using the initial life-table ratios as *exact* survival probabilities. We used as a source of life-table counts the Mortality Table for U.S. White Males 1959-61 reproduced as Table 2 on page 11 of C. W. Jordan's (1967) book on *Life Contingencies*. That is, using this Table with radix $l_0 = 10^5$, with counts l_x given for integer ages x from 1 through 80, we treated the probabilities $p_x = l_{x+1}/l_x$ for $x = 0, \ldots, 79$ as the correct one-year survival probabilities for a second, computer-simulated cohort life-table with radix $l_0^* = 10^5$. Using simulated random variables generated in `R`, we successively generated, as x runs from 1 to 79, random variables $l_{x+1}^* \sim \mathit{Binomial}(l_x^*, p_x)$. In other words, the mechanism of simulation of the sequence $l_0^*, \ldots, l_{79}^*$ was to make the variable l_{x+1}^* depend on previously generated $l_1^*, \ldots, l_x^*$ only through l_x^*, and then to generate l_{x+1}^* as though it counted the heads in l_x^* independent coin-tosses with heads-probability p_x. A comparison of the actual and simulated life-table counts for ages 9 to 79 in 10-year intervals, is given below. The complete simulated life-table was given earlier as Table 1.1.

The implication of Table 3.2 is unsurprising: with radix as high as 10^5, the agreement between the initial and randomly generated life-table counts is extremely good. The Law of Large Numbers guarantees good agreement, with very high probability, between the ratios l_{x+10}/l_x (which here play the role of the probability ${}_{10}p_x$ of success in l_x^* *Bernoulli* trials) and the corresponding simulated random relative frequencies of success l_{x+10}^*/l_x^*. For example, with $x = 69$, the final simulated count of $l_{79}^* = 28657$ lives aged 79 is the success-count in $l^*_{69} = 56186$ Bernoulli trials with success-probability $28814/56384 = .51103$. With this success-probability, the normal approximation (3.3) says that the simulated count l^*_{79} will differ from $.51103 \cdot 56186 = 28712.8$ by 300 or more in either direction with probability approximately 0.0115. (The exact binomial probabilty of the same event is 0.0113.).

The `R` code used to generate Table 3.2 is very simple. If `lvec` denotes a vector of values $(l_0,\, l_{x(1)},\, l_{x(2)},\, \ldots,\, l_{x(K)})$ of numbers of surviving lives in a cohort life-table with radix l_0, where the integer ages $0, x(1), x(2), \ldots, x(K)$ are not necessarily evenly spaced, then the statements

```
K = length(lvec)-1 ;  pvec = lvec[2:(K+1)]/lvec[1:K]
```

create the vector of hypothetical survival probabilities, `pvec[j]` $= l_{x(j+1)}/l_{x_j}$, $j = 0, \ldots, K-1$, and here is a small function to generate the $(K+1)$-vector tt lstar consisting of $l^*_0 \equiv l_0$ together with the output simulated values $l^*_{x(1)},\, \ldots,\, l_{x(K)}$:

```
LifTabSim = function(lvec) {
        K = length(lvec)-1
        lstar = c(lvec[1],rep(0,K))
        for (j in 1:K) lstar[j+1] =
               rbinom(1,lstar[j],lvec[j+1]/lvec[j])
        lstar }
```

The syntax to generate a vector like the third column of Table 3.2 from the second, where `lvec` consists of the radix $l_0 = 10^5$ concatenated with column 2, is: `LifTab(lvec)[2:9]`. As a further example of such a simulation, suppose that 1000 individuals aged 40 have successive probabilities ${}_{10}p_x = 0.85, 0.77, 0.70, 0.65, 0.4$ for $x = 40, 50, 60, 70, 80$, then we can simulate twice, independently and output in `R` the numbers of surviving lives at these ages, as follows:

```
pvec = c(0.85, 0.77, 0.70, 0.65, 0.4)
Lvec = 1000*cumprod(c(1,pvec))
matrix(c(seq(40,80,10), pvec, LifTabSim(Lvec)[2:6],
   LifTabSim(Lvec)[2:6]), nrow=4, ncol=5, byrow=T,
```

```
    dimnames=list(c("Ages","10_p_x","Sim#1 l_x",
                    "Sim#2 l_x"), NULL))
             [,1]   [,2]  [,3]   [,4]  [,5]
Ages        40.00  50.00  60.0  70.00  80.0
10_p_x       0.85   0.77   0.7   0.65   0.4
Sim#1 l_x  842.00 638.00 450.0 296.00 133.0
Sim#2 l_x  854.00 662.00 483.0 344.00 129.0
```

From small experiments like this, we can see that the variability in the simulated numbers l_x^* is considerable for l_0 of 1000 or less.

Exercise 3.A. With the same probabilities ${}_{10}p_x$ use R to simulate 10 times independently the numbers of survivors at ages $40, 50, \ldots, 80$.

(a). What is the spread between the smallest and largest number surviving at each age across your 10 simulations ?

(b). Regarding your 10 sets of simulated numbers of survivors as independent *datasets*, if the underlying life-table probabilities were unknown, what would be your best estimate of the probability ${}_{20}p_{40}$?

(c). Combining the 10 simulated datasets you generated in (b), what is your best estimate of the probability ${}_{20}p_{40}$?

3.3 Expectation of Discrete Random Variables

The Binomial random variables discussed in this Chapter are examples of so-called *discrete random variables*, that is, random variables Z with a discrete (usually finite) list of possible outcomes z, with a corresponding list of probabilities or *probability mass function* values $p_Z(z)$ with which each of those possible outcomes occur. These probabilities $p_Z(z)$ must be positive numbers which summed over all possible values z add to 1. In an insurance context, think for example of Z as the unforeseeable future damage or liability upon the basis of which an insurer has to pay some scheduled *claim amount* $c(Z)$ to fulfill a specific property or liability insurance policy. The Law of Large Numbers says that we can have a *frequentist* operational interpretation of each of the probabilities $p_Z(z)$ with which a claim of size $c(z)$ is presented. In a large population of N independent policyholders, each governed by the same probabilities $p_Z(\cdot)$ of liability occurrences, for each fixed damage-amount z we can imagine a series of N *Bernoulli*$(p_Z(z))$ trials, in which the j_{th} policyholder is said to result in a 'success' if he sustains a damage amount equal to z, and to result in a 'failure' otherwise. The Law of Large Numbers (Theorem 3.7) for these Bernoulli trials says that the number out of these N policyholders who do sustain damage z is for large N extremely likely to differ by no more than δN from $N\, p_Z(z)$.

Returning to a general discussion, suppose that Z is a discrete random variable with a finite list of possible values $z_1, \ldots, z_m$, and let $c(\cdot)$ be a real-valued (nonrandom) cost function such that $c(Z)$ represents an economically meaningful cost incurred when the random variable value Z is given. Suppose that a large number N of independent individuals give rise to respective values Z_j, $j = 1, \ldots, N$ and costs $c(Z_1), \ldots, c(Z_N)$. Here *independent* means that the mechanism causing different individual Z_j values is such that information about the values $Z_1, \ldots, Z_{j-1}$ does not change the (conditional) probabilities with which Z_j takes on its values, so that for all j, i, and $b_1, \ldots, b_{j-1}$,

$$P(Z_j = z_i \,|\, Z_1 = b_1, \ldots, Z_{j-1} = b_{j-1}\,) = p_Z(z_i)$$

Then the Law of Large Numbers, applied as above, says that out of the large number N of individuals it is extremely likely that approximately $p_Z(k) \cdot N$ will have their Z variable values equal to k, where k ranges over $\{z_1, \ldots, z_m\}$. It follows that the average costs $c(Z_j)$ over the N

independent individuals — which can be expressed exactly as

$$N^{-1} \sum_{j=1}^{N} c(Z_j) \;=\; N^{-1} \sum_{i=1}^{m} c(z_i) \cdot \#\{j = 1, \ldots, N :\; Z_j = z_i\}$$

— is approximately given by

$$N^{-1} \sum_{i=1}^{m} c(z_i) \cdot (N\, p_Z(z_i)) = \sum_{i=1}^{m} c(z_i)\, p_Z(z_i)$$

In other words, the Law of Large Numbers implies that the **average cost per trial** among the N independent trials resulting in random variable values Z_j and corresponding costs $c(Z_j)$ has a well-defined approximate (actually, a limiting) value for very large N

$$\textbf{Expectation of cost} \;=\; E(c(Z)) = \sum_{i=1}^{m} c(z_i)\, p_Z(z_i) \tag{3.4}$$

As an application of the formula for expectation of a discrete random variable, consider the expected value of a cost-function $g(T)$ of a lifetime random variable which is assumed to depend on T only through the function $g([T])$ of the integer part of T. This expectation was interpreted earlier as the average cost over all members of the specified life-table cohort. Now the expectation can be verified to coincide with the life-table average previously given, if the probabilities $S(j)$ in the following expression are replaced by the life-table estimators l_j/l_0. Since $P([T] = k) = S(k) - S(k+1)$, the general expectation formula (3.4) yields

$$E(g(T)) = E(g([T]) = \sum_{k=0}^{\omega-1} g(k)\, (S(k) - S(k+1)) \tag{3.5}$$

which, after replacing $S(k) - S(k+1) = \int_k^{k+1} f(t)\, dt$ and $[t] = k$ for $k \le t < k+1$, becomes

$$\sum_{k=0}^{\omega-1} g(k) \int_k^{k+1} f(t)\, dt \;=\; \sum_{k=0}^{\omega-1} \int_k^{k+1} g([t])\, f(t)\, dt \;=\; \int_0^{\omega} g([t])\, f(t)\, dt$$

agreeing precisely with formula (1.3). Similarly, evaluating the discrete conditional expectation given $T \ge x$ means applying the formula (3.4) to the function $c(Z) = g(Z)$ of the discrete random variable $Z = [T]$ using the conditional probability mass function $P(Z = k) = P([T] = k \,|\, T \geq x) = (S(k) - S(k+1))/S(x)$ for all integers $k \geq x$ (and with probability 0 assigned to all integers $k < x$.) Then the conditional expectation is

$$E(g([T]) \,|\, T \geq x) \;=\; \sum_{k=x}^{\omega-1} \frac{S(k) - S(k+1)}{S(x)}\, g(k) \;=\; \sum_{k=x}^{\omega-1} \frac{g(k)}{S(x)} \int_k^{k+1} f(t)\, dt$$

or

$$E(g([T] \,|\, T \geq x) \;=\; \sum_{k=x}^{\omega-1} \int_k^{k+1} \frac{g([t])}{S(x)}\, f(t)\, dt \;=\; \int_0^{\omega} g([t])\, \frac{f(t)}{S(x})\, dt$$

agreeing precisely with formula (1.5).

The preceding discussion shows that expectations or conditional expectations of functions of whole-year ages can equivalently be calculated using the expectation formulas for discrete or continuous random variables. In the discrete case, however, the expressions require knowledge only of the probabilities $S(y)$ of survival for whole-year or integer ages y. Indeed, in the preceding (discrete-version) formula, for $E(g([T]) \,|\, T \geq x)$, let $k \geq x$ be replaced by $k = x + j$, and express

$$\frac{S(k) - S(k+1)}{S(x)} \;=\; \frac{S(x+j)}{S(x)} \left(1 \;-\; \frac{S(x+j+1}{S(x+j)}\right) \;=\; {}_jp_x\,(1 - p_{x+j})$$

Then

$$E(g([T]) \,|\, T \geq x) \;=\; \sum_{k=x}^{\omega-1} \frac{S(k) - S(k+1)}{S(x)}\, g(k) \;=\; \sum_{j=0}^{\omega-x-1} {}_jp_x\,(1 - p_{x+j})\, g(x+j) \tag{3.6}$$

Just as we did in the context of expectations of functions of the life-table waiting-time random variable T , we can interpret the *Expectation* as a weighted average of values (costs, in this discussion) which can be incurred in each trial, weighted by the probabilities with which they occur. There is an analogy in the continuous-variable case, where Z would be a random variable whose approximate probabilities of falling in tiny intervals $[z, z + dz]$ are given by $f_Z(z)dz$, where $f_Z(z)$ is a nonnegative density function integrating to 1. In this case, the weighted average of cost-function values $c(z)$ which arise when $Z \in [z, z + dz]$, with approximate probability-weights $f_Z(z)dz$, is written as a limit of sums or an integral, namely $\int c(z)\, f(z)\, dz$.

3.3.1 Rules for Manipulating Expectations

We have separately defined *expectation* for continuous and discrete random variables. In the continuous case, we treated the expectation of a specified function $g(T)$ of a lifetime random variable governed by the survival function $S(x)$ of a cohort life-table, as the approximate numerical average of the values $g(T_i)$ over all individuals i with data represented through observed lifetime T_i in the life-table. The discrete case was handled more conventionally, along the lines of a 'frequentist' approach to the mathematical theory of probability. First, we observed that our calculations with $Binomial(n,p)$ random variables justified us in saying that the sum $X = X_n$ of a large number n of independent coin-toss variables $\epsilon_1, \ldots, , \epsilon_n$, each of which is 1 with probability p and 0 otherwise, has a value which with very high probability differs from $n \cdot p$ by an amount smaller than δn, where $\delta > 0$ is an arbitrarily small number not depending upon n. The *Expectation* p of each of the variables ϵ_i is recovered approximately as the numerical average $X/n = n^{-1} \sum_{i=1}^n \epsilon_i$ of the independent outcomes ϵ_i of independent trials. This Law of Large Numbers extends to arbitrary sequences of independent and identical finite-valued discrete random variables, saying that

> if $Z_1, Z_2, \ldots$ are independent random variables, in the sense that for all $k \geq 2$ and all numbers r,
>
> $$P(Z_k \leq r \,|\, Z_1 = z_1, \ldots, Z_{k-1} = z_{k-1}) = P(Z_1 \leq r)$$
>
> regardless of the precise values $z_1, \ldots, z_{k-1}$, then for each $\delta > 0$, as n gets large
>
> $$P\Big(|n^{-1} \sum_{i=1}^{n} c(Z_i) - E(c(Z_1))| \geq \delta \Big) \longrightarrow 0 \qquad (3.7)$$
>
> where, in terms of the finite set S of possible values of Z ,
>
> $$E(c(Z_1)) = \sum_{z \in S} c(z)\, P(Z_1 = z) \qquad (3.8)$$

We do not give any further proof here, but the motivating arguments given, together with straightforward manipulations using the result of Theorem 3.7, are an essentially complete proof of (3.7). It is also a fact that the Law of Large Numbers given in equation (3.7) continues to hold if the definition of *independent* sequences of random variables Z_i is suitably generalized, as long as either

> Z_i are discrete with infinitely many possible values defining a set S, and the expectation is as given in equation (3.8) above whenever the function $c(z)$ is such that
>
> $$\sum_{z \in S} |c(z)| \, P(Z_1 = z) \; < \; \infty$$

or

> the independent random variables Z_i are *continuous*, all with the same density $f(t)$ such that $P(q \le Z_1 \le r) = \int_q^r f(t)\, dt$, and expectation is defined by
>
> $$E(c(Z_1)) \; = \; \int_{-\infty}^{\infty} c(t)\, f(t)\, dt \tag{3.9}$$
>
> whenever the function $c(t)$ is such that
>
> $$\int_{-\infty}^{\infty} |c(t)|\, f(t)\, dt \; < \; \infty$$

All of this shows that there really is no choice in devising an appropriate definition of expectations of cost-functions defined in terms of random variables Z, whether discrete or continuous. For the rest of this book, and more generally in applications of probability within actuarial science, we are interested in evaluating expectations of various functions of random variables related to the contingencies and uncertain duration of life. Many of these expectations concern superpositions of random amounts to be paid out after random durations. The following rules for the manipulation of expectations arising in such superpositions considerably simplify the calculations. Assume in what follows that all random payments and times are functions of a single lifetime random variable T.

(1). If a payment consists of a nonrandom multiple (e.g., face-amount F) times a random amount $c(T)$, then the expectation of the payment is the product of F and the expectation of $c(T)$:

$$\textit{Discrete case:} \quad E(F c(T)) = \sum_t F\, c(t)\, P(T=t)$$

$$= F \sum_t c(t)\, P(T=t) = F \cdot E(c(T))$$

$$\textit{Continuous case:} \quad E(F c(T)) = \int F\, c(t) f(t)\, dt = F \int c(t) f(t)\, dt = F \cdot E(c(T))$$

(2). If a payment consists of the sum of two separate random payments $c_1(T)$, $c_2(T)$ (which may occur at different times, taken into account by treating both terms $c_k(T)$ as present values as of the same time), then the overall payment has expectation which is the sum of the expectations of the separate payments:

$$\textit{Discrete case:} \quad E(c_1(T) + c_2(T)) = \sum_t (c_1(t) + c_2(t))\, P(T=t)$$

$$= \sum_t c_1(t)\, P(T=t) + \sum_t c_2(t)\, P(T=t) = E(c_1(T)) + E(c_2(T))$$

$$\textit{Continuous case:} \quad E(c_1(T) + c_2(T)) = \int (c_1(t) + c_2(t))\, f(t)\, dt$$

$$= \int c_1(t)\, f(t)\, dt + \int c_2(t)\, f(t)\, dt = E(c_1(T)) + E(c_2(T))$$

Thus, if an uncertain payment under an insurance-related contract, based upon a continuous lifetime variable T with density f_T, occurs only if $a \le T < b$ and in that case consists of a payment of a fixed amount F occurring at a fixed time h, then the expected present value under a fixed nonrandom interest-rate i with $v = (1+i)^{-1}$, becomes by rule **(1)** above,

$$E(v^h\, F\, I_{[a \le T < b]}) = v^h\, F\, E(I_{[a \le T < b]})$$

where the indicator-notation $I_{[a \le T < b]}$ denotes a random quantity which is 1 when the condition $[a \le T < b]$ is satisfied and is 0 otherwise. Since an indicator random variable has the two possible outcomes $\{0,1\}$ like the coin-toss variables ϵ_i above, we conclude that $E(I_{[a\le T<b]}) = P(a \le T < b) = \int_a^b f_T(t)\,dt$, and the expected present value above is

$$E(v^h\,F\,I_{[a\le T<b]}) = v^h\,F \int_a^b f_T(t)\,dt$$

(3). The expectation of a nonnegative-integer-valued random variable can sometimes be simplified considerably by means of the following useful Lemma.

Lemma 3.1 *Let Z be a nonnegative-integer-valued random variable. Then*

$$EZ \;=\; \sum_{j=1}^{\infty} P(Z \ge j) \qquad (3.10)$$

The Lemma is proved using the rule (Fubini-Tonelli theorem for double summation) that the order of a double summation of nonnegative summands can always be reversed:

$$EZ \;=\; \sum_{k=0}^{\infty} k\,p_Z(k) \;=\; \sum_{k=1}^{\infty}\sum_{j=1}^{k} p_X(k) \;=\; \sum_{j=1}^{\infty}\sum_{k=j}^{\infty} p_Z(k) \;=\; \sum_{j=1}^{\infty} P(Z \ge j)$$

3.3.2 Curtate Expectation of Life

One example of a function of the number $[T]$ of whole years of life, whose conditional expectation is useful and interpretable, is the whole-year residual life $[T]-x$ for a life aged x, where x is an integer age. The expectation, necessarily conditional on the attained age x, is called **curtate mean residual life** or **curtate life expectancy**,

$$\mathrm{e}_x \;=\; E(\,[T]-x\,|\,T \ge x\,) \;=\; \sum_{t=x}^{\omega-1} \frac{P(t \le T < t+1)}{P(T \ge x)}\,(t-x) \qquad (3.11)$$

Substituting the formula (3.6) with $g(t) = t - x$ gives this formula in the alternative form

$$\mathrm{e}_x \;=\; E(\,[T] - x \,|\, T \ge x\,) \;=\; \sum_{j=0}^{\omega-x-1} {}_j p_x \,(1 - p_{x+j})\, j \tag{3.12}$$

A third useful version of this formula can be found by applying formula (3.10) of Lemma 3.1 to the nonnegative integer valued random variable $Z = [T] - x$ with probability masses calculated conditionally given $T \ge x$. This yields

$$\mathrm{e}_x \;=\; E(\,[T] - x \,|\, T \ge x\,) \;=\; \sum_{j=1}^{\omega-x-1} P([T] - x \ge j \,|\, T \ge x) \;=\; \sum_{j=1}^{\omega-x-1} {}_j p_x \tag{3.13}$$

The extension of these expectation formulas to give mean residual lifetimes which are *not* truncated to whole years rests on survival function and density formulas which specify mortality rates between birthdays. The following two sections are devoted to a deeper study of continuous mortality models and interpolation approximations.

3.4 Interpreting Force of Mortality

This Section consists of remarks, relating the force of mortality for a continuously distributed lifetime random variable T (with continuous density function f) to conditional probabilities for discrete random variables. Indeed, for m large (e.g. as large as 4 or 12), the discrete random variable $[Tm]/m$ gives a close approximation to T and represents the attained age at death measured in whole-number multiples of fractions $h =$ one m^{th} of a year. (Here $[\cdot]$ continues to denote the greatest integer less than or equal to its real argument.) Since surviving an additional time $t = nh$ can be viewed as successively surviving to reach times $h, 2h, 3h, \ldots, nh$, and since (by the definition of conditional probability)

$$P(A_1 \cap \cdots \cap A_n) \;=\; P(A_1) \cdot P(A_2 | A_1) \cdots P(A_n | A_1 \cap \cdots \cap A_{n-1})$$

we have (with the interpretation $A_k = \{T \ge x + kh\}$)

$${}_{nh}p_x \;=\; {}_h p_x \,\cdot\, {}_h p_{x+h} \,\cdot\, {}_h p_{x+2h} \,\cdots\, {}_h p_{x+(n-1)h}$$

The form in which this formula is most often useful is the case $h = 1$: for integers $k \geq 2$,

$$ {}_kp_x \;=\; p_x \,\cdot\, p_{x+1} \,\cdot\, p_{x+2} \cdots p_{x+k-1} \tag{3.14}$$

Every continuous waiting-time random variable can be approximated by a discrete random variable with possible values which are multiples of a fixed small unit h of time, and therefore the random survival time can be viewed as the (first failure among a) succession of results of a sequence of independent coin-flips with successive probabilities ${}_hp_{kh}$ of heads. By the Mean Value Theorem applied up to second-degree terms on the function $S(x + h)$ expanded about $h = 0$,

$$S(x{+}h) \;=\; S(x) + hS'(x) + \frac{h^2}{2}\,S''(x{+}\tau h) \;=\; S(x) - hf(x) - \frac{h^2}{2}\,f'(x{+}\tau h)$$

for some $0 < \tau < 1$, if f is continuously differentiable. Therefore, using the definition of $\mu(x)$ as $f(x)/S(x)$ given on page 49,

$${}_hp_x = 1 \,-\, h \cdot \Big[\frac{S(x) - S(x+h)}{hS(x)}\Big] \;=\; 1 \,-\, h\,\Big(\mu(x) \,+\, \frac{h}{2}\,\frac{f'(x+\tau h)}{S(x)}\Big)$$

Going in the other direction, the previously derived formula

$${}_hp_x = \exp\left(-\int_x^{x+h} \mu(y)\,dy\right)$$

can be interpreted by considering the fraction of individuals observed to reach age x who thereafter experience hazard of mortality $\mu(y)\,dy$ on successive infinitesimal intervals $[y, y{+}dy]$ within $[x, x{+}h)$. The lives aged x survive to age $x + h$ with probability equal to a limiting product of infinitesimal terms $(1{-}\mu(y)\,dy) \sim \exp(-\mu(y)\,dy)$, yielding an overall conditional survival probability equal to the negative exponential of accumulated hazard over $[x, x + h)$.

