Math 140, Jeffrey Adams
Test II, March 10, 2010 SOLUTIONS

Question 1.
(a) \[ f'(x) = \lim_{t \to x} \frac{(t^2 + 2) - (x^2 + 2)}{t - x} \]
\[ = \lim_{t \to x} \frac{(t^2 - x^2)}{t - x} \]
\[ = \lim_{t \to x} \frac{(t - x)(t + x)}{t - x} \]
\[ = \lim_{t \to x} (t + x) = 2x \] \hspace{1cm} (1)

(b) When \( x \) is near 1 \( f(x) = 2x + 1 \), so
\[ f'(x) = \lim_{x \to 1} \frac{(2x + 1) - 3}{x - 1} = \lim_{x \to 1} \frac{2x - 2}{x - 1} = \lim_{x \to 1} \frac{2(x - 1)}{x - 1} = 2 \] \hspace{1cm} (2)

(c) Let \( g(x) = |x| \) and \( h(x) = x^2 - 1 \). Then \( f(x) = g(h(x)) \) so by the chain rule \( f'(x) = g'(h(x))h'(x) \) provided \( g'(h(x)) \) and \( h'(x) \) exists. Obviously \( h'(x) \) exists everywhere, so the only question is \( g'(h(x)) \). Since \( g(x) = |x| \) is differentiable except at \( x = 0 \), \( g'(h(x)) \) exists unless \( h(x) = 1 \), i.e. \( x^2 - 1 = 0 \), i.e. \( x = \pm 1 \). So \( f(x) \) is differentiable for all \( x \neq \pm 1 \).

This is also easy to see from the picture: the graph has corners at \( \pm 1 \).

Question 2.
(a) \[ \frac{d}{dx} \sin(x^2 + x) = \cos(x^2 + x)(2x + 1) \]

(b) \[ \frac{d}{dx} \ln(\cos(x^2)) = \frac{1}{\cos(x^2)}(-\sin(x^2)(2x)) \]

(c) \[ \frac{d}{dx} \left( \frac{x^2 + 1}{x+1} \right) = \frac{(2x)(x^2+1)-(x^2+1)(3x^2)}{(x^2+1)^2} \]

(d) \[ \frac{d^2}{dx^2}(xe^x): \frac{d}{dx}(xe^x) = e^x + xe^x, \text{ and } \frac{d^2}{dx^2}(xe^x) = \frac{d}{dx}(e^x + xe^x) = e^x + e^x + xe^x \]

Question 3.
(a) By implicit differentiation:
\[ \cos(y) + x(-\sin(y)) \frac{dy}{dx} + \frac{dy}{dx} = 0 \] \hspace{1cm} (3)
so
\[ \cos(y) + \frac{dy}{dx}(-x \sin(y) + 1) = 0 \] \hspace{1cm} (4)
or
\[ \frac{dy}{dx} = \frac{-\cos(y)}{-x \sin(y) + 1} \] \hspace{1cm} (5)
and plugging in \( x = 0, y = \pi \) gives \( \frac{dy}{dx} = -\cos(\pi) = 1 \).
(b) Label the distance from the wall to the foot of the latter $x(t)$, and the position of the top by $y(t)$. Then \( \frac{dx}{dt} = 8 \), and we want \( \frac{dy}{dt} \). Taking \( \frac{d}{dt} \) of \( x^2 + y^2 = 5 \) gives \( 2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 0 \).

\[
\frac{dy}{dt} = -\frac{2x \frac{dx}{dt}}{2y} \tag{6}
\]

Take \( \frac{dx}{dt} = 8 \). Also when \( y = 4 \) then \( x = 3 \) since \( x^2 + y^2 = 5^2 \), so

\[
\frac{dy}{dt} = -\frac{3 \times 8}{4} = -6.
\]

**Question 4.**

(a) \[
c_2 = c_1 - \frac{f(c_1)}{f'(c_1)}
= 1 - \frac{3}{3+2}
= \frac{2}{5}
\]

and

\[
c_3 = c_2 - \frac{f(c_2)}{f'(c_2)}
= \frac{2}{5} - \frac{\left(\frac{2}{5}\right)^2 + \left(\frac{3}{5}\right) + 1}{3\left(\frac{2}{5}\right)^2 + 2\left(\frac{3}{5}\right)}
\]

(b) Let \( f(x) = \sqrt{x} = x^{\frac{1}{4}} \), so \( f'(x) = \frac{1}{4}x^{-\frac{3}{4}} \). Take \( a = 16 \). Then \( f(a) = 2 \) and \( f'(a) = \frac{1}{4}16^{-\frac{3}{4}} = \frac{1}{4 \times 16^{\frac{3}{4}}} = \frac{1}{4 \times 2^{6/4}} = \frac{1}{32} \). So, with \( x = 17 \),

\[
\sqrt{17} \sim f(a) + f'(a)(x-a) = 2 + \frac{1}{32}(1) = 2 \frac{1}{32}. \tag{9}
\]