Math 140, Jeffrey Adams
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SOLUTIONS

Question 1.

(a) \( f(0) - f(-3) = \frac{(-2) - (-27 - 3 - 2)}{3} = \frac{30}{3} = 10 \), so \( f'(x) = 3x^2 + 1 \), so \( 3x^2 + 1 = 10 \), or \( 3x^2 = 9 \), i.e. \( x^2 = 3 \), i.e. \( x = \pm \sqrt{3} \).

The value which is between \(-3\) and \(0\) is \( x = -\sqrt{3} \).

(b) \( f(x) = \frac{1}{2}x^2 + \frac{1}{3}e^{3x} + C \) Then \( f(0) = 0 + \frac{1}{3} + C \), and setting this equal to \(1\) gives \( C = \frac{2}{3} \).

Question 2.

(a) Rough idea: ignore the constant terms to get \( \frac{2x}{\sqrt{2x^2}} = \frac{2x}{\sqrt{2x}} = \sqrt{2} \).

This is correct: the limit is \( \sqrt{2} \).

More carefully: multiply by \( \frac{1/x}{1/\sqrt{2x}} \) to get \( \frac{2x + 1}{\sqrt{2x^2 + 1}} = \frac{2 + \frac{1}{x}}{\sqrt{2 + \frac{1}{x^2}}} \). As \( x \to \infty \) this clearly goes to \( 2/\sqrt{2} = \sqrt{2} \).

(b) Let \( t = 0 \) be 3 PM. The first ship’s position is given by \( x(t) \). The velocity is 3, so \( x(t) = 3t + C \) for some \( C \). Since \( x(0) = 0 \), \( C = 0 \), and \( x(t) = 3t \).

The second ship’s position is \( y(t) = -4t + D \). Since \( y(0) = 5 \) this gives \( y(0) = -4(0) + D = 5 \), and \( D = 5 \). So \( y(t) = -4t + 5 \).

The distance \( D(t) \) is \( D(t) = \sqrt{(3t)^2 + (-4t + 5)^2} \). This equals \( \sqrt{9t^2 + 16t^2 - 40t + 25} = \sqrt{25t^2 - 40t + 25} = \sqrt{5^2 - 8t + 5} \).

Take the derivative: \( f'(t) = \sqrt{5^2 - 8t + 5} \). Setting this equal to \(0\), we can ignore the \( \sqrt{\text{ terms}}, and set \( 10t - 8 = 0 \), or \( t = 4/5 \). This says the time is 3 PM plus \(4/5\) of an hour, i.e. \( 3:48 \ PM \).
Question 3.

(a) The population is $P(t) = Ce^{kt}$. Since the doubling time is 3 years, this gives $P(3) = 2P(0)$, i.e. $Ce^{3k} = 2C$, or $e^{3k} = 2$, so $3k = \ln(2)$, or $k = \ln(2)/3$.

Setting $P(t) = 500,000 = 5C$ gives $P(t) = Ce^{kt} = 5C$, or $e^{kt} = 5$. Then $kt = \ln(5)$, so $t = \ln(5)/k$, i.e. $t = \ln(5)/((\ln(2))/3) = 3\ln(5)/\ln(2)$.

(b) $f'(x) = 3x^2 - 3$, and setting this equal to 0 gives $3x^2 = 3$, or $x = \pm 1$. Also check the endpoints:

$f(-2) = -8 + 6 = -2$, $f(3) = 27 - 9 = 18$, $f(1) = -2$ and $f(-1) = -1 + 3 = 2$. The max is $f(3) = 18$, and the mins are $f(-2) = f(1) = -2$.

Question 4. Consider the function $f(x) = \frac{1 + x}{1 - x}$.

(a) The limit at $x \to \pm \infty$ is $-1$, so it has horizontal asymptote $-1$ to the left and right. It has a vertical asymptote at $x = 1$.

(b) $f'(x) = \frac{(1-x)(1+x)(-1)}{(1+x)^2} = \frac{-2}{(1-x)^2}$, which is always positive: the function is increasing everywhere.

(c) The second derivative is $-4(1-x)^{-3}(-1) = \frac{4}{(1-x)^3}$. This is positive if $1 - x < 0$, i.e. $x < 1$, and negative if $x > 1$. So $f(x)$ is concave up for $x < 1$, and concave down for $x > 1$.

(d) Sketch the function.