3.5 Interpolation Between Integer Ages

Cohort life-table data l_x and the probability quantities ${}_jp_x$ derived from them (for integers j) depend on and are determined by the survival function $S(k)$ values only at integer arguments k. Yet many expectations of functions important in actuarial applications necessarily involve the survival function values between integer ages. It is possible to approximate these only because, for all but the very youngest and oldest ages, the survival function for human lives is very smooth within years of age, with derivatives that are not dramatically large and themselves do not change rapidly. In terms of calculus concepts, the function value $S(x+t)$ for integer x and $0 \le t < 1$ is given approximately by the Taylor series formula

$$\begin{aligned} S(x+t) &= S(x) + t\,S'(x) + \frac{1}{2}t^2\,S''(x+\theta t) \\ &= S(x) - t\,f'(x) - \frac{1}{2}t^2\,f'(x+\theta t) \end{aligned} \tag{3.15}$$

where $\theta \in (0,1)$ in the argument of S'' in the third term (the mean-value type remainder) of the first line, and where the second line uses the definition $f(x+t) = -S'(x+t)$ valid at all nonnegative integers x and $t \in [0,1)$.

While we will later use Taylor expansions like (3.15) to approximate expectations $E(g(T))$ and conditional expectations $E(g(T)\,|\,T \ge x)$, for now we focus on understanding what (3.15) says about the approximate probability distribution of the lifetime variable T within years of age. If $S'' = -f'$ is small, as is undoubtedly true for human lifetime between ages 2 and 75 in modern public health conditions, then it is tempting to impose the direct modelling assumption $S(x+t) \equiv S(x) - tf(x)$ for integer x and $t \in [0,1)$, together with continuity of S at all integer points. This assumption, often called the *actuarial approximation*, says that for any $0 \le a < b < 1$,

$$P(T \in [x+a, x+b)\,|\,[T] = x) = \frac{S(x+a) - S(x+b)}{(S(x) - S(x+1)} = \frac{(b-a)\,f(x)}{f(x)} = b-a$$

In other words, the 'actuarial approximation' says that failures known to occur within the year between the x and $x+1$ birthdays are actually uniformly distributed (have constant conditional density of 1) within that year.

The 'actuarial-approximation' assumption can be understood either as piecewise linearity, on exact-age intervals $[x, x+1)$, of the continuous survival function $S(y)$ or equivalently as piecewise constancy of the density function $f(y) = -S'(y)$. This assumption is by far the most commonly used one in actuarial work. Two other related possible approximations can be obtained by Taylor expanding not $S(x+t)$ itself but rather the functions $\log S(x+t)$ or $1/S(x+t)$. It turns out that the first of these alternative assumptions, piecewise linearity of $\log S(x+t)$ or equivalently piecewise constancy within intervals $x, x+1)$ of $\frac{d}{dy}\log S(y) = -f(y)/S(y) \equiv \mu(y)$ is well known and has historically been widely used by biostatisticians. Many models in biostatistics or reliability have been formulated with *piecewise constant hazards* (recall that biostatisticians call $\mu(y)$ the *hazard* function while actuaries call it *force of mortality*). The third assumption introduced here, that of piecewise linearity of $1/S(y)$, is called the **Balducci hypothesis**, and is studied by actuarial students largely for historical reasons and as a source of examination problems, since it will be seen immediately below formula (3.22) to have properties which make it unsuitable as a realistic model for survival.

To proceed formally, assume that values $S(x)$ for $x = 0, 1, 2, \ldots$ have been specified or estimated. Approximations to $S(y)$, $f(y)$ and $\mu(y)$ between integers are usually based on one of the following assumptions:

(i) *(Piecewise-uniform density)* $f(x+t)$ is constant for $0 \le t < 1$;

(ii) *(Piecewise-constant hazard)* $\mu(x+t)$ is constant for $0 \le t < 1$;

(iii) *(Balducci hypothesis)* $1/S(x+t)$ is linear for $0 \le t < 1$.

For integers x and $0 \le t \le 1$,

$$\left.\begin{array}{c} S(x+t) \\ -\ln S(x+t) \\ 1/S(x+t) \end{array}\right\} \text{ is linear in } t \text{ under } \left\{\begin{array}{c} \text{assumption (i)} \\ \text{assumption (ii)} \\ \text{assumption (iii)} \end{array}\right. \tag{3.16}$$

Under assumption (i), the slope of the linear function $S(x+t)$ at $t = 0$ is $-f(x)$, which implies easily that $S(x+t) = S(x) - tf(x)$, i.e.,

$$f(x) = S(x) - S(x+1)\ , \quad \text{and} \quad \mu(x+t) = \frac{f(x)}{S(x) - tf(x)} \tag{3.17}$$

so that under (i),

$$\mu(x + \frac{1}{2}) \ = \ f_T(x + \frac{1}{2}) \Big/ S_T(x + \frac{1}{2}) \tag{3.18}$$

Under (ii), where $\mu(x+t) = \mu(x)$, (3.18) also holds, and

$$S(x+t) = S(x)\, e^{-t\,\mu(x)}\ , \quad \text{and} \quad p_k = \frac{S(x+1)}{S(x)} = e^{-\mu(x)}$$

Under (iii), for $0 \le t < 1$,

$$\frac{1}{S(x+t)} = \frac{1}{S(x)} + t\left(\frac{1}{S(x+1)} - \frac{1}{S(x)}\right) \tag{3.19}$$

When equation (3.19) is multiplied through by $S(x+1)$ and terms are rearranged, the result is

$$\frac{S(x+1)}{S(x+t)} = t + (1-t)\,\frac{S(x+1)}{S(x)} = 1 - (1-t)\,q_x \tag{3.20}$$

Recalling that ${}_tq_x = 1 - (S(x+t)/S(x))$, reveals assumption (iii) to be equivalent to

$${}_{1-t}q_{x+t} = 1 - \frac{S(x+1)}{S(x+t)} = (1-t)\Big(1 - \frac{S(x+1)}{S(x)}\Big) = (1-t)\,q_x \tag{3.21}$$

Next differentiate the logarithm of the formula (3.20) with respect to t, to show (still under (iii)) that

$$\mu(x+t) = -\frac{\partial}{\partial t}\ \ln S(x+t) = \frac{q_x}{1-(1-t)q_x} \tag{3.22}$$

Apart from any other property which the Balducci interpolation assumption (iii) might have, formula (3.19) immediately shows that the within-year force of mortality $\mu(x+t)$, $0 \le t < 1$, is actually a decreasing function a feature which seems particularly unrealistic from middle to advanced ages within human lifetimes. By contrast, the within-year force of mortality under assumption (i) as given in (3.17) is evidently increasing, and almost by definition the piecewise-constant hazard assumption (ii) entails within-year constancy of the force of mortality.

The most frequent insurance application for the interpolation assumptions (i)-(iii) and associated survival-probability formulas is to express probabilities of survival for fractional years in terms of probabilities of whole-year survival. In terms of the notations ${}_tp_x$ and q_x for integers x and $0 < t < 1$, the formulas are:

$${}_tp_x = 1 - \frac{(S(x) - t(S(x+1) - S(x))}{S(x)} = 1 - t\,q_x \qquad \text{under (i)} \tag{3.23}$$

$${}_tp_x = \frac{S(x+t)}{S(x)} = \left(e^{-\mu(x)}\right)^t = (1-q_x)^t \qquad \text{under (ii)} \tag{3.24}$$

$$_tp_x \;=\; \frac{S(x+t)}{S(x+1)}\,\frac{S(x+1)}{S(x)} \;=\; \frac{1-q_x}{1-(1-t)q_x} \qquad \text{under (iii)} \qquad (3.25)$$

The application of all of these formulas can be understood in terms of the formula for expectation of a function $g(T)$ of the lifetime random variable T. (For a concrete example, think of $g(T) = (1+i)^{-T}$ as the present value to an insurer of the payment of \$1 which it will make instantaneously at the future time T of death of a newborn which it undertakes to insure.) Then assumptions (i), (ii), or (iii) via respective formulas (3.23), (3.24), and (3.25) are used to substitute into the final expression of the following formulas:

$$E\Big(g(T)\Big) \;=\; \int_0^\infty g(t)\,f(t)\,dt \;=\; \sum_{x=0}^{\omega-1}\int_0^1 g(t+x)\,f(t+x)\,dt$$
$$=\; \sum_{x=0}^{\omega-1} S(x)\int_0^1 g(t+x)\Big(-\frac{\partial}{\partial t}\,{}_tp_x\Big)\,dt$$

3.5.1 Life Expectancy – Definition and Approximation

In terms of a survival function f and S modelling the distribution of exact age at death within years of integer age, we can extend the notion of expected remaining life from the curtate to the complete expectation for a life aged x of $(T-x)$:

$$\textbf{complete expectation of life} \;=\; \mathring{e}_x \;=\; E(T-x\,|\,T\ge x) \qquad (3.26)$$

This quantity is also called *expected residual life* or, in demography, *life expectancy*. This quantity is larger than the the curtate life expectancy e_x because, for a life just completing its x'th year and surviving to exact age $x+j+t$, with x, j integers and $0 \le t < 1$ the complete residual life is $T-x = j+t$ while the curtate residual life is $[T]-x = j$. Thus by definition

$$\mathring{e}_x - e_x \;=\; E(\,T-[T]\,|\,T\ge x\,) \;\in\; [0,1)$$

since this difference is the expectation or weighted average of a quantity between 0 and 1.

The integral formula for life expectancy can be written in any of the three ways

$$\mathring{e}_x \;=\; \int_x^{\omega} (y-x)\,\frac{f(y)}{S(x)}\,dy \;=\; \int_0^{\omega-x} t\;{}_tp_x\,\mu(x+t)\,dt \;=\; \int_0^{\omega-x} {}_tp_x \qquad (3.27)$$

Of these expressions, the first is the basic conditional expectation formula (1.5) with $g(T) = T - x$. The second is obtained from it by the change of variable $t = y - x$, using the identities

$$f(x+t)/S(x) \;=\; \mu(x+t)\,S(x+t)/S(x) \;=\; \mu(x+t)\;{}_tp_x$$

The third is a continuous-time analogue of Lemma 3.1, obtained from the second expression in (3.27) via integration by parts, using $u = t$ and $dv = \mu(x+t)\,{}_tp_x\,dt \;=\; -(1/S(x))\,d(S(x+t))$.

Under the "actuarial approximation" (assumption (i)) of uniform lifetime distribution within whole years of age, we saw above that $T - [T]$ is a random variable with values in $[0, 1)$ which is uniformly distributed (with constant density 1). Therefore, using the formula (1.4) for expectation, we find

$$\textbf{under (i):} \qquad \mathring{e}_x \;-\; e_x \;=\; \int_0^1 t\,dt \;=\; \frac{1}{2} \qquad (3.28)$$

There are no formulas nearly as simple for the difference between complete and curtate life expectancies under interpolation assumptions (ii) or (iii).

3.6 Some Special Integrals

While actuaries ordinarily do not allow themselves to represent real life-table survival distributions by simple finite-parameter families of theoretical distributions (for the good reason that such distributions never approximate the real large-sample life-table data well enough), it is important for the student to be conversant with several integrals which would arise by substituting some of the theoretical models into formulas for various net single premiums and expected lifetimes.

Consider first the *Gamma* functions and integrals arising in connection with Gamma survival distributions. The *Gamma* function $\Gamma(\alpha)$ is defined by

$$\Gamma(\alpha) \;=\; \int_0^{\infty} x^{\alpha-1}\,e^{-x}\,dx \;\;, \qquad \alpha > 0$$

This integral is easily checked to be equal to 1 when $\alpha = 1$, giving the total probability for an exponentially distributed random variable, i.e., a lifetime with constant force-of-mortality 1. For $\alpha = 2$, the integral is the expected value of such a unit-exponential random variable, and it is a standard integration-by-parts exercise to check that it too is 1. More generally, the integral $Gamma(\alpha + 1)$ for positive integer α is the α^{th} moment of the Exponential distribution with parameters $\lambda = 1$. Integration by parts in the *Gamma* integral with $u = x^\alpha$ and $dv = e^{-x}\, dx$ immediately yields the famous *recursion relation* for the *Gamma* integral, first derived by Euler, and valid for all $\alpha > 0$:

$$\Gamma(\alpha+1) = \int_0^\infty x^\alpha\, e^{-x}\, dx = \left(-x^\alpha\, e^{-x}\right)\Big|_0^\infty + \int_0^\infty \alpha\, x^{\alpha-1}\, e^{-x}\, dx \;=\; \alpha \cdot \Gamma(\alpha)$$

This relation, applied inductively, shows that for all positive integers n,

$$\Gamma(n+1) \;=\; n \cdot (n-1) \cdots 2 \cdot \Gamma(1) \;=\; n!$$

The only other simple-to-derive formula explicitly giving values for (non-integer) values of the *Gamma* function is $\Gamma(\,\frac{1}{2}\,) = \sqrt{\pi}$, obtained as follows:

$$\Gamma(\,\frac{1}{2}\,) \;=\; \int_0^\infty x^{-1/2}\, e^{-x} dx \;=\; \int_0^\infty e^{-z^2/2}\, \sqrt{2}\, dz$$

Here we have made the integral substitution $x = z^2/2,\ x^{-1/2}\, dx = \sqrt{2}\, dz$. The last integral can be given by symmetry, using the change of variable $u = -z$ and the fact that the integrand is an even function, to show that

$$\int_{-\infty}^0 e^{-z^2/2}\, dz \;=\; \int_0^\infty e^{-u^2/2}\, du \;=\; \frac{1}{2} \int_{-\infty}^\infty e^{-x^2/2}\, dx \;=\; \frac{1}{2}\sqrt{2\pi} \;=\; \frac{\sqrt{\pi}}{\sqrt{2}}$$

where the last equality is equivalent to the fact (proved in most calculus texts as an exercise in double integration using change of variable to polar coordinates) that the *standard normal distribution*

$$\Phi(x) \;=\; \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-z^2/2}\, dz \tag{3.29}$$

is a *bona-fide* distribution function with limit equal to 1 as $x \to \infty$. The symmetry of the normal density guarantees that half of its probability is assigned to each of $(-\infty, 0)$ and $[0, \infty)$, so that $\Phi(0) = 1/2$.

One of the integrals which arises in calculating expected remaining lifetimes for Weibull-distributed variables is a *Gamma* integral, after integration-by-parts and a change-of-variable. Recall that the *Weibull* density with parameters $\lambda,\ \gamma$ is

$$f(t) \;=\; \lambda\,\gamma\,t^{\gamma-1}\,e^{-\lambda\,t^\gamma} \;, \qquad t > 0$$

so that $S(x) = \exp(-\lambda\,x^\gamma)$. The expected remaining life for a Weibull-distributed life aged x is calculated, via an integration by parts with $u = t - x$ and $dv = f(t)dt = -S'(t)dt$, as

$$\int_x^\infty (t-x)\,\frac{f(t)}{S(x)}\,dt = \frac{1}{S(x)}\,\Big[-(t-x)\,e^{-\lambda t^\gamma}\Big|_x^\infty + \int_x^\infty e^{-\lambda t^\gamma}\,dt\Big]$$

The first term in square brackets evaluates to 0 at the endpoints, and the second term can be re-expressed via the change-of-variable $w = \lambda\,t^\gamma$ with $(1/\gamma)\,w^{1/\gamma-1}\,dw \;=\; \lambda^{1/\gamma}\,dt$, to give in the Weibull example,

$$E(T-x\,|\,T \ge x) = e^{\lambda x^\gamma}\,\frac{1}{\gamma}\,\lambda^{-1/\gamma}\int_{\lambda\,x^\gamma}^\infty w^{(1/\gamma)-1}\,e^{-w}\,dw$$

$$= \;\Gamma(\,\frac{1}{\gamma}\,)\,e^{\lambda\,x^\gamma}\,\frac{1}{\gamma}\,\lambda^{-1/\gamma}\,\Big(1 - G_{1/\gamma}(\lambda\,x^\gamma)\Big)$$

where we denote by $G_\alpha(z)$ the *Gamma distribution function* with shape parameter α,

$$G_\alpha(z) \;=\; \frac{1}{\Gamma(\alpha)}\int_0^z v^{\alpha-1}\,e^{-v}\,dv$$

and the integral on the right-hand side is called the *incomplete Gamma function*. Values of $G_\alpha(z)$ can be found either in published tables which are now quite dated, or among the standard functions of many mathematical/statistical computer packages, such as **Matlab** or **R**. One particular case of these integrals, the case $\alpha = 1/2$, can be re-cast in terms of the standard normal distribution function $\Phi(\cdot)$. We change variables by $v = y^2/2$ to obtain for $z \ge 0$,

$$G_{1/2}(z) \;=\; \frac{1}{\Gamma(1/2)}\int_0^z v^{-1/2}\,e^{-v}\,dv \;=\; \frac{1}{\sqrt{\pi}}\int_0^{\sqrt{2z}} \sqrt{2}\,e^{-y^2/2}\,dy$$

$$= \;\sqrt{\frac{2}{\pi}}\cdot\sqrt{2\pi}\cdot\big(\Phi(\sqrt{2z}) - \Phi(0)\big) \;=\; 2\Phi(\sqrt{2z}) - 1$$

One further expected-lifetime calculation with a common type of distribution gives results which simplify dramatically and become amenable to numerical calculation. Suppose that the lifetime random variable T is assumed *lognormally distributed* with parameters m, σ^2. Then the expected remaining lifetime of a life aged x is

$$E(T-x\,|\,T\ge x) = \frac{1}{S(x)}\int_x^\infty t\,\frac{d}{dt}\Phi(\frac{\log(t)-\log(m)}{\sigma})\,dt \;-\; x$$

Now change variables by $y = (\log(t)-\log(m))/\sigma = \log(t/m)/\sigma$, so that $t = m\,e^{\sigma y}$, and define in particular

$$x' \;=\; \frac{\log(x)-\log(m)}{\sigma}$$

Recalling that $\Phi'(z) = \exp(-z^2/2)/\sqrt{2\pi}$, we find

$$E(T-x\,|\,T\ge x) \;=\; \frac{1}{1-\Phi(x')}\int_{x'}^\infty \frac{m}{\sqrt{2\pi}}\,e^{\sigma y - y^2/2}\,dy \;-\; x$$

The integral simplifies after completing the square $\sigma y - y^2/2 = \sigma^2/2 - (y-\sigma)^2/2$ in the exponent of the integrand and changing variables by $z = y - \sigma$. The result is:

$$E(T-x\,|\,T\ge x) \;=\; \frac{me^{\sigma^2/2}}{1-\Phi(x')}\big(1-\Phi(x'-\sigma)\big) \;-\; x \quad , \qquad x' \;=\; \frac{\log(x/m)}{\sigma}$$

3.7 *Exercise Set 3*

(1). Show that: $\frac{\partial}{\partial x}\,{}_tp_x \;=\; {}_tp_x\cdot(\mu_x - \mu_{x+t})$.

(2). For a certain value of x, it is known that ${}_tq_x = kt$ over the time-interval $t\in[0,3]$, where k is a constant. Express μ_{x+2} as a function of k.

(3). Suppose that an individual aged 20 has random lifetime (= exact age at death) T with continuous density function

$$f_T(t) \;=\; 0.02\,(t-20)\,e^{-(t-20)^2/100} \quad , \qquad t > 20$$

(a) If this individual has a contract with your company that you must pay his heirs $\$10^6 \cdot (1.4 - T/50)$ on the date of his death between ages 20 and 70, then what is the expected payment ?

(b) If the value of the death-payment described in (a) should properly be discounted by the factor $\exp(-0.08(T-20))$ (i.e. by the effective interest rate of $e^{.08} - 1$ per year) to calculate the present value of the payment, then what is the expected present value of the insurance contract ?

Hint for both parts: After a change of variables, the integral in (a) can be evaluated in terms of incomplete Gamma integrals $\int_c^\infty s^{\alpha-1} e^{-s} ds$, where the complete Gamma integrals (for c=0) are known yield the **Gamma function** $\Gamma(\alpha) = (\alpha - 1)!$, for integer $\alpha > 0$.

(4). Suppose that a life-table mortality pattern is this: from ages 20 through 60, twice as many lives die in each 5-year period as in the previous five-year period. Find the probability that a life aged 20 will die between exact ages 40 and 50. If the force of mortality can be assumed constant over each five-year age period (20-24, 25-29, etc.), and if you are told that $l_{60}/l_{20} = 0.8$, then find the probability that a life aged 20 will survive at least until exact age 48.0 .

(5). Obtain an expression for μ_x if $l_x = k\, s^x\, w^{x^2}\, g^{c^x}$, where k, s, w, g, c are positive constants.

(6). Show that: $\int_0^\infty l_{x+t}\, \mu_{x+t}\, dt \;=\; l_x$.

(7). A man wishes to accumulate $\$50,000$ in a fund at the end of 20 years. If he deposits $\$1000$ in the fund at the end of each of the first 10 years and $\$1000 + x$ in the fund at the end of each of the second 10 years, then find x to the nearest dollar, where the fund earns an effective interest rate of 6% .

(8). Express in terms of annuity-functions $a^{(m)}_{\overline{N}|}$ the present value of an annuity of $100 per month paid the first year, $200 per month for the second year, up to $1000 per month the tenth year. Find the numerical value of the present value if the effective annual interest rate is 7% .

(9). Find upper bounds for the following Binomial probabilities, and compare them with the exact values calculated via computer (e.g., using a spreadsheet or exact mathematical function such as **pbinom** in **Splus**) :

(a). The probability that in 1000 *Bernoulli* trials with success-probability 0.4, the number of successes lies outside the (inclusive) range [364, 446].

(b). The probability that of 1650 lives aged exactly 45, for whom ${}_{20}p_{45} = 0.72$, no more than 1075 survive to retire at age 65.

(10). If the force of mortality governing a cohort life-table is such that

$$\mu_t = \frac{2}{1+t} + \frac{2}{100-t} \quad \text{for real } t\,, \qquad 0 < t < 100$$

then find the number of deaths which will be expected to occur between ages 1 and 4, given that the radix l_0 of the life-table is $10,000$.

(11). Find the expected present value at 5% APR of an investment whose proceeds will with probability 1/2 be a payment of \$10,000 in exactly 5 years, and with the remaining probability 1/2 will be a payment of \$20,000 in exactly 10 years.
Hint: calculate the respective present values V_1, V_2 *of the payments in each of the two events with probability* 0.5, *and find the expected value of a discrete random variable which has values* V_1 *or* V_2 *with probabilities* 0.5 *each.*

(12). Derive the formula for the 2'nd and 3'rd moments (that is, $\int f(t)\, g(t)\, dt$ for $g(t) = t^r$, $r = 2, 3$,) of the Gamma(α, λ) density

$$f(t) = (\lambda^\alpha / \Gamma(\alpha!))\, t^{\alpha-1}\, e^{-\lambda t}\, I_{[t \ge 0]}$$

as a function of parameters α and λ. **Hint:** change variables by $y = \lambda t$.

