1. Let \( \pi \) be a random permutation \( \pi \) of the set \( \{1, 2, \ldots, n\} \). Let \( p_n \) denote the probability that \( \pi \) has a cycle of length at least \( n/2 \).
   
   (a) Compute \( p_n \).
   (b) Compute \( \lim_{n \to \infty} p_n \).
   (c) Does your formula in part (a) work if \( n/2 \) is replaced by \( n/3 \)? Explain.

2. Let \( G \) be the directed graph with adjacency matrix \( A = \begin{pmatrix} 3 & 5 \\ 4 & 2 \end{pmatrix} \). (Here \( A(i, j) \) is the number of edges in \( G \) from vertex \( i \) to vertex \( j \).) Let \( p_n \) denote the number of paths of length \( n \) from vertex 1 to vertex 2 (these are the paths of \( n \) edges beginning at vertex 1 and ending at vertex 2).
   
   Compute \( \lim_{n \to \infty} 7^{-n} p_n \).

3. A permutation is an involution if every cycle has length 1 or length 2. (In other words, composing the permutation with itself gives the identity permutation.) Let \( a_n \) denote the number of permutations of \( \{1, 2, \ldots, n\} \) which are involutions.
   
   (a) Find a recurrence relation for the sequence \( (a_n) \).
   (b) What is the probability that an involution of \( \{1, 2, 3, 4, 5, 6\} \) fixes 1?
      
      (By definition, this is \( b_6/a_6 \), where \( b_6 \) is the number of involutions \( \pi \) of \( \{1, 2, 3, 4, 5, 6\} \) such that \( \pi(1) = 1 \).)

4. Four married couples have dinner around a circular table. How many ways can they be seated such that spouses are never adjacent?

5. How many ways are there to divide 6 baseballs, 6 footballs, 6 volleyballs, 6 tennis balls and 6 beach balls between two brothers, so that each gets 15 balls?

6. Let \( G \) be a graph with \( n \) vertices and more than \( (n-1)(n-2)/2 \) edges. Prove that \( G \) is connected. (Note, as \( G \) is a graph, for all vertices \( i \) and \( j \), there is no edge from \( i \) to \( i \), and there is at most one edge from \( i \) to \( j \).)

7. Suppose a sphere is tiled by pentagons and hexagons subject to the following conditions:
   where two of the shapes meet, their intersection is a common vertex or a common edge; and each vertex lies on exactly three edges.
   
   Prove that there is only one possibility for the number of pentagons used in such a tiling, and compute that number. (Hint: to check, look at a soccer ball.)