Math 241 Exam 4 Sample 3 Solutions

1. We’re looking for

\[ \int\int_{\Sigma} (0 \hat{i} + x \hat{j} + z \hat{k}) \cdot \hat{n} \ dS \]

We parametrize \( \Sigma \) by

\[ \vec{r}(x, y) = x \hat{i} + y \hat{j} + (9 - x^2) \hat{k} \quad \text{with} \quad 0 \leq x \leq 2, \ 0 \leq y \leq 2 \]

and then

\[ \vec{r}_x = 1 \hat{i} + 0 \hat{j} - 2x \hat{k} \]
\[ \vec{r}_y = 0 \hat{i} + 1 \hat{j} + 0 \hat{k} \]
\[ \vec{r}_x \times \vec{r}_y = 2x \hat{i} + 0 \hat{j} + 1 \hat{k} \]

Note that these vectors have a positive \( \hat{k} \) component which matches the orientation of \( \Sigma \). So we have

\[ \int\int_{\Sigma} (0 \hat{i} + x \hat{j} + z \hat{k}) \cdot \hat{n} \ dS = + \int\int_{R} (0 \hat{i} + x \hat{j} + (9 - x^2) \hat{k}) \cdot (2x \hat{i} + 0 \hat{j} + 1 \hat{k}) \ dA \]

\[ = \int_{R} 9 - x^2 \ dA \]
\[ = \int_{0}^{2} \int_{0}^{2} 9 - x^2 \ dy \ dx \]
\[ = \int_{0}^{2} 9y - x^2 y \bigg|_{0}^{2} \ dx \]
\[ = \int_{0}^{2} 18 - 2x^2 \ dx \]
\[ = 18x - \frac{2}{3} x^3 \bigg|_{0}^{2} \]
\[ = 18(2) - \frac{2}{3}(2)^3 \]
2. (a) Since $\vec{F}$ is conservative with potential function $f(x, y) = \frac{1}{2}x^2y^2 + x$ and so

$$\int_C x^2y + 1 \, dx + xy^2 \, dy = f(3, 3) - f(1, -2)$$

$$= \left[ \frac{1}{2}(3)^2(3)^2 + 3 \right] - \left[ \frac{1}{2}(1)^2(-2)^2 + 1 \right]$$

(b) We parametrize the line segment as

$$\vec{r}(t) = 5t \, \hat{i} + 4t \, \hat{j} \quad \text{with} \quad 0 \leq t \leq 1$$

and then

$$\vec{r}'(t) = 5 \, \hat{i} + 4 \, \hat{j}$$

$$||\vec{r}'(t)|| = \sqrt{41}$$

and then

$$\int_C 2x + y \, ds = \int_0^1 [2(5t) + 4t] \sqrt{41} \, dt$$
3. By Green’s Theorem we have

\[ \int_C 2x \, dx + x^2 \, dy = \iint_R 2x - 0 \, dA \]

where \( R \) is the region inside the curve. This region is parametrized best in polar coordinates so we have

\[ \int_C 2x \, dx + x^2 \, dy = \iint_R 2x - 0 \, dA \]
\[ = \int_0^{\pi/2} \int_0^2 2r \cos \theta \, r \, dr \, d\theta \]
\[ = \int_0^{\pi/2} \left[ \frac{2}{3} r^3 \cos \theta \right]_0^2 \, d\theta \]
\[ = \int_0^{\pi/2} \frac{14}{3} \cos \theta \, d\theta \]
\[ = \frac{14}{3} \left[ \sin(\pi/2) - \sin(0) \right] \]
\[ = \frac{14}{3} \]
4. The best $\Sigma$ would be the portion of the plane $x + y = 5$ inside the cylinder. The orientation of $\Sigma$ would be toward the right.

Since $\vec{F}(x, y, z) = x \hat{i} + 3 \hat{j} + 2y \hat{k}$ we have $\nabla \times \vec{F} = 2 \hat{i} + 0 \hat{j} + 0 \hat{k}$. Then Stokes’s Theorem tells us

$$\int_C x \, dx + 3 \, dy + 2y \, dz = \int_\Sigma (2 \hat{i} + 0 \hat{j} + 0 \hat{k}) \cdot \vec{n} \, dS$$

Since $\Sigma$ is inside the cylinder it’s a good choice to parametrize it as

$$\vec{r}(r, \theta) = r \cos \theta \, \hat{i} + (5 - r \cos \theta) \, \hat{j} + r \sin \theta \, \hat{k} \quad \text{with} \quad 0 \leq r \leq 2, \ 0 \leq \theta \leq 2\pi$$

Then

$$\vec{r}_r = \cos \theta \, \hat{i} - \cos \theta \, \hat{j} + \sin \theta \, \hat{k}$$
$$\vec{r}_\theta = -r \sin \theta \, \hat{i} + r \sin \theta \, \hat{j} + r \cos \theta \, \hat{k}$$
$$\vec{r}_r \times \vec{r}_\theta = -r \, \hat{i} - r \, \hat{j} + 0 \, \hat{k}$$

Note that these vectors point have negative $\hat{k}$ component and hence point left, opposite to that for $\Sigma$. So we have

$$\int_\Sigma (2 \hat{i} + 0 \hat{j} + 0 \hat{k}) \cdot \vec{n} \, dS = -\int_R (2 \hat{i} + 0 \hat{j} + 0 \hat{k}) \cdot (-r \hat{i} - r \hat{j} + 0 \hat{k}) \, dA$$
$$= -\int_R -2r \, dA$$
$$= -\int_0^{2\pi} \int_0^2 -2r \, dr \, d\theta$$
5. If $D$ is the solid cube then the Divergence Theorem gives us

$$\int_{\Sigma} (5x \hat{i} + 2y \hat{j} - 2z \hat{k}) \cdot \hat{n} \, dS = \int\int\int_D (5 + 2 - 2) \, dV$$

$$= 5 \int\int\int_D 1 \, dV$$

$$= 5 \text{ Volume of Cube}$$

$$= 5(8)$$

$$= 40$$