(13). Derive the formula for the 2'nd and 3'rd moments of the

$$\text{Weibull}(\alpha, \beta) \text{ density} \qquad f(t) = \beta\, \alpha\, t^{\alpha-1}\, e^{-\beta t^\alpha}\, I_{[t \ge 0]}$$

as a function of parameters α and β. **Hint:** change variables appropriately and use the Gamma function.

3.8 Worked Examples

Example 1. Assume that a cohort life-table population satisfies $l_0 = 10^4$ *and*

$$d_x = \begin{cases} 200 & \textit{for} \quad 0 \le x \le 14 \\ 100 & \textit{for} \quad 15 \le x \le 48 \\ 240 & \textit{for} \quad 49 \le x \le 63 \end{cases}$$

(a) Suppose that an insurer is to pay an amount $\$100 \cdot (64 - X)$ *(without regard to interest or present values related to the time-deferral of the payment) for a newborn in the life-table population, if* X *denotes the attained integer age at death. What is the expected amount to be paid ?*

(b) Find the expectation requested in (a) if the insurance is purchased for a life currently aged exactly 10 *.*

(c) Find the expected present value at 4% *interest of a payment of* \$1000 *to be made at the end of the year of death of a life currently aged exactly* 20.

The first task is to develop an expression for survival function and density governing the cohort life-table population. Since the numbers of deaths are constant over intervals of years, the survival function is piecewise linear, and the life-distribution is *piecewise uniform* because the the density is piecewise constant. Specifically for this example, at integer values y,

$$l_y = \begin{cases} 10000 - 200y & \text{for } 0 \le y \le 15 \\ 7000 - 100(y - 15) & \text{for } 16 \le y \le 49 \\ 3600 - 240(y - 49) & \text{for } 50 \le y \le 64 \end{cases}$$

It follows that the terminal age for this population is $\omega = 64$ for this population, and $S(y) = 1 - 0.02\,y$ for $0 \le y \le 15$, $0.85 - 0.01\,y$ for $15 \le y \le 49$, and $1.536 - .024\,y$ for $49 \le y \le 64$. Alternatively, extending the function S linearly, we have the survival density $f(y) = -S'(y) = 0.02$ on $[0, 15)$, $= 0.01$ on $[15, 49)$, and $= 0.024$ on $[49, 64]$.

Now the expectation in (a) can be written in terms of the random lifetime variable with density f as

$$\int_0^{15} 0.02 \cdot 100 \cdot (64 - [y])\, dy \; + \int_{15}^{49} 0.01 \cdot 100 \cdot (64 - [y])\, dy$$

$$+ \int_{49}^{64} 0.024 \cdot 100 \cdot (64 - [y])\, dy$$

The integral has been written as a sum of three integrals over different ranges because the analytical form of the density f in the expectation-formula $\int g(y)f(y)dy$ is different on the three different intervals. In addition, observe that the integrand (the function $g(y) = 100(64-[y])$ of the random lifetime Y whose expectation we are seeking) itself takes a different analytical form on successive one-year age intervals. Therefore the integral just displayed can immediately be seen to agree with the summation formula for the expectation of the function $100(64-X)$ for the integer-valued random variable X whose probability mass function is given by

$$P(X = k) \;=\; d_k/l_0$$

The formula is

$$E(g(Y)) \;=\; E(100(64 - X)) \;=\; \sum_{k=0}^{14} 0.02 \cdot 100 \cdot (64 - k) \,+$$

$$\sum_{k=15}^{48} 0.01 \cdot 100 \cdot (64 - k) \,+\, \sum_{k=49}^{63} 0.024 \cdot 100 \cdot (64 - k)$$

Thus the solution to (a) is given (after the change-of-variable $j = 64 - k$), by

$$2.4 \sum_{j=1}^{15} j \;+\; \sum_{j=16}^{49} j \;+\; 2 \sum_{j=50}^{64} j$$

The displayed expressions can be summed either by a calculator program or by means of the easily-checked formula $\sum_{j=1}^{n} j \;=\; j(j+1)/2$ to give the numerical answer \$3103 .

The method in part (b) is very similar to that in part (a), except that we are dealing with conditional probabilities of lifetimes given to be at least 10 years long. So the summations now begin with $k = 10$, or alternatively end with $j = 64 - k = 54$, and the denominators of the conditional probabilities $P(X = k | X \geq 10)$ are $l_{10} = 8000$. The expectation in (b) then becomes

$$\sum_{k=10}^{14} \frac{200}{8000} \cdot 100 \cdot (64 - k) \,+\, \sum_{k=15}^{48} \frac{100}{8000} \cdot 100 \cdot (64 - k) \,+\, \sum_{k=49}^{63} \frac{240}{8000} \cdot 100 \cdot (64 - k)$$

which works out to the numerical value

$$3.0 \sum_{1}^{15} j \,+\, 1.25 \sum_{16}^{49} j \,+\, 2.5 \sum_{50}^{54} j \;=\; \$2391.25$$

Finally, we find the expectation in (c) as a summation beginning at $k = 20$ for a function $1000 \cdot (1.04)^{-X+19}$ of the random variable X with conditional probability distribution $P(X = k | X \geq 20) = d_k / l_{20}$ for $k \geq 20$. (Note that the function 1.04^{-X+19} is the present value of a payment of 1 at the end of the year of death, because the end of the age- X year for an individual currently at the 20^{th} birthday is $X - 19$ years away.) Since $l_{20} = 6500$, the answer to part (c) is

$$1000 \Big\{ \sum_{k=20}^{48} \frac{100}{6500} \, (1.04)^{19-k} \,+\, \sum_{k=49}^{63} \frac{240}{6500} \, (1.04)^{19-k} \Big\}$$

$$=\; 1000 \Big(\frac{1}{65} \, \frac{1 - 1.04^{-29}}{0.04} \,+\, \frac{24}{650} \, 1.04^{-29} \, \frac{1 - (1.04)^{-15}}{0.04} \Big) \;=\; 392.92$$

Example 2. *Find the change in the expected lifetime of a cohort life-table population governed by survival function* $S(x) = 1 - (x/\omega)$ *for* $0 \leq x \leq \omega$ *if* $\omega = 80$ *and*

(a) the force of mortality $\mu(y)$ *is multiplied by* 0.9 *at all exact ages* $y \geq 40$, *or*

(b) the force of mortality $\mu(y)$ *is decreased by the constant amount* 0.1 *at all ages* $y \geq 40$.

The force of mortality here is

$$\mu(y) \;=\; -\frac{d}{dy} \ln(1 - y/80) \;=\; \frac{1}{80 - y}$$

So multiplying it by 0.9 at ages over 40 changes leaves unaffected the density of $1/80$ for ages less than 40, and for ages y over 40 changes the density from $f(y) = 1/80$ to

$$f^*(y) \;=\; -\frac{d}{dy}\Big(S(40) \, \exp(-0.9 \int_{40}^{y} (80 - z)^{-1} \, dz) \Big)$$

$$= \; -\frac{d}{dy}\Big(0.5\, e^{0.9\, \ln((80-y)/40)} \Big) \; = \; -0.5\, \frac{d}{dy}\left(\frac{80-y}{40}\right)^{0.9}$$

$$= \; \frac{0.9}{80}\,(2-y/40)^{-0.1}$$

Thus the expected lifetime changes from $\int_0^{80}(y/80)\,dy = 40$ to

$$\int_0^{40}(y/80)\,dy \; + \; \int_{40}^{80} y\,\frac{0.9}{80}\,(2-y/40)^{-0.1}\,dy$$

Using the change of variable $z = 2 - y/40$ in the last integral gives the expected lifetime $= 10 + .45(80/.9 - 40/1.9) = 40.53$.

Example 3. Suppose that you have available to you two investment possibilities, into each of which you are being asked to commit \$5000. *The first investment is a risk-free bond (or bank savings-account) which returns compound interest of* 5% *for a* 10*-year period. The second is a 'junk bond' which has probability* 0.6 *of paying* 11% *compound interest and returning your principal after 10 years, probability* 0.3 *of paying yearly interest at* 11% *for* 5 *years and then returning your principal of* \$5000 *at the end of the* 10^{th} *year with no further interest payments, and probability* 0.1 *of paying yearly interest for* 3 *years at* 11% *and then defaulting, paying no more interest and not returning the principal. Suppose further that the going rate of interest with respect to which present values should properly be calculated for the next* 10 *years will either be* 4.5% *or* 7.5%, *each with probability* 0.5. *Also assume that the events governing the junk bond's paying or defaulting are independent of the true interest rate's being* 4.5% *versus* 7.5% *for the next* 10 *years. Which investment provides the better expected return in terms of current (time-*0*) dollars ?*

There are six relevant events, named and displayed along with their probabilities in the following table, corresponding to the possible combinations of true interest rate (Low versus High) and payment scenarios for the junk bond (Full payment, Partial interest payments with return of principal, and Default after 3 years' interest payments):

Event Name	Description	Probability
A_1	Low $\cap$ Full	0.30
A_2	Low $\cap$ Partial	0.15
A_3	Low $\cap$ Default	0.05
A_4	High $\cap$ Full	0.30
A_5	High $\cap$ Partial	0.15
A_6	High $\cap$ Default	0.05

Note that because of independence (first defined in Section 1.1), the probabilities of intersected events are calculated as the products of the separate probabilities, e.g.,

$$P(A_2) \;=\; P(Low) \,\cdot\, P(Partial) \;=\; (0.5) \,\cdot\, (0.30) \;=\; 0.15$$

Now, under each of the events $A_1, A_2, A_3,$ the present value of the first investment (the risk-free bond) is

$$5000 \left\{ \sum_{k=1}^{10} 0.05\,(1.045)^{-k} \;+\; (1.045)^{-10} \right\} \;=\; 5197.82$$

On each of the events $A_4, A_5, A_6,$ the present value of the first investment is

$$5000 \left\{ \sum_{k=1}^{10} 0.05\,(1.075)^{-k} \;+\; (1.075)^{-10} \right\} \;=\; 4141.99$$

Thus, since

$$P(Low) \;=\; P(A_1 \cup A_2 \cup A_3) \;=\; P(A_1) + P(A_2) + P(A_3) \;=\; 0.5$$

the overall expected present value of the first investment is

$$0.5 \,\cdot\, (5197.82 + 4141.99) \;=\; 4669.90$$

Turning to the second investment (the junk bond), denoting by PV the present value considered as a random variable, we have

$$
\begin{aligned}
E(PV \,|\, A_1)/5000 &= 0.11 \sum_{k=1}^{10} (1.045)^{-k} + (1.045)^{-10} = 1.51433 \\
E(PV \,|\, A_4)/5000 &= 0.11 \sum_{k=1}^{10} (1.075)^{-k} + (1.075)^{-10} = 1.24024 \\
E(PV \,|\, A_2)/5000 &= 0.11 \sum_{k=1}^{5} (1.045)^{-k} + (1.045)^{-10} = 1.12683 \\
E(PV \,|\, A_5)/5000 &= 0.11 \sum_{k=1}^{5} (1.075)^{-k} + (1.075)^{-10} = 0.93024 \\
E(PV \,|\, A_3)/5000 &= 0.11 \sum_{k=1}^{3} (1.045)^{-k} = 0.302386 \\
E(PV \,|\, A_6)/5000 &= 0.11 \sum_{k=1}^{3} (1.075)^{-k} = 0.286058
\end{aligned}
$$

Therefore, we conclude that the overall expected present value $E(PV)$ of the second investment is

$$
\sum_{i=1}^{6} E(PV \cdot I_{A_i}) = \sum_{i=1}^{6} E(PV|A_i)\, P(A_i) = 5000 \cdot (1.16435) = 5821.77
$$

So, although the first-investment is 'risk-free', it does not keep up with inflation in the sense that its present value is not even as large as its starting value. The second investment, risky as it is, nevertheless beats inflation (i.e., the expected present value of the accumulation after 10 years is greater than the initial face value of \$5000) although with probability $P(\mathit{Default}) = 0.10$ the investor may be so unfortunate as to emerge (in present value terms) with only 30% of his initial capital.

3.9 Appendix to Chapter 3: Large Deviation Probabilities

Theorem 3.2 *(Large Deviation Inequalities) Suppose that* X *is a Binomial*(N,p) *random variable, denoting the number of successes in* N *Bernoulli*(p) *trials. If* $1 > b > p > c > 0$, *then*

$$P(X \geq Nb) \leq \exp\Big\{ -N\Big[b \ln\Big(\frac{b}{p}\Big) \ + \ (1-b)\ln\Big(\frac{1-b}{1-p}\Big)\Big]\Big\}$$

$$P(X \leq Nc) \leq \exp\Big\{ -N\Big[c \ln\Big(\frac{c}{p}\Big) \ + \ (1-c)\ln\Big(\frac{1-c}{1-p}\Big)\Big]\Big\}$$

Proof. After the first inequality in (a) is proved, the second inequality will be derived from it. Since the event $[X \geq Nb]$ is the union of the disjoint events $[X = k]$ for $k \geq Nb$, which in turn consist of all outcome-strings $(a_1, \ldots, a_N) \in \{0,1\}^N$ for which $\sum_{j=1}^N a_j = k \geq Nb$, a suitable subset of the binomial probability mass function values $p_X(k)$ are summed to provide

$$P(X \geq Nb) = \sum_{k: Nb \leq k \leq N} P(X = k) = \sum_{k \geq Nb} \binom{N}{k} p^k \,(1-p)^{N-k}$$

For every $s > 1$, this probability is

$$\leq \sum_{k \geq Nb} \binom{N}{k} p^k \,(1-p)^{N-k}\, s^{k-Nb} \ = \ s^{-Nb} \sum_{k \geq Nb} \binom{N}{k} (ps)^k \,(1-p)^{N-k}$$

$$\leq \ s^{-Nb} \sum_{k=0}^{N} \binom{N}{k} (ps)^k \,(1-p)^{N-k} \ = \ s^{-Nb}\,(1-p+ps)^N$$

Here extra terms (corresponding to $k < Nb$) have been added in the next-to-last step, and the binomial theorem was applied in the last step. The trick in the proof comes now: since the left-hand side of the inequality does not involve s while the right-hand side does, and since the inequality must be valid for every $s > 1$, it remains valid if the right-hand side is minimized over s. The calculus minimum does exist and is unique, as you can check by calculating that the second derivative in s is always positive. The minimum occurs where the first derivative of the logarithm of the last expression is 0, i.e., at $s = b(1-p)/(p(1-b))$. Substituting this value for s yields

$$P(X \geq Nb) \;\leq\; \left(\frac{b\,(1-p)}{p\,(1-b)}\right)^{-Nb} \left(\frac{1-p}{1-b}\right)^{N}$$

$$=\; \exp\left(-N\Big[b\,\ln\left(\frac{b}{p}\right) + (1-b)\,\ln\left(\frac{1-b}{1-p}\right)\Big]\right)$$

as desired.

The second inequality follows from the first. Replace X by $Y = N - X$. Since Y also is a count of 'successes' in *Bernoulli*$(1-p)$ trials, where the 'successes' counted by Y are precisely the 'failures' in the *Bernoulli* trials defining X, it follows that Y also has a *Binomial*(N, q) distribution, where $q = 1 - p$. Note also that $c < p$ implies $b = 1 - c > 1 - p = q$. Therefore, the first inequality applied to Y instead of X with $q = 1-p$ replacing p and $b = 1 - c$, gives the second inequality for $P(Y \geq Nb) = P(X \leq Nc)$.

Note that for all r between 0, 1, the quantity $r \ln \frac{r}{p} + (1-r) \ln \frac{1-r}{1-p}$ as a function of r is convex and has a unique minimum of 0 at $r = p$. Therefore when $b > p > c$, the upper bound for $N^{-1} \ln P([X \geq bN] \cup [X \leq cN])$ is strictly negative and does not involve N. For an improved estimate of the probability bounded in Theorem 3.1, let $\delta \in (0,\, \min(p, 1-p))$ be arbitrarily small, choose $b = p + \delta$, $c = p - \delta$, and combine the inequalities of part (a) to give the precise estimate:

$$P(|\frac{X}{N} - p| \geq \delta) \leq 2 \cdot \exp(-Na) \tag{3.30}$$

where

$$\begin{aligned} a \;&=\; \min\Big((p+\delta)\,\ln(1+\tfrac{\delta}{p}) + (1-p-\delta)\,\ln(1-\tfrac{\delta}{1-p})\,, \\ &\quad (p-\delta)\ln(1-\tfrac{\delta}{p}) + (1-p+\delta)\ln(1+\tfrac{\delta}{1-p})\Big) \;>\; 0 \end{aligned} \tag{3.31}$$

This last inequality gives a much stronger and numerically more useful upper bound than Theorem 3.1 on the probability with which th *relative frequency of success* X/N differs from the true probability p of success by as much as δ. The probabilities of such *large deviations* between X/N and δ are in fact exponentially small as a function of the number N. □

If the probabilities $P(|X/N - p| \geq \delta)$ in Theorem 3.1 are generally much smaller than the upper bounds given for them, then why are those bounds of interest ? (These are 1 minus the probabilities illustrated in Table 1.) First, they provide relatively quick hand-calculated estimates showing that large batches of independentcoin-tosses are extremely unlikely to yield relative frequencies ofheads much different from the true probability or limiting relative frequency of heads. Another, more operational, way to render this conclusion of Theorem 3.1 is that two very large insured cohorts with the same true survival probabilities are very unlikely to have materially different survival experience. However, as Table 1 illustrates, for practical purposes the normal approximation to the binomial probabilities of large discrepancies from the expectation is generally much more precise than the large deviation bounds of Theorem 3.2.

The bounds given in Theorem 3.2 get small with large N *much* more rapidly than the simpler bounds based on the *Chebychev inequality* used in proving Theorem 3.1 (*cf.* Hogg and Tanis 1997). We can tolerate the apparent looseness in the bounds because in actuarial applications involving really extreme tail probabilities (e.g. Slud and Hoesman 1989), it can be shown that the exponential rate of decay as a function of N in the true tail-probabilities $P_N = P(X \geq Nb)$ or $P(X \leq Nc)$ in Theorem 3.2 (i.e., the constants appearing in square brackets in the exponents on the right-hand sides of the bounds) are exactly the right ones: no larger constants replacing them could give correct bounds.

3.10 Useful Formulas from Chapter 3

Binomial(N,p) probability $\quad P(X=k) = \binom{N}{k} p^k\,(1-p)^{N-k}$

p. 75

Discrete r.v. Expectation $\quad E(c(Z)) = \sum_{i=1}^{m} c(z_i)\, p_Z(z_i)$

p. 85

Non-neg. integer-valued r.v. Expectation $\quad E(Z) \;=\; \sum_{j=1}^{\infty} P(Z \ge j)$

p. 90

Curtate life expectancy $\quad \mathrm{e}_x \;=\; \sum_{j=1}^{\omega-x-1} {}_jp_x$

p. 91

$${}_kp_x \;=\; p_x\; p_{x+1}\; p_{x+2}\; \cdots\; p_{x+k-1} \quad , \qquad k \ge 1 \quad \text{integer}$$

p. 92

$${}_{k/m}p_x \;=\; \prod_{j=0}^{k-1} {}_{1/m}p_{x+j/m} \quad , \qquad k \ge 1 \quad \text{integer}$$

p. 92

(i) Piecewise Unif. $\quad S(x+t) \;=\; tS(x+1)+(1-t)S(x) \;\;, \quad x$ integer , $\quad t \in [0,1]$

p. 94

(ii) Piecewise Const. μ : $\ln S(x+t) \;=\; t \ln S(x+1) + (1-t) \ln S(x)$

p. 94

(iii) Balducci hypothesis $\dfrac{1}{S(x+t)} \;=\; \dfrac{t}{S(x+1)} \;+\; \dfrac{1-t}{S(x)}$

p. 94

$${}_tp_x \;=\; \frac{S(x) - t(S(x+1) - S(x))}{S(x)} \;=\; 1 - t\, q_x \qquad \text{under (i)}$$

p. 95

$${}_tp_x \;=\; \frac{S(x+t)}{S(x)} \;=\; \left(e^{-\mu(x)}\right)^t \;=\; (1-q_x)^t \qquad \text{under (ii)}$$

p. 95

$${}_tp_x \;=\; \frac{S(x+t)}{S(x+1)}\,\frac{S(x+1)}{S(x)} \;=\; \frac{1-q_x}{1-(1-t)q_x} \qquad \text{under (iii)}$$

p. 96

Complete life expectancy $\mathring{e}_x \;=\; \int_0^{\omega - x} {}_sp_x\, ds$

p. 97

$$\Gamma(\alpha) \;=\; \int_0^\infty x^{\alpha-1}\, e^{-x}\, dx$$

p. 98

$$\Phi(x) \;=\; \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-z^2/2}\, dz$$

p. 98

Chapter 4

Expected Present Values of Insurance Contracts

We are now ready to draw together the main strands of the development so far: (A) expectations of discrete and continuous random variables defined as functions of a life-table waiting time T until death, and (B) discounting of future payment (streams) based on interest-rate assumptions. We first define the contractual terms of and discuss relations between the major sorts of insurance, endowment and life annuity contracts, and next to use interest theory to define the present value of the contractual payment stream *by the insurer* as a nonrandom function of the random individual lifetime T. In each case, this leads to a formula for the *expected* present value of the payout by the insurer, an amount called the **net single premium** or **net single risk premium** of the contract because it is the single cash payment by the insured at the beginning of the insurance period which would exactly compensate for the average of the future payments which the insurer will have to make.

The details of the further mathematical discussion fall into two parts: first, the specification of formulas in terms of cohort life-table quantities for net single premiums of insurances and annuities which pay only at whole-year intervals; and second, the application of the various survival assumptions concerning interpolation between whole years of age, to obtain the corresponding formulas for insurances and annuities which have m payment times per year. We close this Chapter with a discussion of instantaneous-payment insurance, continuous-payment annuity, and mean-residual-life formulas, all of which involve continuous-time expectation integrals. We also relate these expectations with their m-payment-per-year discrete analogues, and compare the corresponding integral and summation formulas.

Similar discussions can be found in the books *Life Contingencies* by Jordan (1967) and *Actuarial Mathematics* by Bowers et al. (1997). The approach here differs in unifying concepts by discussing together all of the different contracts, first in the whole-year case, next under the interpolation assumption (i) with m payment periods per year, and finally in the instantaneous case.

4.1 Preliminaries

The topic of study in this Chapter is contracts resulting in contingent payment streams depending on the age at death T of a single individual. There are three major types of contracts to consider: insurance, life annuities, and endowments. More complicated kinds of contracts — which we do not discuss in detail — can be obtained by combining (superposing or subtracting) these in various ways. Of course, other types of insurances on lives do exist, which pay only when a single life terminates due to a specified cause or set of causes (insurances based on *multiple decrement tables*), or which contractually involve more than one life (for example husband-wife pairs), insurances and annuities on *joint lives*. For these further topics, we refer the reader to Bowers et al. (1997). Only single life contracts without distinctions between causes of mortality are discussed here.

In what follows, we adopt several uniform notations and assumptions. Let x denote the initial integer age of the holder of the insurance, life annuity, or endowment contract, assuming for convenience that the contract is initiated on the holder's birthday. Fix a nonrandom effective interest rate i , and retain the notation $v = (1+i)^{-1}$, together with the other notations previously discussed for annuities of nonrandom duration. Next, denote by m the number of payment-periods per year, all times being measured from the date of policy initiation. Thus, for given m, an insurance will pay off at the end of the fraction $1/m$ of a year during which death occurs, and life-annuities pay regularly m times per year until the annuitant dies.

The *term* or *duration* n of the contract will always be assumed to be an integer multiple of $1/m$. Note that policy durations are all measured from policy initiation, and therefore are smaller by x than the exact age of the policyholder at termination. Thus, we refer to **policy time** for the life aged x as the time scale with origin at policy initiation, assumed to be the x birthday of the policy-holder, and at chronological age t for the policyholder, we say the **policy age** is $t - x$.

The random exact age at which the policyholder dies is denoted by T, and all of the contracts under discussion have the property that T is the only random variable upon which either the amount or time of payment can depend. In examples based on m payment periods per year, the amount of the payment will be assumed to depend on T only through the greatest integer less than or equal to mT.

$$\text{If} \quad \frac{k}{m} \le T - x < \frac{k+1}{m} \quad , \qquad \text{then} \qquad T_m \equiv \frac{[mT]}{m} = x + \frac{k}{m} \tag{4.1}$$

denotes the attained age at death measured in completed $(1/m)$'th years.

As before, the survival function of T is denoted $S(t)$, and the density by $f(t)$. The probabilities of the various possible occurrences under the policy are therefore calculated using the conditional probability distribution of T given that $T \ge x$, which has continuous probability density $f(t)/S(x)$ at all times $t \ge x$. When the amounts and times of payments under a contract depend only on the whole-year age at death ($m = 1$), all probabilities and conditional expectations refer only to the discrete random variable $[T] = T_1$ and are calculated in terms of the conditional probability mass function, given for nonnegative integers x, k by

$$\begin{aligned} P([T] = x + k \,|\, [T] \ge x) &= P(k \le T - x < k + 1 \,|\, T \ge x) \\ &= {}_k p_x - {}_{k+1} p_x = {}_k p_x \, q_{x+k} \end{aligned} \tag{4.2}$$

and depends only on the cohort life-table entries, since the displayed conditional probability is precisely d_{x+k}/l_x.

In the setting where the insurance and annuity contracts are formulated in terms of m possible death and payment periods per year, the probability calculations necessarily involve interpolations of the survival function $S(t)$ within whole years of age, but only to values t of the form $x + k/m$, k

an integer. Then all conditional expecations are calculated in terms of the probability mass function of the random variable T_m given as in (4.1):

$$P(T_m = x + \frac{k}{m} \Big| T_m \ge x) \;=\; P(\frac{k}{m} \le T - x < \frac{k+1}{m} \Big| T \ge x)$$

$$= \frac{1}{S(x)} \left[S(x + \frac{k}{m}) - S(x + \frac{k+1}{m}) \right] \;=\; {}_{k/m}p_x \;-\; {}_{(k+1)/m}p_x$$

$$= P(T \ge x + \frac{k}{m} \Big| T \ge x) \cdot P(T < x + \frac{k+1}{m} \Big| T \ge x + \frac{k}{m})$$

$$= \; {}_{k/m}p_x \;\cdot\; {}_{1/m}q_{x+k/m} \tag{4.3}$$

4.2 Insurance & Life Annuity Contracts

An **Insurance** contract is an agreement to pay a *face amount* — perhaps modified by a specified function of the time until death — if the insured, a life aged x, dies at any time during a specified period, the *term* of the policy, with payment to be made at the end of the $1/m$ year within which the death occurs. Usually the payment will simply be the face amount $F(0)$, but for example in *decreasing term* policies the payment will decrease linearly as a function of T_m over the term of the policy. The insurance is said to be a *whole-life* policy if the duration $n = \infty$, and a *term insurance* otherwise.) The general form of this contract, for a specified term $n \le \infty$, payment-amount function $F(\cdot)$, and number m of possible payment-periods per year, is to

pay $F(T - x)$ at $T_m - x + \frac{1}{m}$ time units following policy initiation, if $T \in [x, x + n)$. **(Ins$_m$)**

Specializing to the case $m = 1$, so that $T_m = [T]$ is the whole-year age at death, the present value of the insurance company's payment under the contract **(Ins)** or **(Ins$_1$)** is

$$\begin{cases} F([T - x])\, v^{[T-x]+1} & \text{if} \quad x \le T < x + n \\ 0 & \text{otherwise} \end{cases} \tag{4.4}$$

The simplest and most common case of this contract and formula arise when the face-amount $F(0) = F$ is the constant amount paid whenever a death within the term occurs. Then the payment is F, with present value $F\,v^{[T]-x+1}$, if $x \le T < x+n$, and both the payment and present value are 0 otherwise. In this case, with $F \equiv 1$, the *net single premium* has the standard notation $\mathrm{A}^1_{x:\overline{n}|}$. When the insurance is **whole-life** $(n = \infty)$, the subscript n and bracket $_{\overline{n}|}$ and superscript ^{1} over x are dropped, so that $\mathrm{A}^1_{x:\overline{\infty}|} \equiv \mathrm{A}_x$.

A **Life Annuity** contract is an agreement to pay a scheduled payment to the policyholder at every interval $1/m$ of a year while the annuitant is alive, up to a maximum number of nm payments. Again the payment amounts are ordinarily constant, but in principle any nonrandom time-dependent schedule of payments $F(k/m)$ can be used, where $F(s)$ is a fixed function and s ranges over multiples of $1/m$. To avoid ambiguity, we adopt the convention that in the finite-term or *temporary* life annuities, *either* $F(0) = 0$ *or* $F(n) = 0$. In this general setting, the life annuity contract requires the insurer to

pay amounts $F(k/m)$ at policy times $\frac{k}{m}$,
$0 \le \frac{k}{m} \le T - x$, at most nm payments. **(LifAnn$_m$)**

As in the case of annuities certain (the nonrandom annuities discussed in Chapter 1), we refer to life annuities with first payment at time 0 as (life) *annuities-due* and to those with first payment at time $1/m$ (and therefore last payment at time n in the case of a finite term n over which the annuitant survives) as (life) *annuities-immediate*.

Again specialize to the case $m = 1$: under the contract **(LifAnn)** or **(LifAnn$_1$)**, up to n payments are made (since $F(0) = 0$ or $F(n) = 0$), and the present value of the insurance company's payment under the life annuity contract is

$$\sum_{k=0}^{[T-x]} F(k)\,v^k \tag{4.5}$$

Here the situation is definitely simpler in the case where the payment amounts $F(k)$ are *level* or constant, either $F(k) \equiv F$ for $k = 0, 1, \ldots, n-1$, or $F(k) \equiv F$ for $k = 1, 2, \ldots, n$. In the first of these cases, the life-annuity-due payment stream becomes an annuity-due certain (the kind discussed previously under the Theory of Interest) as soon as the random variable T is fixed. Indeed, if we replace $F(k)$ by 1 for $k = 0, 1, \ldots, n-1$, and by 0 for larger indices k, then the present value in equation (4.5) is $\ddot{a}_{\overline{\min([T]-x+1,\, n)}|}$, and its expected present value (= net single premium) is denoted $\ddot{a}_{x:\overline{n}|}$. In the case of temporary life annuities-immediate, which have payments commencing at policy time 1 and continuing annually either until death or for a total of n payments, the present value formula as a function of $[T]$ is the certain annuity immediate $a_{\overline{\min([T]-x,\, n)}|}$, since this is the present value of the pattern of annual unit payments starting at policy time 1 up to $[T-x]$ or n, whichever comes first. The expected-present value notation for temporary life annuities immediate is $a_{x:\overline{n}|}$.

The third major type of insurance contract is the **Endowment**, which pays a contractual face amount $F = F(0)$ at the end of n policy years if the policyholder initially aged x survives to age $x+n$. This contract is the simplest, since neither the amount nor the time of payment, only whether the payment is made at all, is uncertain. The pure endowment contract commits the insurer to

$$\text{pay an amount } F \text{ at policy time } n \text{ if } T \geq x+n \qquad \textbf{(Endow)}$$

The present value of the pure endowment contract payment is

$$F\, v^n \qquad \text{if} \quad T \geq x+n, \quad 0 \qquad \text{otherwise} \tag{4.6}$$

The net single premium or expected present value for a pure endowment contract with face amount $F = 1$ is denoted $A_{x:\overline{n}|}^{\;\;\,1}$ or ${}_nE_x$ and is evidently equal to

$$A_{x:\overline{n}|}^{\;\;\,1} \;=\; {}_nE_x \;= v^n\, {}_np_x \tag{4.7}$$

The other contract frequently referred to in actuarial texts is the **Endowment Insurance**, which for a life aged x and term n is simply the sum of the pure endowment and the term insurance, both with term n and the same face amount 1. Here the standard contract with m payment periods per year and unit level face amount calls for the insurer to

$$\text{pay } 1 \text{ at policy time} \begin{cases} T_m - x + 1/m & \text{if} \quad T < x+n \\ n & \text{if} \quad T \geq x+n \end{cases} \quad (\textbf{EndIns}_m)$$

Simplifying to the case of a single payment per year ($m = 1$), we express the present value of this contract as v^n on the event $[T \geq n]$ and as $v^{[T-x]+1}$ on the complementary event $[T < n]$. Note that $[T - x] + 1 \leq n$ whenever $T - x < n$. Thus, in both cases, the present value is given by

$$v^{\min([T-x]+1,\, n)} \tag{4.8}$$

The expected present value of the unit endowment insurance (still in the case $m = 1$) is denoted $A_{x:\overline{n|}}$. The notations for the net single premium of the term insurance and of the pure endowment are intended to be mnemonic, respectively denoting the portions of the endowment insurance net single premium respectively triggered by the expiration of life — in which case the superscript 1 is positioned above the x —or by the expiration of the fixed term, in which case the superscript 1 is positioned above the term n.

Another example of an insurance contract which does not need separate treatment, because it is built up simply from the contracts already described, is the *n-year deferred insurance*. This policy pays a constant face amount at the end of the current fraction $1/m$ year containing the policy time $T - x$, but only if death occurs after the deferral time n , i.e., after age $x + n$ for a new policyholder aged precisely x. When the face amount is 1, the contractual payout is precisely the difference between the unit whole-life insurance and the n-year unit term insurance. When $m = 1$, the notation and formula for the net single premium is

$${}_nA_x \;=\; A_x \;-\; A^1_{x:\overline{n|}} \tag{4.9}$$

4.2.1 Formal Relations between Risk Premiums, $m = 1$

In this subsection, we collect a few useful identities connecting the different types of risk premiums for contracts with $m = 1$ payment period per year. These identities therefore hold and can be used in computational formulas without regard to particular life-table interpolation assumptions. The first, which we have already seen, is the definition of endowment insurance as the superposition of a constant-face-amount term insurance with a pure endowment of the same face amount and term. In terms of net single premiums, this identity is

$$\mathrm{A}_{x:\overline{n|}} \;=\; \mathrm{A}^1_{x:\overline{n|}} + \mathrm{A}_{x:\overset{1}{\overline{n|}}} \tag{4.10}$$

Another important identity concerns the relation between expected present values of endowment insurances and life annuities. The great generality ofthe identity arises from the fact we saw in the discussion following (4.5) that, for a fixed value of the random lifetime T, the present value of the life annuity-due payout coincides with the annuity-due certain, and is given by

$$\ddot{a}_{\overline{\min([T-x]+1,n)}|} = \frac{1 - v^{\min([T-x]+1,\, n)}}{d}$$

where the second expression follows from the first via formula (2.4). Thus, the unit life annuity-due has present value which is a simple linear function of $v^{\min([T-x]+1,n)}$ which we saw in (4.8) is the present value of the unit endowment insurance. Taking expectations (over values of the random variable T, conditionally given $T \geq x$) in the present value formula, and substituting $\mathrm{A}^{(m)}_{x:\overline{n}|}$ as expectation of (4.8), then yields:

$$\ddot{a}_{x:\overline{n}|} = E_x\Big(\frac{1 - v^{\min([T-x]+1,\, n)}}{d}\Big) = \frac{1 - A_{x:\overline{n}|}}{d} \tag{4.11}$$

where recall that $E_x(\,\cdot\,)$ denotes the conditional expectation $E(\,\cdot\,|\, T \geq x)$. A more common and algebraically equivalent form of the identity (4.11) is

$$d\, \ddot{a}_{x:\overline{n}|} + \mathrm{A}_{x:\overline{n}|} = 1 \tag{4.12}$$

To obtain a corresponding identity relating net single premiums for life annuities-immediate to those of endowment insurances, we need to relate the risk premiums for life annuities-immediate to those of life annuities-due. Unlike the case of annuities-certain (i.e., nonrandom-duration annuities), one cannot simply multiply the present value of the life annuity-due for fixed T by the discount-factor v in order to obtain the corresponding present value for the life annuity-immediate with the same term n. The difference arises because the payment streams (for the life annuity-due deferred 1 year and the life-annuity immediate) end at the same *time* rather than with the same number of payments when death occurs before time n. The correct conversion-formula is obtained by treating the life annuity-immediate of term n as paying, in all circumstances, a present value of 1 (equal to the cash payment at policy initiation) less than the life annuity-due with term $n+1$. Taking expectations leads to the formula

$$a_{x:\overline{n}|} = \ddot{a}_{x:\overline{n+1}|} - 1 \tag{4.13}$$

Now, combining this conversion-formula with the identity (4.11), we find

$$a_{x:\overline{n}|} = \ddot{a}_{x:\overline{n+1}|} - 1 = \frac{1 - A_{x:\overline{n+1}|}}{d} - 1 = \frac{1}{i} - \frac{1}{d} A_{x:\overline{n+1}|} \tag{4.14}$$

and

$$d\, a_{x:\overline{n}|} + \mathrm{A}_{x:\overline{n+1}|} = \frac{d}{i} = v \tag{4.15}$$

In these formulas, we have made use of the definition $1/d = (1+i)/i$, leading to the simplifications

$$1/d = 1/i + 1 \quad , \qquad i/d = 1 + i = v^{-1}$$

Since the n-year deferred insurance with risk premium (4.9) pays a benefit only if the insured survives at least n years, it can alternatively be viewed as an endowment with benefit equal to a whole life insurance to the insured (then aged $x+n$) after n years if the insured lives that long. With this interpretation, the n-year deferred insurance has net single premium $= {}_nE_x \cdot A_{x+n}$. This expected present value must therefore be equal to (4.9), providing the identity:

$$A_x - A^1_{x:\overline{n}|} = v^n\, {}_np_x \cdot A_{x+n} \tag{4.16}$$

4.2.2 Formulas for Net Single Premiums

All of the net single premiums (or risk premiums) considered so far are computable completely in terms of of life-table quantities ${}_jp_x$ and q_{x+j}. To emphasize the fact that these risk premiums depend on cohort life table quantities alone, this subsection collects the formulas for risk premiums of the insurance, annuity, and endowment contracts defined above, written explicitly as sums for the case $m = 1$. Recall for this purpose the conditional probability mass function (4.2) of $[T - x]$ given $T \geq x$.

Here and from now on, for an event B depending on the random lifetime T, the notation I_B denotes the so-called *indicator random variable* which is equal to 1 whenever T has a value such that the condition B is satisfied and is equal to 0 otherwise.

$$I_B \; = \; 1 \qquad \text{if condition } B \text{ holds} \; , \qquad = \; 0 \qquad \text{if not}$$

First, the expectation of the present value (4.4) of the random term insurance payment (with level face value $F(0) \equiv 1$) is

$$\mathrm{A}^{1}_{x:\overline{n}|} \; = \; E_x\left(v^{[T-x]+1}\, I_{\{T \le x+n\}}\right) \; = \; \sum_{k=0}^{n-1} v^{(k+1)/m}\; {}_kp_x \; q_{x+k} \tag{4.17}$$

The index k in the summation formula denotes the year of policy time within which death occurs, $k \le T - x < k+1$. The summation itself is the weighted sum, over all indices k such that $k < n$, of the present values v^{k+1} to be paid by the insurer in the event that the policy age at death falls in $[k,\, k+1)$, multiplied by the probability, given in formula (4.2), that this event occurs.

Putting together the formula (4.17) with the previous identity (4.10) provides us with a formula for the net single premium of the endowment insurance,

$$\mathrm{A}_{x:\overline{n}|} \; = \; \sum_{k=0}^{n-1} v^{k+1}\left({}_kp_x - {}_{k+1}p_x\right) + \; v^n \; {}_np_x \tag{4.18}$$

Next, to figure the expected present value of the life annuity-due with term n, note that payments of 1 occur at all policy ages k, $k = 0, \ldots, n-1$, for which $T - x \ge k$. Therefore, since the present values of these payments are v^k and the payment at policy time k is made with probability ${}_kp_x$,

$$\ddot{a}_{x:\overline{n}|} \; = \; E_x\Big(\sum_{k=0}^{n-1} v^k\, I_{\{T-x \ge k\}}\Big) \; = \; \sum_{k=0}^{n-1} v^k\; {}_kp_x \tag{4.19}$$

Finally the pure endowment has present value

$${}_nE_x \; = \; E_x\left(v^n\, I_{[T-x \ge n]}\right) \; = \; v^n \; {}_xp_n \tag{4.20}$$

Most generally of all, a contract which pays $G(k)$ at policy time k if the insured life initially aged x survives to age $x+k$, and which pays $F(k)$ at policy time $k+1$ if the insured life aged x dies at age $T \in [x+k,\, x+k+1)$, has net single (risk) premium equal to

$$E_x\Big(\sum_{k=0}^{\omega-x-1} \Big[G(k)\, v^k\, I_{\{T \geq x+k\}} \;+\; F(k)\, v^{k+1}\, I_{\{[T-x]=k\}} \Big] \Big)$$

$$= \sum_{k=0}^{\omega-x-1} {}_kp_x \Big[G(k)\, v^k \;+\; F(k)\, v^{k+1}\, q_{x+k} \Big] \tag{4.21}$$

where $P([T-x]=k \,|\, T \geq x)$ has been expressed as in (4.2). This setting, where both functions $G(k)$ and $F(k)$ could depend on a finite term parameter n, encompasses all of the insurances, life annuities, and endowments introduced so far in this Chapter.

The formulas (4.17) and (4.19) and (4.21) are benchmarks in the sense that they represent a complete solution to the problem of determining net single premiums without the need for interpolation of the life-table survival function between integer ages. However the insurance, life-annuity, and endowment-insurance contracts payable only at whole-year intervals are all slightly impractical as insurance vehicles. In the next section, we approach the calculation of net single premiums for the more realistic context of m-period-per-year insurances and life annuities, using only the standard cohort life-table data collected by integer attained ages.

4.3 Risk Premiums & Relations, $m > 1$

At this point, we return to the basic definitions of the standard insurance, annuity, and endowment contracts defined above, in order to extend the theoretical formulas, and identities to cover the case of general m-payment-period per year contracts.

The pure endowment contract **(Endow)** with present value formula (4.6), and net single premium notation and formula (4.7), does not require any separate discussion here, since it involved only a single potential payment at an integer policy time. It is therefore no different for general m than for $m = 1$.

Next we consider the pure term insurance **(Ins$_m$)** with term n and m payment periods per year, and level face amount. Recall that this contract, with unit face amount, pays 1 at the end of the $1/m$ year of death, if death occurs before policy time n. That is, the payment of 1 occurs at policy time k/m if $k/m \le T - x < (k+1)/m$, $k/m < n$. Accordingly, the present value of the insurer's payment is

$$\sum_{k=0}^{nm-1} v^{(k+1)/m} \, I_{\{k/m \le T-x<(k+1)/m\}} \tag{4.22}$$

and the net single premium or expected present value is

$$\mathrm{A}^{(m)\,1}_{x:\overline{n}|} \;=\; \sum_{k=0}^{nm-1} v^{(k+1)/m} \;{}_{k/m}p_x \;{}_{1/m}q_{x+k/m} \tag{4.23}$$

Here k/m in the summation formula denotes the beginning of the $1/m$ year of policy time within which death is to occur. Again the risk premium summation is the weighted sum, over all indices k such that $k/m < n$, of the present values $v^{(k+1)/m}$ to be paid by the insurer in the event that the policy age at death falls in $[k/m, \, (k+1)/m)$ multiplied by the probability, given in formula (4.3), that this event occurs.

To figure the expected present value of the life annuity-due with term n, note that payments of $1/m$ occur at all policy ages k/m, $k = 0, \ldots, nm-1$, for which $T - x \ge k/m$. Therefore, since the present values of these payments are $(1/m)\, v^{k/m}$ and the payment at k/m is made with probability ${}_{k/m}p_x$,

$$\ddot{a}^{(m)}_{x:\overline{n}|} \;=\; E_x \left(\sum_{k=0}^{nm-1} \frac{1}{m} \, v^{k/m} \, I_{[T-x \ge k/m]} \right) \;=\; \frac{1}{m} \sum_{k=0}^{nm-1} v^{k/m} \;{}_{k/m}p_x \tag{4.24}$$

The useful identities described in Section 4.2.1 above, connecting the different types of risk premiums for contracts with $m = 1$ payment period per year, all have extensions to general m payment per year contracts.

The first extension, analogous to (4.10), is the definition of endowment insurance as the superposition of a constant-face-amount term insurance with a pure endowment of the same face amount and term. Recall that the m-payment-period per year endowment insurance with term n and unit face amount pays 1 at policy time $(k+1)/m$ if $k/m \le T-x < (k+1)/m$ for $k = 0, 1, \ldots, nm-1$, and pays 1 at policy time n if $T \ge x+n$. The present value of the payout clearly has the single epxression $v^{\min(T_m - x + 1/m,\, n)}$. In terms of net single premiums, the notational identity is

$$\mathrm{A}^{(m)}_{x:\overline{n}|} \;=\; \mathrm{A}^{(m)\,1}_{\;x:\overline{n}|} + \mathrm{A}^{(m)}{}_{x:\overline{n}|}^{\;\;\;1} \;=\; \sum_{k=0}^{m(\omega-x)-1} v^{\min((k+1)/m,\, n)} \;{}_{k/m}p_x \;{}_{1/m}q_{x+k/m}$$

$$= \sum_{k=0}^{nm-1} v^{(k+1)/m} \;{}_{k/m}p_x \;{}_{1/m}q_{x+k/m} \;+\; v^n \;{}_np_x \tag{4.25}$$

Again we find a formula for the endowment insurance by a combining the identity (4.25) with the formula (4.23) for Insurance:

$$\mathrm{A}^{(m)}_{x:\overline{n}|} \;=\; \sum_{k=0}^{nm-1} v^{(k+1)/m} \left({}_{k/m}p_x - {}_{(k+1)/m}p_x\right) + \; v^n \;{}_np_x \tag{4.26}$$

The general identity (4.11) concerning the relation between expected present values of endowment insurances and life annuities also extends straightforwardly. With m payments per year, and the individual payments of $1/m$ again totalling 1 per year, the term-n life annuity-due payout is given via formula (2.4) by

$$\ddot{a}^{(m)}_{\overline{\min(T_m - x + 1/m,\, n)}|} \;=\; (1 - v^{\min(T_m - x + 1/m,\, n)}) \,/\, d^{(m)}$$

Again the unit life annuity-due has present value which is a simple linear function of the present value $v^{\min(T_m - x + 1/m,\, n)}$ of the unit endowment insurance. Taking expectations (over values of the random variable T, conditionally given $T \ge x$) in the present value formula, and substituting the notation $\mathrm{A}^{(m)}_{x:\overline{n}|}$ then yields:

$$\ddot{a}^{(m)}_{x:\overline{n}|} \;=\; E_x\Big(\frac{1 - v^{\min(T_m - x + 1/m,\, n)}}{d^{(m)}}\Big) = \frac{1 - A^{(m)}_{x:\overline{n}|}}{d^{(m)}} \tag{4.27}$$

where recall that $E_x(\,\cdot\,)$ denotes the conditional expectation $E(\,\cdot\,|\,T \ge x)$. An algebraically equivalent form of the identity (4.27) is

$$d^{(m)}\;\ddot{a}^{(m)}_{x:\overline{n}|} \,+\, \mathrm{A}^{(m)}_{x:\overline{n}|} \;=\; 1 \tag{4.28}$$

For multiple payment periods per year, the idea for converting from risk premiums of life annuities-due to life annuities immediate is very similar to the idea behind the conversion formula (4.13) for $m = 1$. The payment stream for the unit annuity-immediate to a life aged x with payment of 1 per year, term n years, and m payments per year consists of payments $1/m$ at each of the policy times k/m such that $1 \le k \le nm$ and $k/n \le T - x$. The corresponding payment stream for an annuity-due with term $n + 1/m$ is exaxtly the same, except that the latter omits the initial payment of $1/m$ at time 0. Therefore the respective expected present values $a^{(m)}_{x:\overline{n}|}$ and $\ddot{a}^{(m)}_{x:\overline{n+1/m}|}$ differ by exactly the present value of that initial payment of $1/m$, establishing the identity

$$a^{(m)}_{x:\overline{n}|} \;=\; \ddot{a}^{(m)}_{x:\overline{n+1/m}|} - \frac{1}{m} \;=\; E_x\Big(\, \ddot{a}^{(m)}_{\overline{\min(T_m - x, n) + 1/m}|} \Big) \tag{4.29}$$

From this m-payment-period conversion formula, we directly obtain an identity relating the net single premium for life annuities-immediate with m payment periods per year to that of the m payment period endowment insurances. The result is

$$a^{(m)}_{x:\overline{n}|} = \ddot{a}^{(m)}_{x:\overline{n+1/m}|} - \frac{1}{m} \;=\; \frac{1 - A^{(m)}_{x:\overline{n+1/m}|}}{d^{(m)}} - \frac{1}{m} \;=\; \frac{1}{i^{(m)}} - \frac{1}{d^{(m)}}\, A^{(m)}_{x:\overline{n+1/m}|} \tag{4.30}$$

and

$$d^{(m)}\; a^{(m)}_{x:\overline{n}|} + A^{(m)}_{x:\overline{n+1/m}|} \;=\; \frac{d^{(m)}}{i^{(m)}} \;=\; v^{1/m} \tag{4.31}$$

In these formulas, we have made use of the definition

$$m/\, d^{(m)} \;=\; (1 + \, i^{(m)}/m) \Big/ \, (i^{(m)}/m)$$

leading to the simplifications

$$\frac{m}{d^{(m)}} \;=\; \frac{m}{i^{(m)}} + 1 \quad , \qquad \frac{i^{(m)}}{d^{(m)}} \;=\; 1 + \frac{i^{(m)}}{m} \;= v^{-1/m}$$

We conclude this section with a general formula extending (4.21). A contract which, for all integers $k = 0,\, 1,\, \ldots,\, m(\omega - x) - 1$, pays $\frac{1}{m}\, G_x(k/m)$ at policy time k/m if the insured life initially aged x survives to age $x + k/m$, and which pays $F_x(k/m)$ at policy time $(k+1)/m$ if the insured life aged x dies within the exact-age interval $T \in [x+k/m,\, x+(k+1)/m)$, has net single (risk) premium equal to

$$E_x\Big(\sum_{k=0}^{m(\omega-x)-1} \Big[\frac{1}{m}\, G_x(k/m)\, v^{k/m}\, I_{\{T-x\geq k/m\}} + F_x(k/m)\, v^{(k+1)/m}\, I_{\{T_m - x = k/m\}}\Big]\Big)$$

$$= \sum_{k=0}^{m(\omega-x)-1} {}_{k/m}p_x \Big[\frac{1}{m}\, G_x(k)\, v^{k/m} \;+\; F_x(k/m)\, v^{(k+1)/m}\; {}_{1/m}q_{x+k/m}\Big] \qquad (4.32)$$

See the Worked Examples (numbers 3 and 4) for illustrations of numerical calculations with the standard formulas (4.22) and (4.24), as well as the general formula (4.32).

The idea behind equation (4.32) can also be used to express the net single premium of a life insurance or annuity in a varying interest rate environment. Following the ideas of Sections 1.2.4 and 1.2.5, we know that the present valueof a unit payment at policy time t under a time-varying instantaneous interest rate $r(s) \equiv \exp(\delta(s)) - 1$ (expressed in terms of the policy time-argument s) is $1/A(t) = \exp(-\int_0^t \delta(s)\, ds)$. Then the present value of a term insurance of duration n with level payment amount F at policy time $(k+1)/m$ if the insured life aged x dies within the exact-age interval $T \in [x+k/m,\, x+(k+1)/m)$, $0 \leq k < nm$, is given by

$$F \cdot \exp\Big(-\int_0^{(k+1)/m} \delta(s)\, ds\Big)\, I_{\{T_m = x+k/m\}} \;=\; \frac{F}{A((k+1)/m)}\, I_{\{k/m \leq T-x < (k+1)/m\}}$$

Similarly, the present value of a temporary life annuity due of duration n which makes level payments of amount $\frac{1}{m}G$ at all policy times $k/m \leq \min(T-x,\, n)$, is

$$\frac{G}{m} \cdot \sum_{j=0}^{nm-1} \exp\Big(-\int_0^{j/m} \delta(s)\, ds\Big)\, I_{\{j/m \leq T-x\}} \;=\; \frac{G}{m} \sum_{j=0}^{nm-1} \frac{1}{A(k/m)}\, I_{\{j/m \leq T-x\}}$$

Then the net single premium of a contract which pays G/m at policy time $k/m \leq n$ if the insured life initially aged x survives to age $x+k/m$, and which pays F at policy time $(k+1)/m \leq n$ if the insured life aged x

dies within the exact-age interval $T \in [x + k/m,\, x + (k + 1)/m)$, is the expectation of the sum of the last two displayed expressions, and is given by

$$\sum_{k=0}^{nm-1} {}_{k/m}p_x \left[\frac{G}{m \cdot A(k/m)} + \frac{F}{A((k+1)/m)}\, {}_{1/m}q_{x+k/m}\right] \tag{4.33}$$

4.4 Interpolation Formulas in Risk Premiums

A key issue in understanding the special nature of life insurances and annuities with multiple payment periods is the calculation of these probabilities from the underlying probabilities ${}_jp_y$ (for integers $j,\ y$) which can be deduced or estimated from life-tables. In the present Section, we combine the Actuarial Assumption — (i) of Chapter 3, saying that deaths are uniformly distributed within whole year of age — with the insurance and (temporary) life annuity-due risk premium formulas. In this setting, the number m of payment periods per year is greater than 1, and by formula (3.23) for all integers $j = 0,\, 1,\, \ldots, m-1$:

$${}_{j/m}p_x \;=\; 1 - {}_{j/m}q_x \;=\; 1 \;-\; (j/m)\, q_x$$

so that

$${}_{j/m}p_x\ {}_{1/m}q_{x+j/m} \;=\; {}_{j/m}p_x \;-\; {}_{(j+1)/m}p_x \;=\; (1/m)\, q_x$$

For any integer $k = bm + j$, where $0 \le l < m - 1$ and b is an integer,

$${}_{k/m}p_x\ {}_{1/m}q_{x+k/m} \;=\; {}_bp_x \left({}_{j/m}p_{x+b} \;-\; {}_{(j+1)/m}p_{x+b}\right) \;=\; (1/m)\, {}_bp_x\, q_{x+b} \tag{4.34}$$

Substituting (4.34) into (4.23) with summation indices $k = bm+j$, gives

$$\mathrm{A}^{(m)1}_{x:\overline{n}|} \;=\; \sum_{b=0}^{n-1}\sum_{j=0}^{m-1} v^{b+(j+1)/m}\ {}_{b+j/m}p_x\ {}_{1/m}q_{x+b+j/m}$$

$$=\; \frac{1}{m}\sum_{b=0}^{n-1} v^b\, {}_bp_x\, q_{x+b} \sum_{j=0}^{m-1} v^{(j+1)/m}$$

$$=\; \left(\frac{1}{m}\sum_{j=0}^{m-1} v^{(j+1)/m}\right) \cdot\ (1+i) \left(\sum_{b=0} v^{b+1}\, {}_bp_x\, q_{x+b}\right)$$

The two factors in parentheses in the final displayed expression are respectively the one-year annuity-immediate present value $a^{(m)}_{\overline{1|}}$ and the one-payment-per-year term insurance risk-premium $A^1_{x:\overline{n|}}$. Since

$$(1+i)\, a^{(m)}_{\overline{1|}} \;=\; (1+i)\,(1-v)/i^{(m)} \;=\; i/i^{(m)}$$

it follows that under interpolation assumption (i),

$$A^{(m)\,1}_{x:\overline{n|}} \;=\; (i/i^{(m)}) \sum_{b=0}^{n-1} v^{b+1}\,{}_bp_x\, q_{x+b} \;=\; (i/i^{(m)})\, A^1_{x:\overline{n|}} \tag{4.35}$$

Similarly, formula (4.34) substituted into the temporary life annuity formula (4.24) with summation index $k = bm + j$ gives

$$\ddot{a}^{(m)}_{x:\overline{n|}} \;=\; \frac{1}{m} \sum_{b=0}^{n-1} \sum_{j=0}^{m-1} v^{b+j/m}\,{}_{j/m}p_{x+b} \cdot {}_bp_x \;=\; \frac{1}{m} \sum_{b=0}^{n-1} v^b\,{}_bp_x \sum_{j=0}^{m-1} v^{j/m}\,(1 - \frac{j}{m}\, q_{x+b})$$

$$=\; \frac{1}{m} \sum_{j=0}^{m-1} v^{j/m}\, \ddot{a}_{x:\overline{n|}} \;-\; \frac{1+i}{m^2} \Big(\sum_{j=0}^{m-1} j\, v^{j/m} \Big)\, A^1_{x:\overline{n|}} \tag{4.36}$$

This formula can be reduced further in either of two ways. First, one can appeal to the definition of increasing temporary annuity-due and refer to the formula given in paragraph (iv) of Section 2.1.1:

$$\frac{1}{m^2} \sum_{j=0}^{m-1} j\, v^{j/m} \;=\; \frac{1}{m^2} \sum_{j=0}^{m-1} (j+1)\, v^{j/m} \;-\; \frac{1}{m}\, \ddot{a}^{(m)}_{\overline{1|}}$$

$$=\; (I^{(m)}\ddot{a})^{(m)}_{\overline{1|}} \;-\; \frac{1-v}{m\, d^{(m)}} \;=\; \frac{1}{d^{(m)}} \Big(\frac{1-v}{d^{(m)}} \;-\; v \;-\; \frac{1-v}{m} \Big) \tag{4.37}$$

Alternatively, using the identity

$$A^{(m)}_{x:\overline{n|}} \;=\; A^{(m)\,1}_{x:\overline{n|}} \;+\; v^n\,{}_np_x$$

within (4.27), and directly substituting (4.35), yields **under (i)**

$$\ddot{a}^{(m)}_{x:\overline{n|}} \;=\; \frac{1}{d^{(m)}} \Big(1 \;-\; \frac{i}{i^{(m)}}\, A^1_{x:\overline{n|}} \;-\; v^n\,{}_np_x \Big) \tag{4.38}$$

We leave as an Exercise for the interested reader to verify algebraically, using (4.37) together with (4.11), that the formulas (4.36) and (4.38) are equal.

4.5 Continuous Risk Premium Formulas

The present chapter has developed formulas for the net single premiums or risk premiums of the principal life insurance and annuity contracts, first in the setting of one payment period per year ($m = 1$) and then in the case of multiple payment periods ($m > 1$) per year. In the limit as m gets large, the risk premium formulas become expected present values calculated as continuous integrals with respect to survival densities. To recall why this limit exists, note that for any function $g(T)$ which depends on T only through the last completed $1/m$'th year $T_m = [Tm]/m$,

$$E_x(g(T)) = \sum_{k=0}^{m(\omega-x)-1} g(x+k/m) \, {}_{k/m}p_x \, {}_{1/m}p_{x+k/m}$$

where we have used (4.3) as the probability mass function for T_m. The displayed expectation formula is then also valid with m replaced by any integer multiple $M = mn$, $n \ge 1$, and has the equivalent expression

$$\sum_{k=0}^{M(\omega-x)-1} \frac{g(x+k/M)}{S(x)} \int_{x+k/M}^{x+(k+1)/M} f(t)\, dt$$

$$= \frac{1}{S(x)} \sum_{k=0}^{M(\omega-x)-1} \int_{x+k/M}^{x+(k+1)/M} g(t)\, f(t)\, dt = \int_0^{\omega-x} g(x+s)\, \frac{f(x+s)}{S(x)}\, ds$$

Assume now that the function $g(t)$ is continuous (and therefore uniformly continuous) on the bounded lifetime interval $[0,\omega]$, so that $g(t)-g([tM]/M$ can be made uniformly small by choice of a sufficiently large multiple M of m. Since the displayed expectation formulas are exactly valid when applied to the function $g([tM]/M$, it follows also for the general continuous function g that

$$E_x(g(T)) = \lim_{M\to\infty} E_x(g(T_M)) = \int_0^{\omega-x} g(x+s)\, \frac{f(x+s)}{S(x)}\, ds \qquad (4.39)$$

or, with the substitution $f(x+s) = \mu(x+s)\, S(x+s)$,

$$E_x(g(T)) = \int_0^{\omega-x} g(x+s)\, \mu(x+s)\, \frac{S(x+s)}{S(x)}\, ds \qquad (4.40)$$

Similar justifications can be given for these expectation formulas also whenever g is piecewise continuous. These integral formulas can be used either to calculate the limiting values of expected present values for insurance contracts with large m, or to calculate other expectations of demographic and biostatistical interest, such as life expectancies.

4.5.1 Continuous Insurance Contracts

So far in this Chapter, all of the expectations considered have been associated with the discretized random lifetime variables $[T]$ and $T_m = [mT]/m$. However, Insurance and Annuity contracts can also be defined with respectively instantaneous and continuous payments, as follows. First, an **instantaneous-payment** or **continuous insurance** with face-value F is a contract which pays an amount F at the instant of death of the insured. (In practice, this means that when the actual payment is made at some later time, the amount paid is F together with interest compounded from the instant of death.) As a function of the random lifetime T for the insured life initially with exact integer age x, the present value of the amount paid is $F \cdot v^{T-x}$ for a whole-life insurance and $F \cdot v^{T-x} \cdot I_{[T<x+n]}$ for an n-year term insurance. The expected present values or net single premiums on a life aged x are respectively denoted $\overline{\mathrm{A}}_x$ for a whole-life contract and $\overline{\mathrm{A}}^1_{x:\overline{n}|}$ for an n-year term insurance. The **continuous life annuity** is a contract which provides continuous payments at rate 1 per unit time for duration equal to the smaller of the remaining lifetime of the annuitant or the term of n years. Here the present value of the contractual payments, as a function of the exact age T at death for an annuitant initially of exact integer age x, is $\overline{a}_{\overline{\min(T-x,\,n)}|}$ where n is the (possibly infinite) duration of the life annuity. Recall that

$$\overline{a}_{\overline{K}|} \;=\; \int_0^\infty v^t \, I_{[t\le K]}\, dt \;=\; \int_0^K v^t \, dt \;=\; (1-v^K)/\delta$$

is the present value of a continuous payment stream of 1 per unit time of duration K units, where $v = (1+i)^{-1}$ and $\delta = \ln(1+i)$. The actuarial notation for the net single premium of the temporary continuous life annuity is $\bar{a}_{x:\overline{n}|}$ which simplifies to $\bar{a}_x$ when $n = \infty$.

The objective of this section is to develop and interpret formulas for the continuous-time net single premiums, along with one further quantity which has been defined as a continuous-time expectation of the lifetime variable T, namely the **mean residual life** (also called **complete life expectancy**) $\mathring{e}_x = E_x(T - x)$ for a life aged x.

4.5.2 Integral Formulas

We apply the continuous conditional expectation formulas (4.39) or (4.40) directly for the three choices

$$g(y) \quad = \quad y - x \quad , \quad v^{y-x} \quad , \quad \text{or} \quad v^{y-x} \cdot I_{\{y-x<n\}}$$

which respectively have the conditional $E_x(\cdot)$ expectations

$$\mathring{e}_x \quad , \quad \overline{A}_x \quad , \quad \overline{A}^{\,1}_{x:\overline{n}|}$$

For easy reference, the integral formulas for these three cases are:

$$\mathring{e}_x \; = \; E_x(T - x) \; = \; \int_0^\infty s \; \mu(x+s) \; {}_sp_x \, ds \tag{4.41}$$

$$\overline{A}_x \; = \; E_x(v^{T-x}) \; = \; \int_0^\infty v^s \; \mu(x+s) \; {}_sp_x \, ds \tag{4.42}$$

$$\overline{A}^{\,1}_{x:\overline{n}|} \; = \; E_x\left(v^{T-x} \, I_{\{T-x \le n\}}\right) \; = \; \int_0^n v^s \; \mu(x+s) \; {}_sp_x \, ds \tag{4.43}$$

Next, we obtain two additional formulas, for continuous life annuities-due

$$\overline{a}_x \quad \text{and} \quad \overline{a}_{x:\overline{n}|}$$

which correspond to $E_x\{g(T)\}$ for the two choices

$$g(t) \; = \; \int_0^{\omega - x} v^y \, I_{\{y+x \le t\}} \, dy \qquad \textbf{or} \qquad \int_0^n v^y \, I_{\{y+x \le t\}} \, dy$$

where we naturally assume that $x + n \le \omega$ in the case of the temporary life annuity.

After switching the order of the integrals and the conditional expectations, and evaluating the conditional expectation of an indicator as a conditional probability, in the form

$$E_x\left(I_{\{y\le T-x\}}\right) \;=\; P(T\ge x+s\,|\,T\ge x) \;=\; \frac{S(x+y)}{S(x)} \;=\; {}_yp_x$$

the resulting two equations become

$$\overline{a}_x \;=\; E_x\left(\int_0^{\omega-x} v^y\, I_{\{y\le T-x\}}\,dt\right) \;=\; \int_0^{\omega-x} v^y\;{}_yp_x\,dy \qquad (4.44)$$

$$\overline{a}_{x:\overline{n}|} \;=\; E_x\left(\int_0^{n} v^y\, I_{\{y\le T-x\}}\,dy\right) \;=\; \int_0^{n} v^y\;{}_yp_x\,dy \qquad (4.45)$$

As seen above in (4.39), risk premiums for continuous insurance and annuity contracts have a close relationship to the corresponding contracts with m payment periods per year for large m. That is, the term insurance net single premiums

$$\mathrm{A}^{(m)1}_{\;x:\overline{n}|} \;=\; E_x\left(v^{T_m-x+1/m}\right)$$

approach the continuous insurance value (4.42) as a limit when $m\to\infty$. A simple direct proof can be given because the payments at the end of the fraction $1/m$ of year of death are at most $1/m$ years later than the continuous-insurance payment at the instant of death, so that the following obvious inequalities hold:

$$\overline{\mathrm{A}}^{1}_{x:\overline{n}|} \;\le\; \mathrm{A}^{(m)1}_{\;x:\overline{n}|} \;\le\; v^{1/m}\,\overline{\mathrm{A}}^{1}_{x:\overline{n}|} \qquad (4.46)$$

Since the right-hand term in the inequality (4.46) also converges for large m to the leftmost term, the middle term which is sandwiched in between must converge to the same limit (4.43).

For the continuous annuity, (4.45) can be obtained as a limit of formulas (4.19) either from (4.39) or by using Riemann sums, as the number m of payments per year goes to ∞, i.e.,

$$\overline{a}_{x:\overline{n}|} \;=\; \lim_{m\to\infty}\, \ddot{a}^{(m)}_{x:\overline{n}|} \;=\; \lim_{m\to\infty}\sum_{k=0}^{nm-1}\frac{1}{m}\,v^{k/m}\;{}_{k/m}p_x \;=\; \int_0^n v^t\;{}_tp_x\,ds$$

The final formula coincides with (4.45), according with the intuition that the limit as $m\to\infty$ of the payment-stream which pays $1/m$ at intervals of time

$1/m$ between 0 and $T_m - x$ inclusive is the continuous payment-stream which pays 1 per unit time throughout the policy-age interval $[0, T-x)$.

The limiting argument of the previous paragraph shows immediately that under interpolation assumption (i), there are simple formulas relating $\bar{\mathrm{A}}_{x:\overline{n}|}$ and $\bar{a}_{x:\overline{n}|}$ to $\mathrm{A}^1_{x:\overline{n}|}$. Indeed, by (4.35), under (i)

$$\bar{A}_{x:\overline{n}|} \;=\; \lim_{m\to\infty} \mathrm{A}^{(m)\,1}_{\;\;x:\overline{n}|} \;=\; \lim_{m} \frac{i}{i^{(m)}}\, \mathrm{A}^1_{x:\overline{n}|} \;=\; \frac{i}{\delta}\, \mathrm{A}^1_{x:\overline{n}|} \tag{4.47}$$

and by (4.38), also under (i),

$$\bar{a}_{x:\overline{n}|} \;=\; \lim_{m\to\infty} \ddot{a}^{(m)}_{x:\overline{n}|} \;=\; \lim_{m} \frac{1}{d^{(m)}} \Big(1 \;-\; \frac{i}{i^{(m)}}\, \mathrm{A}^1_{x:\overline{n}|} \;-\; v^n\; {}_np_x\Big)$$

or

$$\bar{a}_{x:\overline{n}|} \;=\; \frac{1}{\delta} \Big(1 \;-\; \frac{i}{\delta}\, \mathrm{A}^1_{x:\overline{n}|} \;-\; v^n\; {}_np_x\Big) \tag{4.48}$$

Finally, it is easy to see by passing to the limit $m \to \infty$ in (4.28) that

$$\delta\; \bar{a}_{x:\overline{n}|} + \bar{\mathrm{A}}_{x:\overline{n}|} \;=\; 1 \tag{4.49}$$

More elaborate relations will be given in the next Chapter between net single premium formulas which do require interpolation-assumptions for probabilities of survival to times between integer ages to formulas for $m = 1$, which do not require such interpolation.

The expressions in formulas (4.41), (4.43), and (4.45) can be contrasted with the respective expectations (3.12), (4.17) and (4.19) for a function of the integer-valued random variable $[T]$ (taking $m = 1$). In particular, the complete life expectancy $\mathring{\mathrm{e}}_x$ in (4.41) is compared to the curtate life expectancy e_x under interplation assumption (i) in (3.28). The comparison between complete and curtate life expectancies under more general within-year survival distributions in subsection 4.5.4 below.

4.5.3 Risk Premiums under Theoretical Models

Let us work out examples of the multiple time-period per year and continuous formulas analytically and numerically, under a particular theoretical survival model.

Consider first the slightly artificial (but still useful) case where the residual life $T - x$ for a life aged x is precisely Weibull(γ, λ) distributed, with force of mortality for T given by $\mu(x+s) \;=\; \lambda\,\gamma\,s^{\gamma-1}$, and

$$ {}_sp_x \;=\; e^{-\lambda\, s^\gamma} \qquad \text{for all} \quad s \ge 0 \;\;, \quad q_{x+k} \;=\; 1 - \exp(-\lambda\{(k+1)^\gamma - k^\gamma\}) $$

Then respectively according to formulas (4.17), (4.23), and (4.42), we find formulas for term-insurance finite-m net single premiums under Weibull survival as follows:

$$ \mathrm{A}^{1}_{x:\overline{n}|} \;=\; \sum_{k=0}^{n-1} v^{k+1}\,(e^{-\lambda k^\gamma} - e^{-\lambda(k+1)^\gamma}) $$

$$ \mathrm{A}^{(m)\,1}_{\;\;x:\overline{n}|} \;=\; \sum_{k=0}^{n-1} v^{k+1/m} \sum_{j=0}^{m-1} v^{j/m}\,(e^{-\lambda(k+j/m)^\gamma} \;-\; e^{-\lambda(k+(j+1)/m)^\gamma}) $$

According to formulas (4.19), (4.24), and (4.44), the special Weibull-lifetime temporary life annuity-due risk premiums for finite m are:

$$ \ddot{a}_{x:\overline{n}|} \;=\; \sum_{k=0}^{n-1} v^k\,e^{-\lambda\,k^\gamma} \quad , \qquad \ddot{a}^{(m)}_{x:\overline{n}|} \;=\; \sum_{k=0}^{n-1}\sum_{j=0}^{m-1} v^{k+j/m}\,e^{-\lambda\,(k+j/m)^\gamma} $$

The continuous cases ($m = \infty$) of these formulas are as follows:

$$ \bar{\mathrm{A}}^{1}_{x:\overline{n}|} \;=\; \int_0^n v^s\,\lambda\gamma\,s^{\gamma-1}\,e^{-\lambda s^\gamma}\,ds \quad , \qquad \bar{a}_{x:\overline{n}|} \;=\; \int_0^n v^s\,e^{-\lambda s^\gamma}\,ds $$

Finally, according to formulas (3.11) and (4.41) the curtate and complete life expectancies at integer ages $y \ge x$ for Weibull (residual) lifetimes are:

$$ \mathrm{e}_x \;=\; \sum_{k=0}^{\infty} k\,(e^{-\lambda k^\gamma} - e^{-\lambda(k+1)^\gamma}) \quad , \qquad \mathring{\mathrm{e}}_x \;=\; \int_0^\infty s\,\lambda\gamma\,s^{\gamma-1}\,e^{\lambda s^\gamma}\,ds $$

We next give `R` code and a table showing some numerical comparisons of these insurance and life-annuity risk premiums, for $m = 1,\,4,\,\infty$. First, we give an `R` function for Weibull-lifetime term insurance risk premiums:

```
> WeibIns = function(lambda, gamma, m, n, i) {
    ## Function to calculate term-n insurance risk prem
  ### for 1 paymt per year, m paymts, and continuous ins
     v = 1/(1+i)
     xk = 0:(n-1)
     Ins1 = v*sum( v^xk*(exp(-lambda*xk^gamma)-
                     exp(-lambda*(xk+1)^gamma)) )
     xkjm = (0:(n*m-1))/m
     Insm = v^(1/m)* sum( v^xkjm*(exp(-lambda*xkjm^gamma) -
               exp(-lambda*(xkjm+1/m)^gamma)) )
     InsC = lambda*gamma*integrate(function(s,.v,.lam,.gam)
               .v^s*s^(.gam-1)*exp(-.lam*s^(.gam)),0, n,
               .v=v, .lam=lambda, .gam=gamma)$val
     c(termIns.1 = Ins1, termIns.m = Insm, termIns.cont = InsC)
   }
```

To illustrate the use of this function, we create a small Table of values for a few different values of n, with parameters (λ, γ) similar to those used in Chapter 2, but chosen successively so that (a) ${}_{32}p_{40} = .5/.925 = .5405$, ${}_{50}p_{40} = .04/.925 = .04324$ as in Figure 2.5, (b) ${}_{32}p_{40} = .6$, ${}_{50}p_{40} = .047$, or (c) ${}_{32}p_{40} = .65$, ${}_{50}p_{40} = .05$. In each case, the nominal age x is 40, in the R code to produce this Table, we begin by coding a function to solve for $\lambda, \gamma)$ when $\pi_1 = {}_{32}p_{40}$, $\pi_2 = {}_{50}p_{40}$ are given. The interest rates considered here are 0.05 or 0.06, and $n = 20$.

```
> LamGam = function(pi1,pi2, age1, age2) {
     ### Function to solve for lambda and gamma Weibull
  ### parameters, when S(age1)=pi1, S(age2)=pi2 are fixed.
     haz1 = -log(pi1) ;   haz2 = -log(pi2)
     gam = log(haz2/haz1)/log(age2/age1)
     lam = haz1/age1^gam
     c(lambda=lam, gamma=gam) }
> LamGam(.5/.925,.04/.925,32,50)
      lambda     gamma
   1.952e-06 3.653e+00
```

```
>  WeibIns(1.952e-6,3.653,4,20,.05)
   termIns.1    termIns.m termIns.cont
     0.04844      0.04930      0.04960

> InsArr = array(0, dim=c(6,6), dimnames=list(NULL, c("lambda",
    "gamma", "i", "Ins.1", "Ins.m", "Ins.C")))
  intrat = c(.05, .06)
  sprobs = rbind(c(.5/.925,.04/.925),c(.6,.047),c(.65,.05))
  for(a in 1:2) for (b in 1:3) {
   k = 2*(b-1)+a
>  lamgam=LamGam(sprobs[b,1],sprobs[b,2],32,50)
   InsArr[k,] = c(lamgam[1],lamgam[2],intrat[a],
        WeibIns(lamgam[1],lamgam[2],4,20,intrat[a]))  }
   InsArr
        lambda gamma    i   Ins.1   Ins.m   Ins.C
[1,] 1.952e-06 3.653 0.05 0.04846 0.04932 0.04962
[2,] 1.952e-06 3.653 0.06 0.04189 0.04278 0.04309
[3,] 4.715e-07 4.009 0.05 0.03396 0.03457 0.03478
[4,] 4.715e-07 4.009 0.06 0.02924 0.02986 0.03008
[5,] 1.241e-07 4.345 0.05 0.02436 0.02479 0.02494
[6,] 1.241e-07 4.345 0.06 0.02091 0.02135 0.02151
```

The variations among the parameters and interest rates make much more difference to the results than does the number m of payment periods per year. Note that the overall size of the risk premium for a unit 20-year term insurance is reasonable, because the probability it will pay anything at all is ${}_{20}q_{40}$, which (for odd-numbered rows) takes the three values

```
> 1-exp(-AnnuArr[c(1,3,5),1]*20^AnnuArr[c(1,3,5),2])
[1] 0.10461 0.07467 0.05435
```

A similar Table of risk premiums for temporary life annuities-due, with exactly the same parameters, is given below along with the `R` code for the life-annuity function calculation.

```
> WeibAnnD = function(lambda, gamma, m, n, i) {
    ## Function to calculate term-n insurance risk prem
  ### for 1 paymt per year, m paymts, and continuous ins
     v = 1/(1+i)
     xk = 0:(n-1)
     Annu1 = sum( v^xk*exp(-lambda*xk^gamma) )
     xkjm = (0:(n*m-1))/m
     Annum = sum( v^xkjm*exp(-lambda*xkjm^gamma) )/m
     AnnuC = integrate(function(s,.v,.lam,.gam)
               .v^s*exp(-.lam*s^(.gam)),0, n,
               .v=v, .lam=lambda, .gam=gamma)$val
     c(tmpAnn.1 = Annu1, tmpAnn.m = Annum, tmpAnn.cont = AnnuC)
   }
 > AnnuArr = InsArr
  dimnames(AnnuArr)[[2]][4:6]=c("Annu.1","Annu.m","Annu.C")
  for(a in 1:2) for (b in 1:3) {
   k = 2*(b-1)+a
   lamgam=LamGam(sprobs[b,1],sprobs[b,2],32,50)
   AnnuArr[k,4:6] = WeibAnnD(lamgam[1],lamgam[2],4,20,intrat[a]) }
   AnnuArr
        lambda gamma    i Annu.1 Annu.m Annu.C
[1,] 1.952e-06 3.653 0.05  12.90  12.65  12.56
[2,] 1.952e-06 3.653 0.06  11.99  11.72  11.63
[3,] 4.715e-07 4.009 0.05  12.96  12.72  12.64
[4,] 4.715e-07 4.009 0.06  12.05  11.78  11.69
[5,] 1.241e-07 4.345 0.05  13.00  12.76  12.68
[6,] 1.241e-07 4.345 0.06  12.09  11.82  11.73
```

Life expectancy calculations and comparisons are done in the next subsection, for Gompertz survival. Examples of formula development for other special parametric distributional families are contained in the Exercises.

4.5.4 Numerical Calculations of Life Expectancies

Formulas (4.41) or (3.12) respectively provide the complete and curtate age-specific life expectancies, in terms respectively of survival densities and life-table data. Formula (3.28) provides the actuarial approximation for complete life expectancy in terms of life-table data, based upon interpolation-assumption (i) (Uniform mortality within year of age). In this Section, we illustrate these formulas using the Illustrative simulated and extrapolated life-table data of Table 1.1.

Life expectancy formulas necessarily involve life table data and/or survival distributions specified out to arbitrarily large ages. While life tables may be based on large cohorts of insured for ages up to the seventies and even eighties, beyond that they will be very sparse and very dependent on the particular small group(s) of aged individuals used in constructing the particular table(s). On the other hand, the fraction of the cohort at moderate ages who will survive past 90, say, is extremely small, so a reasonable extrapolation of a well-established table out to age 105 or so may give sufficiently accurate life-expectancy values at ages not exceeding 105. Life expectancies are in any case forecasts based upon an implicit assumption of future mortality following exactly the same pattern as recent past mortality. Life-expectancy calculations necessarily ignore likely changes in living conditions and medical technology which many who are currently alive will experience. Thus an assertion of great accuracy for a particular method of calculation would be misplaced.

All of the numerical life-expectancy calculations produced for the Figure of this Section are based on the extrapolation (2.10) of the illustrative life table data from Table 1.1. According to that extrapolation, death-rates q_x for all ages 78 and greater are taken to grow exponentially, with $log(q_x/q_{78}) = (x-78)\ln(1.0885)$. This exponential behavior is approximately but not precisely compatible with a Gompertz-form force-of-mortality function

$$\mu(78+t) \;=\; \mu(78)\; c^t$$

in light of the approximate equality $\mu(x) \approx q_x$, an approximation which progressively becomes less valid as the force of mortality gets larger. To see this, note that under a Gompertz survival model,

$$\mu(x) = Bc^x \quad , \qquad q_x = 1 - \exp\left(-Bc^x \, \frac{c-1}{\ln c}\right)$$

and with $c = 1.0885$ in our setting, $(c-1)/\ln c = 1.0436$.

Since curtate life expectancy (3.12) relies directly on (extrapolated) life-table data, its calculation is simplest and most easily interpreted. Figure 4.1 presents, as plotted points, the age-specific curtate life expectancies for integer ages $x = 0, 1, \ldots, 78$. Since the complete life expectancy at each age is larger than the curtate by exactly $1/2$ under interpolation assumption (i), we calculated for comparison the complete life expectancy at all (real-number) ages, under assumption (ii) (from Chapter 3, as in equation 3.16) of piecewise-constant force of mortality within years of age. Under this assumption, by formula (3.24), mortality within year of age $(0 < t < 1)$ is ${}_tp_{x+k} = (p_{x+k})^t$, and force of mortality is

$$\mu(x+k+t) \;=\; -\frac{d}{dt}\,\ln\,{}_tp_{x+k} \;=\; -\ln\,p_{x+k}$$

Using formulas (4.41), (3.12), and interpolation assumption (ii), the exact formula for the difference between complete and curtate life expectancy becomes

$$\mathring{e}_x \;-\; e_x \;=\; \sum_{k=0}^{\omega-x-1} {}_kp_x \left\{ \int_k^{k+1} s\,\mu(x+s)\,{}_{s-k}p_{x+k}\,ds \;-\; k\,q_{x+k} \right\}$$

$$= \sum_{k=0}^{\omega-x-1} {}_kp_x \left\{ (-\ln\,p_{x+k}) \int_0^1 (k+t)\,e^{t\,\ln p_{x+k}}\,dt \;-\; k\,q_{x+k} \right\}$$

$$= \sum_{k=0}^{\omega-x-1} {}_kp_x \left\{ (-\ln\,p_{x+k}) \left(\frac{(k+1)p_{x+k} - k}{\ln p_{x+k}} \;-\; \frac{p_{x+k}-1}{(\ln p_{x+k})^2} \right) \;-\; k\,q_{x+k} \right\}$$

$$= \sum_{k=0}^{\omega-x-1} {}_kp_x \left(-\,p_{x+k} \;-\; \frac{q_{x+k}}{\ln p_{x+k}} \right) \tag{4.50}$$

The complete minus curtate life expectancies calculated from this formula were found range from 0.493 at ages 40 and below, down to 0.485 at age 78 and 0.348 at age 99. (Contrast this result with the constant difference of $1/2$ under assumption (i).) Thus there is essentially no new information in the calculated complete life expectancies, and they are not plotted.

The aspect of Figure 4.1 which is most startling to the intuition is the large expected numbers of additional birthdays for individuals of advanced ages. Moreover, the large life expectancies shown are comparable to actual US male mortality circa 1959, so would be still larger today.

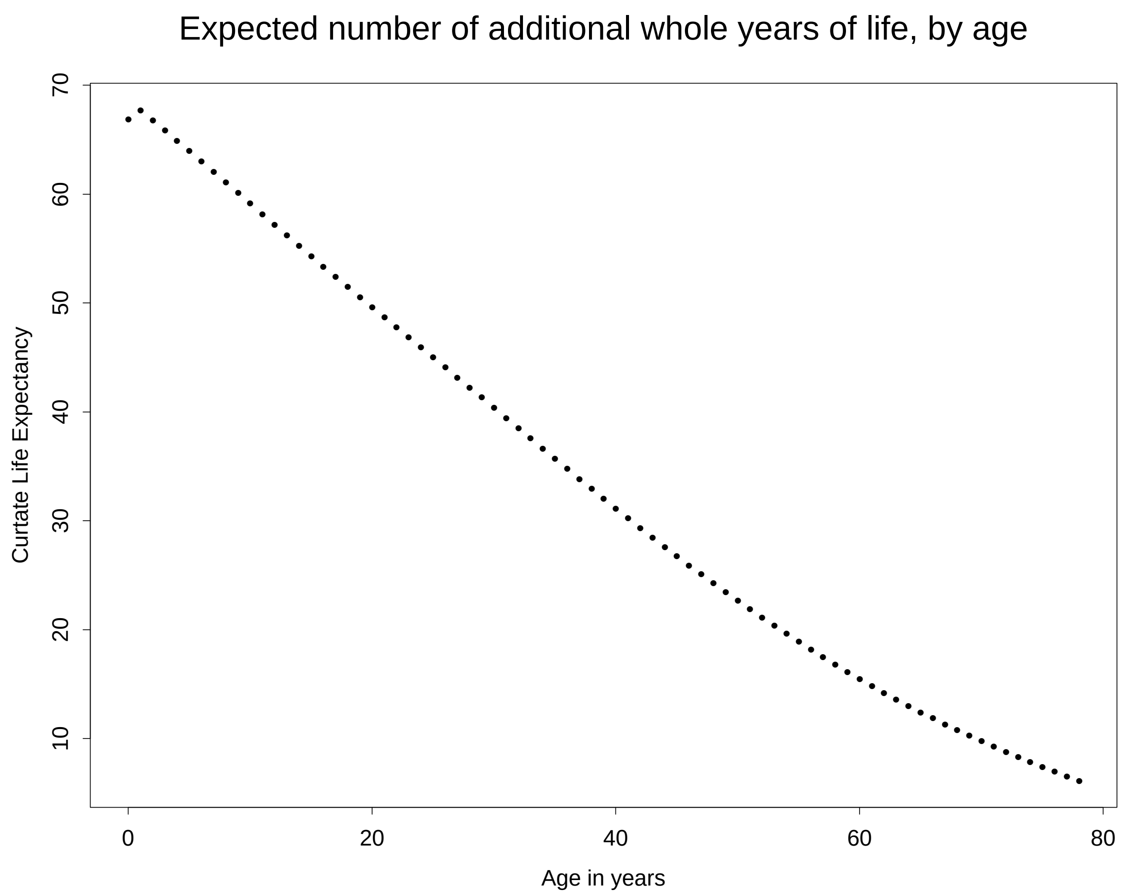

Figure 4.1: Curtate life expectancy e_x as a function of age, calculated from the simulated illustrative life table data of Table 1.1, with age-specific death-rates q_x extrapolated as indicated in formula (2.10).

4.6 *Exercise Set 4*

(1). For each of the following three lifetime distributions, find (a) the expected remaining lifetime for an individual aged 20, and (b) ${}_{7/12}q_{40}/q_{40}$.

(i) *Weibull*$(.00634, 1.2)$, with $S(t) = \exp(-0.00634\, t^{1.2})$,

(ii) *Lognormal*$(\log(50), 0.325^2)$, with $S(t) = 1 - \Phi((\log(t) - \log(50))/0.325)$,

(iii) *Piecewise exponential* with force of mortality given the constant value $\mu_t = 0.015$ for $20 < t \le 50$, and $\mu_t = 0.03$ for $t \ge 50$. *In these integrals, you should be prepared to use integrations by parts, gamma function values, tables of the normal distribution function* $\Phi(x)$, *and/or numerical integrations via calculators or software.*

(2). (a) Find the expected present value, with respect to the constant effective interest rate $r = 0.07$, of an insurance payment of \$1000 to be made at the instant of death of an individual who has just turned 40 and whose remaining lifetime $T - 40 = S$ is a continuous random variable with density $f(s) = 0.05\, e^{-0.05\, s}$, $s > 0$.

(b) Find the expected present value of the insurance payment in (a) if the insurer is allowed to delay the payment to the end of the year in which the individual dies. Should this answer be larger or smaller than the answer in (a) ?

(3). If the individual in Problem 2 pays a life insurance premium P at the **beginning** of each remaining year of his life (including this one), then what is the expected total present value of all the premiums he pays before his death ?

(4). Suppose that an individual has equal probability of dying within each of the next 40 years, and is certain to die within this time, i.e., his age is x and

$$ {}_kp_x \; - \; {}_{k+1}p_x = 0.025 \qquad \text{for} \qquad k = 0, 1, \ldots, 39 $$

Assume the fixed interest rate $r = 0.06$.

(a) Find the net single whole-life insurance premium A_x for this person.

(b) Find the net single premium for the term and endowment insurances $\mathrm{A}^1_{x:\overline{20}|}$ and $\mathrm{A}_{x:\overline{30}|}$.

(5). Show that the expected whole number of years of remaining life for a life aged x is given by

$$c_x \;=\; E([T]-x \mid T \ge x) \;=\; \sum_{k=0}^{\omega-x-1} k\;{}_kp_x\, q_{x+k}$$

and prove that this quantity as a function of integer age x satisfies the recursion equation

$$c_x \;=\; p_x\;(1\,+\,c_{x+1})$$

(6). Show that the expected present value b_x of an insurance of 1 payable at the beginning of the year of death (or equivalently, payable at the end of the year of death along with interest from the beginning of that same year) satisfies the recursion relation (4.51) above.

(7). Prove the identity (4.16) algebraically.

For the next two problems, consider a cohort life-table population for which you know only that $l_{70} = 10,000$, $l_{75} = 7000$, $l_{80} = 3000$, $l_{85} = 0$, and that the distribution of death-times within 5-year age intervals is uniform.

(8). Find (a) $\mathring{e}_{75}$ and (b) the probability of an individual aged 70 in this life-table population dying between ages 72.0 and 78.0.

(9). Find the probability of an individual aged 72 in this life-table population dying between ages 75.0 and 83.0, if the assumption of uniform death-times within 5-year intervals is replaced by:

(a) a constant force of mortality within 5-year age-intervals;

(b) assuming linearity of $1/S(t))$ within 5-year age intervals.

(10). Suppose that a population has survival probabilities governed at all ages by the force of mortality

$$\mu_t \;=\; \begin{cases} .01 & \text{for} \quad 0 \le t < 1 \\ .002 & \text{for} \quad 1 \le t < 5 \\ .001 & \text{for} \quad 5 \le t < 20 \\ .004 & \text{for} \quad 20 \le t < 40 \\ .0001 \cdot t & \text{for} \quad 40 \le t \end{cases}$$

Then (a) find ${}_{30}p_{10}$, and (b) find $\mathring{e}_{50}$.

(11). Suppose that a population has survival probabilities governed at all ages by the force of mortality

$$\mu_t = \begin{cases} .01 & \text{for} \quad 0 \le t < 10 \\ .1 & \text{for} \quad 10 \le t < 30 \\ 3/t & \text{for} \quad 30 \le t \end{cases}$$

Then (a) find ${}_{30}p_{20} =$ the probability that an individual aged 20 survives for at least 30 more years, and (b) find $\mathring{e}_{30}$.

(12). Assuming the same force of mortality as in the previous problem, find $\mathring{e}_{70}$ and $\overline{A}_{60}$ if $i = 0.09$.

(13). The force of mortality for impaired lives is three times the standard force of mortality at all ages. The standard rates q_x of mortality at ages 95, 96, and 97 are respectively 0.3, 0.4, and 0.5 . What is the probability that an impaired life age 95 will live to age 98 ?

(14). You are given a survival function $S(x) = (10-x)^2/100\,, \;\; 0 \le x \le 10$.

(a) Calculate the average number of future years of life for an individual who survives to age 1.

(b) Calculate the difference between the force of mortality at age 1, and the probability that a life aged 1 dies before age 2.

(15). An n-year term life insurance policy to a life aged x provides that if the insured dies within the n-year period an annuity-certain of yearly payments of 10 will be paid to the beneficiary, with the first annuity payment made on the policy-anniversary following death, and the last payment made on the N^{th} policy anniversary. Here $1 < n \le N$ are fixed integers. If $B(x, n, N)$ denotes the net single premium (= expected present value) for this policy, and if mortality follows the law $l_x = C(\omega - x)/\omega$ for some terminal integer age ω and constant C, then find a simplified expression for $B(x, n, N)$ in terms of interest-rate functions, ω, and the integers x, n, N. Assume $x + n \le \omega$.

(16). The father of a newborn child purchases an endowment and insurance contract with the following combination of benefits. The child is to receive \$100,000 for college at her 18^{th} birthday if she lives that long and \$500,000 at her 60^{th} birthday if she lives that long, and the father as beneficiary is to receive \$200,000 at the end of the year of the child's death if the child dies before age 18. Find expressions, **both** in actuarial notations and in terms of $v = 1/(1+i)$ and of the survival probabilities ${}_kp_0$ for the child, for the net single premium for this contract.

(17). Verify algebraically, using (4.37) together with (4.11), that the right-hand sides of formulas (4.36) and (4.38) are equal.

For the next four problems, concerning insurance contract risk premiums in a variable interest rate environment, apply formula (4.33).

(18). Suppose that a life aged x wants to purchase a 20-year term insurance now, and that interest rates over the next twenty years will be $i = .05$ for policy ages in $(0, 10]$ and $i = .06$ for policy ages in $(10, 20]$. Find the net single premium for a unit term insurance (i.e., if the insurance payment is 1) if survival is governed by the Weibull$(2 \cdot 10^{-6}, 4)$ distribution for $T - x$. (You may use the formulas and `R` code of section 4.5.3 to aid in the calculation.)

(19). Pass to the limit $m \to \infty$ in formula (4.33) to derive an analogous formula for the net single premium, in a variable interest rate environment with instantaneous force of interest $\delta(t)$ at policy time t, of a contract for a life aged x paying a continuous-time stream at rate G at all policy times $t < \min(T - x, n)$ and paying a lump-sum amount F at the instant of death if death occurs before age $x + n$.

(20). Suppose that a life aged x wants to purchase a 10-year term insurance or temporary annuity-due, and that interest rates over the next ten years will be $i = .07$ for policy ages in $(0, 6]$ and $i = .04$ for policy ages in $(6, 10]$. Show that the net single premium for a unit 10-year term insurance or a unit temporary life annuity-due, with one payment period per year (i.e., $m = 1$) depends on the survival distribution only through the cohort life table quantities ${}_kp_x$ for integers $k = 1, 2, \ldots, 10$.

(21). Suppose that a life aged x wants to purchase a 10-year term insurance, and that it is believed that interest rates will vary over the 10-year interval according to the rule $\delta(t) = \delta \cdot (1 + 0.002\,t)$. Show that the net single premium for a unit 10-year term insurance depends on the continuous conditional survival probabilities ${}_tp_x$, $0 \le t \le 10$ and not only on the values for integer t.

4.7 Worked Examples

Example 1. Toy Life-Table *(assuming uniform failures)*

Consider the following life-table with only six equally-spaced ages. (That is, assume $l_6 = 0$.) Assume that the rate of interest $i = .09$, so that $v = 1/(1+i) = 0.9174$ and $(1-e^{-\delta})/\delta = (1-v)/\delta = 0.9582$.

x	Age-range	l_x	d_x	e_x	$\overline{A}_x$
0	$[0,1)$	1000	60	4.2	0.704
1	$[1,2)$	940	80	3.436	0.749
2	$[2,3)$	860	100	2.709	0.795
3	$[3,4)$	760	120	2.0	0.844
4	$[4,5)$	640	140	1.281	0.896
5	$[5,6)$	500	500	0.5	0.958

Using the data in this Table, and interest rate $i = .09$, we begin by calculating the expected present values for simple contracts for term insurance, annuity, and endowment. First, for a life aged 0, a 3-year term insurance with payoff amount \$1000 has present value given by formula (4.17) as

$$1000\,\mathrm{A}^{1}_{0:\overline{3}|} \;=\; 1000\left\{0.917\,\frac{60}{1000} \;+\; (0.917)^2\,\frac{80}{1000} \;+\; (0.917)^3\,\frac{100}{1000}\right\} \;=\; 199.60$$

Second, for a life aged 2, a 3-year temporary annuity-due of \$700 per year (with last payment at age 4) has present value computed from (4.19) to be

$$700\,\ddot{a}_{2:\overline{3}|} \;=\; 700\left\{1 \;+\; 0.917\,\frac{760}{860} \;+\; (0.917)^2\,\frac{640}{860}\right\} \;=\; 1705.98$$

For the same life aged 2, the 3-year Endowment for \$700 has present value

$$700\,\mathrm{A}_{2:\overline{3}|}^{\;\;1} \;=\; 700\,\cdot\,(0.9174)^3\,\frac{500}{860} \;=\; 314.26$$

Thus we can also calculate (for the life aged 2) the present value of the 3-year annuity-immediate of \$700 per year as

$$700\,\cdot\,\left(\ddot{a}_{2:\overline{3}|} \,-\, 1 \,+\, \mathrm{A}_{0:\overline{3}|}^{\;\;1}\right) \;=\; 1705.98 - 700 + 314.26 \;=\; 1320.24$$

We next apply and interpret the formulas of Section 4.5.1, together with the observation that

$$ {}_jp_x \cdot q_{x+j} \;=\; \frac{l_{x+j}}{l_x} \cdot \frac{d_{x+j}}{l_{x+j}} \;=\; \frac{d_{x+j}}{l_x} $$

to show how the last two columns of the Table were computed. In particular, by (4.41)

$$ \mathrm{e}_2 = \frac{100}{860}\cdot 0 + \frac{120}{860}\cdot 1 + \frac{140}{860}\cdot 2 + \frac{500}{860}\cdot 3 + \frac{1}{2} = \frac{1900}{860} + 0.5 = 2.709 $$

Moreover: observe that $c_x = \sum_{k=0}^{5-x} k \, {}_kp_x q_{x+k}$ satisfies the "recursion equation" $c_x = p_x \, (1 + c_{x+1})$ (*cf.* Exercise 5 above), with $c_5 = 0$, from which the e_x column is easily computed by: $\mathrm{e}_x = c_x + 0.5$.

Now apply the present value formula for continuous insurance to find

$$ \overline{A}_x \;=\; \sum_{k=0}^{5-x} {}_kp_x \, q_{x+k} \, v^k \, \frac{1-e^{-\delta}}{\delta} \;=\; 0.9582 \sum_{k=0}^{5-x} {}_kp_x \, q_x \, v^k \;=\; 0.9582 \; b_x $$

where b_x is the expected present value of an insurance of 1 payable at the *beginning* of the year of death (so that $A_x = v \, b_x$) and satisfies $b_5 = 1$ together with the recursion-relation

$$ b_x \;=\; \sum_{k=0}^{5-x} {}_kp_x \, q_{x+k} \, v^k \;=\; p_x \, v \, b_{x+1} \;+\; q_x \qquad (4.51) $$

(Proof of this recursion is Exercise 6 above.)

Example 2. Find a simplified expression in terms of actuarial expected present value notations for the net single premium of an insurance on a life aged x*, which pays* $F(k) = C \, \ddot{a}_{\overline{n-k}|}$ *if death occurs at any exact ages between* $x+k$ *and* $x+k+1$*, for* $k = 0, 1, \ldots, n-1$*, and interpret the result.*

Let us begin with the interpretation: the beneficiary receives at the end of the year of death a lump-sum equal in present value to a payment stream of C annually beginning at the end of the year of death and terminating at the end of the n^{th} policy year. This payment stream, if superposed upon an n-year life annuity-immediate with annual payments C, would result in a certain payment of C at the end of policy years $1, 2, \ldots, n$. Thus the expected present value in this example is given by

$$C\, a_{\overline{n}|} \;-\; C\, a_{x:\overline{n}|} \tag{4.52}$$

Next we re-work this example purely in terms of analytical formulas. By formula (4.52), the net single premium in the example is equal to

$$\sum_{k=0}^{n-1} v^{k+1}\, {}_kp_x\, q_{x+k}\, C\, \ddot{a}_{\overline{n-k+1}|} \;=\; C \sum_{k=0}^{n-1} v^{k+1}\, {}_kp_x\, q_{x+k}\, \frac{1-v^{n-k}}{d}$$

$$= \frac{C}{d} \left\{ \sum_{k=0}^{n-1} v^{k+1}\, {}_kp_x\, q_{x+k} \;-\; v^{n+1} \sum_{k=0}^{n-1} \left({}_kp_x \;-\; {}_{k+1}p_x \right) \right\}$$

$$= \frac{C}{d} \left\{ \mathrm{A}^{1}_{x:\overline{n}|} \;-\; v^{n+1}\, (1 - {}_np_x) \right\}$$

$$= \frac{C}{d} \left\{ \mathrm{A}_{x:\overline{n}|} \;-\; v^n\, {}_np_x \;-\; v^{n+1}\, (1 - {}_np_x) \right\}$$

and finally, by substituting expression (4.15) with $m = 1$ for $\mathrm{A}_{x:\overline{n}|}$, we have

$$\frac{C}{d} \left\{ 1 - d\, \ddot{a}_{x:\overline{n}|} \;-\; (1-v)\, v^n\, {}_np_x \;-\; v^{n+1} \right\}$$

$$= \frac{C}{d} \left\{ 1 - d\, (1 + a_{x:\overline{n}|} \;-\; v^n\, {}_np_x) \;-\; d\, v^n\, {}_np_x \;-\; v^{n+1} \right\}$$

$$= \frac{C}{d} \left\{ v \;-\; d\, a_{x:\overline{n}|} \;-\; v^{n+1} \right\} \;=\; C \left\{ \frac{1-v^n}{i} \;-\; a_{x:\overline{n}|} \right\}$$

$$= \quad C \left\{ a_{\overline{n}|} \;-\; a_{x:\overline{n}|} \right\}$$

So the analytically derived answer agrees with the one intuitively arrived at in formula (4.52).

Example 3. Consider the following cohort life table fragment applicable to lives aged from 30 to 36. Find the risk premiums for unit-face-amount 6-year duration annuity-due and term insurance, (a) with m=1, (b) with $m = 4$ and uniform failure density within year of failure, and (c) with $m = 4$ and respective failure probabilities .2, .2, .3, .3 of dying within the 4 quarter-years given the year of failure. Assume interest rate 5% throughout.

x	Age-range	l_x	d_x
30	$[30, 31)$	95000	165
31	$[31, 32)$	94835	150
32	$[32, 33)$	94685	155
33	$[33, 34)$	94530	158
34	$[34, 35)$	94472	172
35	$[35, 36)$	94300	187

To clarify the different calculations in the three parts (a)-(c), we provide R code as well as numerical answers. Parts (a) and (b) respectively make use of formulas (4.17), (4.19) and (4.35) plus (4.36).

```
> probmass = c(165,150,155,158,172,187)/95000
> kpx = 1-cumsum(c(0,probmass[1:5]))
> probmass
[1] 0.001737 0.001579 0.001632 0.001663 0.001811 0.001968
> kpx
[1] 1.0000 0.9983 0.9967 0.9951 0.9934 0.9916
> Ains = sum( 1.05^(-(1:6)) * probmass )
  AnnDue = sum( 1.05^(-(0:5)) * kpx )
  c(Ains=Ains, AnnDue=AnnDue)
      Ains   AnnDue
 0.008751 5.308505                 ### answer to (a)
> i4 = 4*(1.05^.25-1)
  aux1 = sum(1.05^(-(0:3)/4))/4
  aux2 = sum((0:3)*1.05^(-(0:3)/4))/4^2
  c(Ains.m = Ains*i4/.05,
     AnnDue.m = aux1*AnnDue - 1.05*aux2*Ains)
  Ains.m AnnDue.m
0.008592 5.209397                  ### answer to (b)
```

For part (c), we start from formulas (4.23) and (4.24) and calculate by decomposing sums as we did in Section 4.4,

$$\mathrm{A}^{(m)1}_{x:\overline{n}|} = \sum_{b=0}^{n-1} {}_bp_x \sum_{j=0}^{m-1} v^{b+(j+1)/m} \left({}_{(j+1)/m}q_{x+b} - {}_{j/m}q_{x+b} \right)$$

$$\ddot{a}^{(m)}_{x:\overline{n}|} = \sum_{b=0}^{n-1} {}_bp_x \, \frac{1}{m} \sum_{j=0}^{m-1} v^{b+j/m} \, (1 - {}_{j/m}q_{x+b})$$

Now for $m = 4$, we have in this problem the special assumption that for integers b and $0 \le j < 4$,

$$\big({}_{j/4}q_{x+b} - {}_{(j+1)/4}q_{x+b}\big)/q_{x+b} = \begin{cases} 0.2 & \text{if} \quad j = 0, 1 \\ 0.3 & \text{if} \quad j = 2, 3 \end{cases}$$

It follows that ${}_{j/4}q_{x+b} \, / \, q_{x+b}$ has respective values 0, 0.2, 0.4, 0.7 for $j = 0, 1, 2, 3$. Substituting, we find

$$\mathrm{A}^{(4)\,1}_{\;x:\overline{n}|} = \sum_{b=0}^{n-1} {}_bp_x \, q_{x+b} \, v^{b+1} \, (1+i) \, \Big(0.2\, v^{1/4} \, + \, 0.2\, v^{1/2} \, + \, 0.3\, v^{3/4} \, + \, 0.3\, v\Big)$$

$$= \mathrm{A}^1_{x:\overline{n}|} \cdot \Big(0.2\, v^{1/4} + +0.2\, v^{1/2} \, + \, 0.3\, v^{3/4} \, + \, 0.3\, v\Big) \, (1+i)$$

and

$$\ddot{a}^{(4)}_{x:\overline{n}|} = \frac{1}{4} \sum_{b=0}^{n-1} {}_bp_x \, v^b \, \Big(1 \, - \, v\, q_{x+b} \, (0.2\, v^{-3/4} \, + \, 0.4\, v^{-1/2} \, + \, 0.7\, v^{-1/4})\Big)$$

$$= \ddot{a}_{x:\overline{n}|} - \frac{1}{4} \mathrm{A}^1_{x:\overline{n}|} \Big(0.2\, v^{-3/4} \, + \, 0.4\, v^{-1/2} \, + \, 0.7\, v^{-1/4}\Big)$$

The numerics in R for part (c) now follow:

```
> aux3 = .2*(1.05^.75+1.05^.5)+.3*(1.05^.25 + 1)
  aux4 = .2*1.05^.75+.4*1.05^.5+.7*1.05^.25
  c(Ains.ptc = Ains*aux3, AnnDue.ptc = AnnDue - 0.25*Ains*aux4
  Ains.ptc AnnDue.ptc
  0.008892   5.305604
```

So the different within-year distribution of failures made roughly a 2% difference in the term insurance and temporary life annuity-due risk premiums.

Example 4. Find the risk premiums for a 24-year life annuity-due and insurance figured with $m = 1$ and $m = 4$ based on the probability densities, with parameters as given in Figure 2.5 in Chapter 2, (a) Gamma$(14.74, .4383)$*, and (b) Lognormal*$(3.491, (.246)^2)$.

Here the ideas and formulas are simple: the only issue is how to organize the numerical calculations. The Gamma distribution function can be directly called in R, while the Lognormal distribution function values are simply expressed in terms of the normal distribution function. In neither case need we perform numerical integrations.

```
kpx1 = 1-pgamma(0:24, rate=.4383, shape=14.74)
kpx2 = 1-pnorm(log(0:24),mean=3.491, sd =.246)
array(c(sum(-diff(kpx1)/1.05^(1:24)),
        sum(-diff(kpx2)/1.05^(1:24)),
        sum(kpx1[1:24]/1.05^(0:23)),
        sum(kpx2[1:24]/1.05^(0:23)),
        kpx1[12], kpx2[12], kpx1[24], kpx2[24]), dim=c(2,4),
        dimnames=list(c("Gamma","Lognormal"),
          c("Ains","AnnDue","P(T-x>12)","P(T-x>24)")))

             Ains AnnDue P(T-x>12) P(T-x>24)
Gamma     0.04558  14.36    0.9998    0.9001
Lognormal 0.03504  14.41    1.0000    0.9258
```

The large relative difference between the term-insurance risk premiums is due to the more rapid decrease of the Gamma versus the Lognormal survival function between 12 and 24 years, which can be seen also in Figure 2.5.

4.8 Useful Formulas from Chapter 4

$$T_m \;=\; [Tm]/m \;=\; x + \frac{k}{m} \quad \text{if} \quad x + \frac{k}{m} \le T < x + \frac{k+1}{m}$$

p. 117

$$P(T_m = x + \frac{k}{m} \mid T \ge x) \;=\; {}_{k/m}p_x \;-\; {}_{(k+1)/m}p_x \;=\; {}_{k/m}p_x \;\cdot\; {}_{1/m}q_{x+k/m}$$

p. 118

$$\text{Endowment} \quad \mathrm{A}_{x:\overline{n|}}^{\;\;1} \;=\; {}_nE_x \;=\; E_x\Big(v^n\, I_{[T-x\ge n]}\Big) \;=\; v^n\, {}_np_x$$

p. 120

$$\mathrm{A}_{x:\overline{n|}} \;=\; \mathrm{A}^1_{x:\overline{n|}} \;+\; \mathrm{A}_{x:\overline{n|}}^{\;\;1} \;=\; \mathrm{A}^1_{x:\overline{n|}} \;+\; {}_nE_x$$

p. 121

$$\ddot{a}_{x:\overline{n|}} \;=\; E_x\Big(\frac{1 - v^{\min([T-x]+1,\, n)}}{d}\Big) = \frac{1 - \mathrm{A}_{x:\overline{n|}}}{d}$$

p. 122

$$d\, \ddot{a}_{x:\overline{n|}} \;+\; \mathrm{A}_{x:\overline{n|}} \;=\; 1$$

p. 122

$$\text{Term (temporary) life annuity} \quad a_{x:\overline{n|}} \;=\; \ddot{a}_{x:\overline{n+1/m|}} \;-\; 1/m$$

p. 122

$$A_x^{(m)} \;-\; \mathrm{A}^{(m)1}_{\;\;x:\overline{n|}} \;=\; v^n\, {}_np_x \;\cdot\; A_{x+n}$$

p. 123

$$\text{Term Insurance} \quad \mathrm{A}^1_{x:\overline{n|}} \;=\; E_x\left(v^{[T-x]+1}\, I_{\{T<x+n\}}\right) \;=\; \sum_{k=0}^{n-1} v^{k+1}\, {}_kp_x\, q_{x+k}$$

p. 124

$$\ddot{a}_{x:\overline{n}|} \;=\; E_x\Big(\ddot{a}_{\overline{\min([T-x]+1,n)}|}\Big) \;=\; \sum_{k=0}^{n-1} v^k \;{}_kp_x$$

p. 124

$$\mathrm{A}_{x:\overline{n}|} \;=\; \sum_{k=0}^{n-1} v^{k+1}\left({}_kp_x - {}_{k+1}p_x\right) + \; v^n \;{}_np_x$$

p. 124

$$\mathrm{A}^1_{x:\overline{n}|} \;=\; E_x\left(v^{T_m-x+1/m}\, I_{\{T<x+n\}}\right) \;=\; \sum_{k=0}^{nm-1} v^{(k+1)/m}\;{}_{k/m}p_x\;{}_{1/m}q_{x+k/m}$$

p. 126

$$\mathrm{A}^{(m)}_{x:\overline{n}|} \;=\; \mathrm{A}^{(m)1}{}_{x:\overline{n}|} + \mathrm{A}^{(m)}{}_{x:\overline{n}|}^{\;\;\,1} \;=\; \mathrm{A}^{(m)1}{}_{x:\overline{n}|} + {}_nE_x$$

p. 127

$$\mathrm{A}^{(m)}_{x:\overline{n}|} \;=\; \sum_{k=0}^{nm-1} v^{(k+1)/m}\left({}_{k/m}p_x - {}_{(k+1)/m}p_x\right) + \; v^n\;{}_np_x$$

p. 127

$$\ddot{a}^{(m)}_{x:\overline{n}|} \;=\; E_x\Big(\frac{1-v^{\min(T_m-x+1/m,\,n)}}{d^{(m)}}\Big) = \frac{1-\mathrm{A}^{(m)}_{x:\overline{n}|}}{d^{(m)}}$$

p. 127

$$d^{(m)}\;\ddot{a}^{(m)}_{x:\overline{n}|} + \mathrm{A}^{(m)}_{x:\overline{n}|} \;=\; 1$$

p. 127

$$a^{(m)}_{x:\overline{n}|} \;=\; \ddot{a}^{(m)}_{x:\overline{n+1/m}|} - 1/m$$

p. 128

under (i): $$\mathrm{A}^{(m)\,1}_{x:\overline{n}|} = (i/i^{(m)}) \sum_{b=0}^{n-1} v^{b+1}\, {}_bp_x\, q_{x+b} = (i/i^{(m)})\, \mathrm{A}^1_{x:\overline{n}|}$$

p. 131

under (i): $$\ddot{a}^{(m)}_{x:\overline{n}|} = \frac{1}{d^{(m)}} \Big(1 - \frac{i}{i^{(m)}}\, \mathrm{A}^1_{x:\overline{n}|} - v^n\, {}_np_x\Big)$$

p. 131

$$\mathring{\mathrm{e}}_x = E_x(T-x) = \int_0^\infty s\, \mu(x+s)\, {}_sp_x\, ds$$

p. 134

$$\overline{\mathrm{A}}^1_{x:\overline{n}|} = E_x\left(v^{T-x}\, I_{\{T-x\le n\}}\right) = \int_0^n v^s\, \mu(x+s)\, {}_sp_x\, ds$$

p. 134

$$\overline{a}_{x:\overline{n}|} = E_x\left(\int_0^n v^y\, I_{\{y\le T-x\}}\, dy\right) = \int_0^n v^y\, {}_yp_x\, dy$$

p. 135

$$\delta\, \bar{a}_{x:\overline{n}|} + \bar{\mathrm{A}}_{x:\overline{n}|} = 1$$

p. 136

under (ii): $$\mathring{\mathrm{e}}_x - \mathrm{e}_x = \sum_{k=0}^{\omega-x-1} {}_kp_x \Big(- p_{x+k} - \frac{q_{x+k}}{\ln p_{x+k}}\Big)$$

p. 142

Appendix A

Duration Data Structures

In this Chapter, we introduce some of the features of real data structures embodying waiting-time or duration data. Such data arise in a wide variety of disciplines and applied fields, including:

- *Life Insurance*, where payments are made and received as contractually determined functions of the duration of an insured individual's lifetime;
- *Casualty Insurance*, where the durations of interest are the times until accident, health emergency, or other adverse occurrence resulting in liability or loss;
- *Other Insurance*, such as mortgage insurance relating to the waiting time until a specified emergency resulting in
- *Clinical Trials* and other Biomedical studies, where human lives meeting specific criteria are followed between some initiating event (such as diagnosis of a disease or a specific treatment or intervention) and a response of interest (such as alleviation of symptoms, or death, or tumor recurrence or return of other disease condition);
- *Epidemiology*, where larger human populations are followed between recruitment to a study population
- *Reliability*, where the object of study is either cumulative time or cumulative operational loading in an engineered system until failure or specified degradation of performance; and

- *Economics*, where the waiting times of interest are generally times of transition, such as those for individuals from employment to unemployment or vice versa, for businesses from inception to profitability or bankruptcy, for economies between macroeconomic events, etc.

All of these examples involve the analysis of 'lifetime' or 'waiting-time' or 'duration' data, consisting of a **waiting-time random variable** T observed, incompletely, for many individuals of a **study population**. All of them also consider probability distributions and expected values for functions depending on waiting-time random variables, and for many purposes of statistical anlysis and estimation, reduce the complex data as actually recorded into the idealized format of the Life Table.

A.1 Concepts and Terminology of Duration (or Mortality, or Survival) Studies

We next define and discuss some concepts and terminology that will allow us to identify common versus distinct aspects of duration data in the different subject areas listed above. We restrict attention in our discussion to studies and datasets concerning individuals, usually people, being observed over chronological time intervals from entry into the study until the occurrence of an event of interest – the **study endpoint** – or the end of followup – called a **right-censoring time** – whichever comes first.

Study Population. In a formal observational setting, the study population is defined thorugh qualifying characteristics. For example, one might recruit into a clinical trial males with the same disease diagnosis at a designated set of hospital centers, who are between 30 and 60 years of age and otherwise is good health, and who consent to be randomly assigned to an experimental versus standard treatment. In an epidemiologic context, the population might consist of those in certain professions or risky occcupations, age-intervals, and locations, or who have specified existing medical conditions (such as high cholesterol) and consent to participate in a study entailing a number of scheduled medical examinations. In an insurance context, mortality or time-to-event statistics might be compiled for all individuals insured by one or a group of companies, over a specified time-window, or the subset of such people subject to a particular risk – such as ‘cigarette smokers’. In some insurance tabulations, data are gathered on special high- or low-risk populations in order to justify premiums different from those paid by the general population.

Mode of entry. Depending on the purpose for which time-to-event data are gathered, the initiating event for the interval of length T can have different possible relationships to the chronological time at which an individual is brought under observation. The simplest case is where these are the same, or where all individuals in the study are entered simultaneously: this kind of survival or duration study is called a **cohort study**. For example, a study in which babies born in a given year are followed for the next period of (3 or 10 or 20) years would be a cohort study, as would a reliability study in which 100 machines of a given type are set running – possibly under heighted load or stress – and observed until failure. Another example would be a survival study in which the interesting duration variable T is ‘time from diagnosis to death’, and data are to be collected by followup over time on a set of subjects who receive this diagnosis within a short period, say three months.

More broadly, and with a slightly different meaning, the term “cohort study” applies to longitudinal data collected on a set of individuals selected simultaneously at the outset, for example in a survey, and then followed over time. In that usage, ‘cohort’ and ‘longitudinal’ study are roughly synonymous. As used in an actuarial or demographic context, which is the way we use it in this book, ‘cohort’ refers to a set of individuals who have the same whole-number age at the same time and whose waiting time until death or other failure-event is of interest. In this way, one could refer to the ‘cohort’ of US males in the state of New Jersey who were 50 years old in 1977.

On the other hand, most survival studies and insurance portfolios consist of individuals who at any single chronological time have widely differing current ages. Whether in clinical trials or Insurance, entry of individuals into observation occurs by **staggered entry**, at differing chronological times chosen by the individual. In demography or epidemiologic studies, large populations are studied beginning at a specified date, so that all **entry times** into the study are simultaneous, but the individuals’ ages at entry vary.

Usually in Insurance and demography and epidemiology, the time variable of interest is age. Thus, birth is the event initiating the individual’s clock, but at entry into the data-collection, the individual's age is recorded. When the entry age is positive, the individual's data are said to be **left-censored**: the individual could have been observed to experience the study endpoint only at an age greater than the entry age.

Mode of study termination. Survival and other duration studies are often conducted over fixed administrative time windows. Subjects enter either together, in a cohort, or individually, staggered. The study will end and be reported as of a fixed chronological termination time, so individuals under study may have a positive age at entry and age of last followup in the study without ever having experienced the study endpoint. Moreover, in many studies as in Insurance portfolios, individuals can withdraw from observation before the study endpoint, for reasons which may or may not be related to the nearness of that endpoint. For these reasons, data about the individual's variable T may be incomplete and are said to be **right-censored**: within the dataset, the individual's T is known only to be greater than or equal to the last age of followup, which is also called that individual's **right-censoring time**.

Based on the examples and discussion above, we can formulate the following general data structure for a duration or survival study. If the individuals in a study are indexed administratively by $i = 1, 2, \ldots, N$, then each individual must come equipped with at least the following information:

$E_i =$ chronological time of entry of individual i into the study

$A_i =$ age of individual i at entry into the study

$T_i =$ age of individual i at last followup or endpoint under the study

$D_i =$ binary indicator equal to 1 if i experiences endpoint during followup, and equal to 0 otherwise

In terms of these notations, individual i first enters the study at chronological time E_i and is under active observation, or **under followup**, for a total duration of $T_i - A_i$. Thus the chronological interval of followup is $[E_i,\, E_i + T_i - A_i]$. The individual's earliest age in the study is A_i, and the latest is T_i. If $D_i = 1$, the final age T_i is also the age at which the study endpoint is observed to occur for individual i, while if $D_i = 0$, then individual i does not experience the study end-point while under followup.

The terminology 'under followup' is the one used in clinical or epidemiologic settings. Expressed in terms of age during the followup period, individual i would be said to be **on test** — by adoption of an older terminology from Reliability — on the age interval $[A_i, T_i]$. The biomedical term would be that individual i is **at risk** at ages in the interval $(A_i, T_i]$, while the Insurance term is that the individual is **exposed** (or 'exposed to risk', as might be said also in an epidemiologic or demographic context) on that age interval.

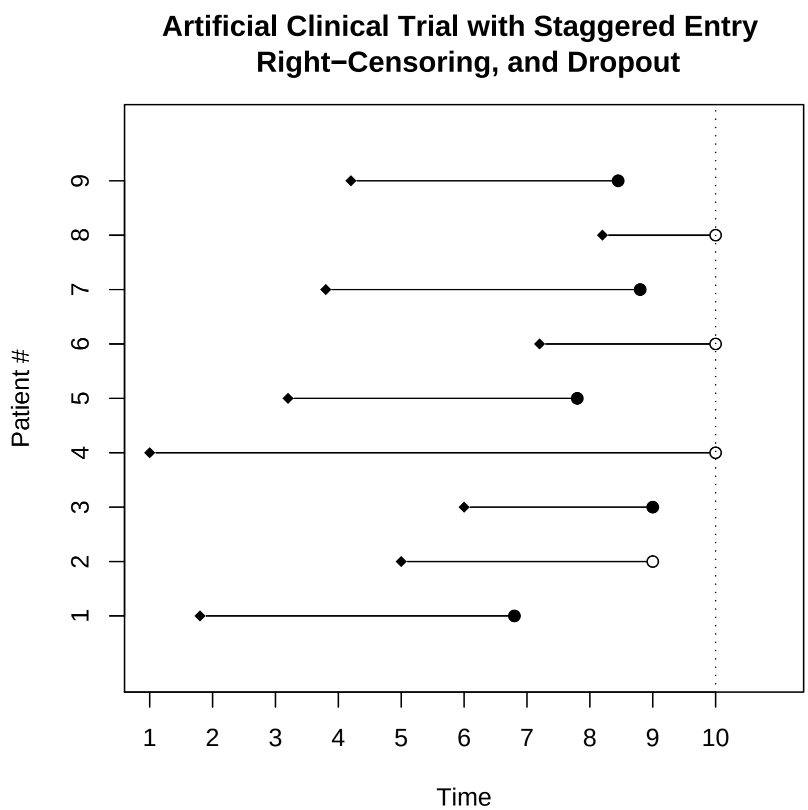

Figure A.1: Representation of entry times (solid diamonds), at-risk intervals (solid line segments), and death (filled circle) or right-censoring (hollow circle) for each of 9 patients in an artificial clinical trial.

The definitions for right-censored staggered-entry survival data are illustrated in Figure A.1, for 9 patients in an artificial clinical trial with stagggered entry. Each patient's interval at risk is a horizontal line segment beginning just after the entry-time depicted with a solid diamond and ending at the time indicated with a circle, solid if an observed death and hollow if a dropout or censoring time. 'Dropout' takes place at the hollow circle for patient 2, while the other censoring events would be called 'administrative right-censoring' because the clinical trial observation periods all end at the chronological time 10 indicated by a vertical dashed line.

The central idea of the **Life Table**, described in Section A.2 below, is to tabulate over equally spaced intervals of age, or time-on-test, the numbers of study subjects at risk and observed to fail (i.e., to experience the study endpoint while under followup). This idea is fundamental to statistical analysis of duration data in all of the fields of study listed at the beginning of this Chapter.

In this Section, we have addressed the most frequently occurring complexities of Insurance and other survival data as actually collected. However, there are still other complexities, some of which we can mention briefly. Survival data, or their underlying study populations are often defined through information collected in sample surveys, in which some demographic groups are given heavier weight than their proportion in the general population. The way in which the data should be analyzed depend on survey inclusion probabilities or weights, and also on what the target sampled population was, including whether the criteria for inclusion depend on time-dependent (for example health-related) variables. Two books describing many forms of biomedical survival data, with examples, are those of Klein and Moeschberger (2003) and Lee (1992).

A.2 Formal Notion of the Life Table

Consider the artificial clinical trial data summarized in Figure A.1. Such data might have been collected for the purpose of understanding the rate of mortality at different ages. The **Life Table**, or more specifically the *cohort life table*, is a simplified representation which summarizes only the numbers 'at risk' and the numbers observed to fail and be censored, in the one-year age intervals $[1, 2)$, $[2, 3)$, $\ldots$, $[9, 10)$. In the notation of the previous Section, the patient labelled i is said to be at risk (i.e. could potentially be observed to die) at time t if $A_i < t \le T_i$, and is actually observed to die at T_i only if $D_i = 1$. If we agree to consider only integer times $t = k$, then we have

$$Y_k \;=\; \#\text{ at risk at age } \; k \;=\; \sum_i I_{[A_i < k \le T_i]}$$

$$d_k \;=\; \#\text{ observed to die in } \; [k, k+1) \;=\; \sum_i D_i \, I_{[k \le T_i < k+1]}$$

$$c_k \;=\; \#\text{ right-censored in } \; [k, k+1) \;=\; \sum_i (1 - D_i) \, I_{[k \le T_i < k+1]}$$

However, it is not true that all individuals dying within the age interval $[k, k+1)$ were necessarily at risk at time k. For this reason, it is important to tabulate an additional quantity, the total number of subject-years in the survival study during which subjects were under followup at exact ages in $[k, k+1)$:

$$\tau_k \;=\; \text{Time on test at ages in } \; [k, k+1) \;=\; \sum_i (\min(T_i, k+1) - \max(k, A_i))$$

Clearly, τ_k is more informative than simply Y_k as a denominator against which to compare the observed number of failures d_k in order to estimate the rate of failures within the successive age-intervals. Indeed, we will see in Chapter 8 that a reasonable estimator of the death-rate within age-intervals $k, k+1)$ are the ratios $\lambda_k \;=\; d_k/\tau_k$.

To complete this brief illustration, we tabulate in Table A.1 the quantities mentioned so far for the data represented in Figure A.1.

A.2.1 The Cohort Life Table

If a *cohort* of individual subjects were entered into a study simultaneously with the same age-variable and followed up until they died, then the life table could have a simpler form, and a simpler interpretation. In that case, the right-censored counts c_k would all be 0, and the table itself would contain all of the information $(E_i, A_i, T_i, D_i)_{i=1}^n$ for the n subjects. If a were the common initial age, then the proportions Y_k/Y_a would estimate

Table A.1: Life table quantities for the staggered-entry survival data used to construct Figure A.1.

k	Age-int	Y_k	d_k	c_k	τ_k	λ_k
1	$[1,2)$	0	0	0	1.2	0.0
2	$[2,3)$	2	0	0	2.0	0.0
3	$[3,4)$	2	0	0	3.0	0.0
4	$[4,5)$	4	0	0	4.8	0.0
5	$[5,6)$	5	0	0	6.0	0.0
6	$[6,7)$	6	1	0	6.8	0.147
7	$[7,8)$	5	1	0	6.0	0.167
8	$[8,9)$	6	1	0	6.05	0.165
9	$[9,10)$	4	3	1	3.0	1.0

the fraction of the population studied which survived to age k, and the identity $Y_{k+1} = Y_k - d_k$ would always hold.

However, only in very special applications, not in Insurance, can data actually be collected in this cohort format. One is a study which follows up a cohort of newborns, or a cohort of people selected somehow either at the same age or with the same initiating event (like diagnosis of a disease whose mortality is of interest). Some longitudinal epidemiologic studies, like the famous Framingham study [ref ?] which monitored various risk factors for heart disease, follow large numbers of people – many of whom are of the same age or fall into narrow age brackets initially – over time. Animal studies can follow cohorts, e.g. of newborn laboratory rats, which are subjected to the same diets or survival stresses. But the one area of application in which this kind of data is very common is Engineering Reliability, where a number of devices are set running at identical (usually accelerated) stresses, in parallel, and observed until they fail.

Despite the fact that mortality data for large human populations are generally not collected in cohorts, the data are often tabulated as though they *were* collected that way. Regardless of how the data in a mortality study were collected, once can first estimate the age-specific death rates directly,

q_k = number of observed deaths at exact ages in the interval $[k, k+1)$, divided by the total number of person-years spent by subjects under followup in the study

This might be done in practice only after approximating or imputing the times on test not directly observable in the study. Moreover, in Demography or Insurance, the estimated death-rates are often altered slightly to enhance smoothness of the estimated death-rates as a sequence indexed by k. Finally, in presenting the death-rates for purposes of calculation of insurance premiums or population projections, the death rates are presented in the tabular form which we now define as the **cohort life table**.

The cohort life table, briefly, displays the (integer-rounded) numbers of expected survivors at each birthday k and numbers of deaths between successive birthdays, for a population of large hypothetical size experiencing exactly the death rates q_k interpreted as conditional probabilities of dying at age $[k, k+1)$ given survival to the k'th birthday. Begin by choosing a large conventional size l_0, called the **radix** of the cohort life table, for a population cohort of newborns. This number is a power of 10, usually 10^5. Denote by $[\cdot]$ the greatest-integer or floor function. Then the cohort life table consists of the columns

$$l_k \;=\; \Big[l_0 \prod_{j=0}^{k-1} q_j \Big] \;=\; \text{number of lives aged } \; k$$

$$d_k \;=\; l_k - l_{k+1} \;=\; \text{number of deaths at ages in } \; [k, k+1)$$

for k ranging from 0 up to and including the largest integer age $\omega - 1$, where ω is the **terminal age**) seen for any subject of the mortality study. Next to these columns may also be displayed the death-rates q_k. Note that, apart from rounding errors, $q_k = d_k / l_k$ for all k.

This 'life table' is an artificial construction, referring directly to no actually observed population, but containing exactly the same information as the column of (smoothed, rounded) death rates q_k. It is the mortality record of a fictitious population cohort with exactly the same death rates after smoothing and rounding as those estimated from some actual population.

A.3 Sample Spaces for Duration Data

The preceding sections have described first the actual setting in which random durations are observed within a realistic mortality study, and then the idealized presentation of the observed mortality in the form of a cohort life table, in which mortality of a fictitious population cohort is recorded. From the viewpoint of Probability Theory, random variables or data are formalized as measurement functions on the sample space Ω of all possible detailed outcomes of a survival experiment. It is instructive to define the sample spaces needed at three levels of complexity of the probability and statistics of survival models.

A.3.1 Sample Space for a Single Newly Insured Life

The simplest case is the one studied in the first two chapters of this book, where only integer ages are recorded in the survival experiment, and all probabilities and expected values related to functions of a single lifetime or integer-age-at-death random variable $[T]$. Here the sample space and underlying probability is very easy to describe: $\Omega = \{1, 2, \ldots, l_0\}$ enumerates the l_0 lives summarized in the cohort life table, with equal assigned probability $Pr(\{i\}) = 1/l_0$ for each individual labelled i, $1 \le i \le l_0$. Recalling that $d_k = l_k - l_{k+1}$ in the cohort life table, for all $k = 0, 1, \ldots, \omega - 1$, we note that $l_0 = \sum_{k=0}^{\omega-1} d_k$. Then the single integer-valued age random variable $[T]$ for a new individual being insured can be explicitly constructed as a function of $i \in \Omega$ as follows: for $k = 0, 1, \ldots, \omega - 1$,

$$[T](i) = k \quad \text{if and only if} \qquad \sum_{j=0}^{k-1} d_j < i \le \sum_{j=0}^{k} d_j \tag{A.1}$$

The interpretation of this rule is that if we number the l_0 individuals i in the cohort life table in order of the whole-number age k at which they die, then the first $\sum_{j=1}^{k-1} d_j = l_0 - l_k$ individuals die at ages less than k, and the next d_k individuals die at integer age k.

The underlying random experiment is to select an individual i equiprobably from the list of all all l_0 individuals in the cohort 'population': that is, in this simplified model the lifetime $[T]$ of the newly to-be-insured individual is modelled as being the same as a randomly selected member of the cohort population. Then the event $[T] = k$ consists precisely of the subset of indices i satisfying $\sum_{j=0}^{k-1} d_j < i \le \sum_{j=0}^{k} d_j$, and therefore has size d_k and probability d_k/l_0 as desired.

Remark A.1 *In the next subsection, we consider the sample space appropriate to a cohort of lives, assumed independent, simultaneously following the same mortality rates summarized in a cohort life table. There are cases of intermediate complexity not discussed in detail in this book, cases where each lifetime is additionally labelled by a cause of death* L *or where events defined in terms of the dependent lifetimes* T^A, T^B *of a pair of related lives (such as a husband-wife pair) have consequences for Insurance. The feature of these intermediate-complexity sample spaces is that a single vector* $(T_1, \ldots, T_K)$ *of finitely many, possibly dependent, lifetime random variables are modelled simultaneously.*

The case of lifetimes (T, L) *labelled by cause arises when for each of* K *distinct types of mortality, such as death from specified disease (as in 'cancer insurance') or accident or from other causes, there is an underlying random variable* T_k, $k = 1, 2, \ldots, K$, *giving the age at which the individual would have died from that cause if not earlier killed from another cause. Then the actual observed failure age* T *is* $\min(T_1, \ldots, T_K)$ *and the random label* L *is the integer in* $\{1, 2, \ldots, K\}$ *for which* $T_L = T$. *This setup is called a* **competing risks** *model (see Gail 1975; David and Moeschberger 1978) in the biostatistical literature, and relates to* **multiple decrement** *(cohort) life tables in an Insurance context (Gerber 1997 Ch. 7; Jordan 1991 Part II), but these topics are not treated further in this book.*

The joint modelling of pairs (T^A, T^B) *or larger multi-life groups of lifetimes is important in the calculation of insurance premiums and annuity or pension values for husband-wife pairs, for example in insurances of both husbands and wives or in annuities — possibly variable, like US Social Security, or with a smaller payment to the survivor — which revert to the surviving member of the pair when one member dies. This topic is treated under the heading of* **contingent multi-life functions** *(Jordan 1991 Part II, or Gerber 1997 Ch. 8).* □

A.3.2 Sample Space for a Full Cohort Population

As mentioned explicitly in Section A.2.1, the cohort population whose mortality is summarized in the cohort life table is generally a complete fiction. Nevertheless, the relative frequency ratios defining the survival functions and death rates which are derived from the life table and used in calculated insurance premiums could also be viewed as statistical estimators of unknown statistical parameters based on a set of n independent identically distributed lifetime random variables T_i, $1 \leq i \leq n$. That is, rather than viewing the cohort data as fixed, we can view them as the realized values of a set of independent identically distributed lifetime random variables representating a realistic underlying mechanism of mortality. In particular, while the sample space described in Section A.3.1 is inherently discrete, it is a little more realistic to treat the possible cohort lifetimes as a set of continuous random variables, which are independent across individuals and can each take values anywhere on the positive age axis.

The sample space described in this Section allows us to consider the intrinsic variability of the estimated death rates q_k and other statistics derived as a function of observed age-at-death random variables *if those variable values were lifetime lengths of individuals under followup for their entire lives*. This aspect of the random mortality experiment is still an artificial idealization, since we have already argued in this Chapter that realistic mortality studies generally have a much more complicated and inconvenient pattern of staggered positive ages at entry and of loss to followup before death for many subjects under study.

The greater realism of cohort-type survival experiments, whose sample sizes we now define, is of particular use in Chapter 3 of this book, where the quality of death-rate estimates is studied and where the simulation of new cohort life-tables with specified survival functions $S(t)$ is described. Yet it is immediately apparent that this realism comes at a price of greater mathematical complexity. The sample space itself must be a set of detailed outcomes not only for a single continuously distributed lifetime, but for a sequence of n independent lifetimes. The most natural space to use is $\Omega = \mathbf{R}_+^n = [0, \infty)^n$, with the vector-valued mapping given by the identity mapping on Ω:

$$\underline{T}(\underline{s}) \;=\; \{T_i(\underline{s})\}_{i=1}^n \;=\; \{s_i\}_{i=1}^n$$

Unlike the situation in Section A.2.1, where the random-lifetime mapping was defined in such a way that the probabilities associated with individual outcomes were equiprobable, now the probabilities are defined by the property that the lifetimes all follow survival function $S(t)$ and are independent of one another, a specification accomplished by the definition

$$Pr\Big(\Big\{\underline{s} \in \mathbf{R}^n \ : \quad a_1 < s_1 \le b_1, \, a_2 < s_2 \le b_2, \, \ldots, \, a_n < s_n \le b_n\Big\}\Big) \ =$$

$$Pr\Big(\Big\{\underline{s} \in \mathbf{R}^n \ : \ \underline{T}(\underline{s}) \in (a_1, b_1] \times (a_2, b_2] \times \cdots \times (a_n, b_n]\Big\}\Big) \ = \prod_{i=1}^{n} (S(a_i) - S(b_i))$$

Probabilities of other events concerning the random-variable components $T_i(\underline{s})$ are implicitly determined from this definition on n-dimensional recangles $\bigotimes_{i=1}^{n} (a_i, b_i]$ by means of the probability axioms (finite or countable additivity), since a very large class of events can be generated by limits of increasing unions and decreasing intersections of unions of such rectangles. Further details of the unique specification of probability laws from a generating collection of open sets can be found in more advanced treatments of Probability Theory, such as Ross (2005) or Billingsley (1995).

A.3.3 Sample Space for the Realistic Mortality Study

The Sample Space Ω needed to accomodate the detailed outcome data $(E_i, A_i, T_i, D_i, \ 1 \le i \le n)$ described in Section A.1 requires a Cartesian product of $\mathbf{R}_+^{3n}$ whose coordinates model the values factors of all of the random variables $E_i, A_i, T_i, \ 1 \le i \le n$, along with a further space $\{0,1\}^n$ to model the values $D_i, \, 1 \le i \le n$. The usual assumption of independence of (E_i, A_i, T_i, D_i) across different individuals i is embodied in a definition of Probability on Ω as a so-called 'product probability' across n spaces $\mathbf{R}_+^3 \times \{0,1\}$. However, the joint probability density of (E_i, A_i, T_i, D_i), can have all sorts of different realistic dependence structures. We refer to texts on Survival Analysis (Cox and Oakes 1994; Klein and Moeschberger 2003; David and Moeschberger 1978) for discussion of such matters. In this book, only in Chapter 8 do we address a simplified although typical setting ('independent death and censoring') to introduce maximum likelihood estimators of survival in models with piecewise constant hazards and Kaplan-Meier estimators in models with general ('nonparametric') hazards.

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