Math 475: Stirling Numbers

April 17, 2012

1 Stirling Numbers

For n a positive integer let

$$x^{\underline{n}} = x(x-1)\dots(x-n+1)$$

Also set $x^{\underline{0}} = 1$, so $x^{\underline{n}}$ is a polynomial of degree n in the indeterminate x, with top order term x^n. For example $x^{\underline{2}} = x(x-1) = x^2 - x$, $x^{\underline{3}} = x^3 - 3x^2 + 2x$.

Lemma 1.1 *Suppose $f(x) = a_0 + a_1 x + \dots a_n x^n = \sum_{i=0}^{n} a_i x^i$ is a polynomial of degree n with $a_i \in \mathbb{Z}$. Then*

$$f(x) = \sum_{i=0}^{n} b_i x^{\underline{i}}$$

for some unique integers b_j.

Let V the real vector space of polynomials of degree less than or equal to n. This is just a formal way of saying

$$V = \{a_0 + a_1 x a_2 x^2 + \dots a_n x^n\} \quad (a_i \in \mathbb{R})$$

with the usual addition of polynomials, multiplication of polynomials by a scalar. This vector space is obviously n-dimensional: it has a basis $\{1, x, x^2, \dots, x^n\}$. This means every polynomial is uniquely a sum of these monomials (which is obvious).

Lemma 1.2 *The polynomials $\{x^{\underline{0}}, x^{\underline{1}}, x^{\underline{2}}, \dots, x^{\underline{n}}\}$ are also a basis of V.*

Proof. Every $x^{\underline{k}}$ is a sum of the x^ℓ. We have to show the converse: you can write x^k as a sum of terms $x^{\underline{\ell}}$. (This shows the $x^{\underline{k}}$ span, and there are exactly $n+1$ of them, so they are a basis.)

Consider the matrix of $\left\{ {n \atop k} \right\}$:

$$\begin{pmatrix} \left\{ {0 \atop 0} \right\} & \left\{ {0 \atop 1} \right\} & \left\{ {0 \atop 2} \right\} & \left\{ {0 \atop 3} \right\} & \left\{ {0 \atop 4} \right\} \dots \\ \left\{ {1 \atop 0} \right\} & \left\{ {1 \atop 1} \right\} & \left\{ {1 \atop 2} \right\} & \left\{ {1 \atop 3} \right\} & \left\{ {1 \atop 4} \right\} \dots \\ \left\{ {2 \atop 0} \right\} & \left\{ {2 \atop 1} \right\} & \left\{ {2 \atop 2} \right\} & \left\{ {2 \atop 3} \right\} & \left\{ {2 \atop 4} \right\} \dots \\ \left\{ {3 \atop 0} \right\} & \left\{ {3 \atop 1} \right\} & \left\{ {3 \atop 2} \right\} & \left\{ {3 \atop 3} \right\} & \left\{ {3 \atop 4} \right\} \dots \\ \left\{ {4 \atop 0} \right\} & \left\{ {4 \atop 1} \right\} & \left\{ {4 \atop 2} \right\} & \left\{ {4 \atop 3} \right\} & \left\{ {4 \atop 4} \right\} \dots \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 3 & 1 & 0 \\ 0 & 1 & 7 & 6 & 1 \end{pmatrix}$$

The fact that $\left\{ {n \atop k} \right\} = 0$ for $k > n$ says this is lower triangular, an d $\left\{ {n \atop n} \right\} = 1$ says it has ones on the diagonal. The determinant of this matrix (of whatever size n) is 1. Therefore the matrix is invertible. This is precisely what is necessary. ☐

In fact we've shown more:

Lemma 1.3 *For each n there is a formula*

$$x^n = b_0 x^{\underline{0}} + b_1 x^{\underline{1}} + \dots b_n x^{\underline{n}}$$

where the b_i are **integers**.

This is because the formula for the (i, j) entry of the inverse of a matrix A is $(-1)^{i-j} A_{j,i} \det(A)^{-1}$ where $A_{j,i}$ is a determinant of a sub-matrix of A. If all entries of A are integers, and the determinant is one, these are integers. Furthermore the inverse is lower diagonal.

Definition 1.4 *The Stirling numbers of the first and second kind are defined* *as follows. For $n \geq 1, k \geq 0$:*

$$x^n = \sum_{i=0}^{n} \left\{ {n \atop i} \right\} x^{\underline{i}}$$
$$x^{\underline{n}} = \sum_{i=0}^{n} (-1)^{n-i} \left[{n \atop i} \right] x^i$$

In particular $\left\{ {n \atop k} \right\} = \left[{n \atop k} \right] = 0$ *for* $k > n$. *Also*

$$\left\{ {0 \atop 0} \right\} = \left[{0 \atop 0} \right] = 1$$
$$\left\{ {0 \atop k} \right\} = \left[{0 \atop k} \right] = 0 \quad (k > 0)$$

From the Definition it is immediate that:

$$\sum_{k=0}^{n} \left\{ {n \atop k} \right\} \left[{n \atop k} \right] (-1)^{n-k} = \delta_{m,n}$$
$$\sum_{k=0}^{n} \left[{n \atop k} \right] \left\{ {n \atop k} \right\} (-1)^{n-k} = \delta_{m,n}$$

where $\delta_{m,n} = 1$ if $m = n$, and 0 otherwise.

There are Pascal-style recurrence relations for these.

Lemma 1.5

$$\left[{n+1 \atop m} \right] = n \left[{n \atop m} \right] + \left[{n \atop m-1} \right]$$
$$\left\{ {n+1 \atop m} \right\} = m \left\{ {n \atop m} \right\} + \left\{ {n \atop m-1} \right\}$$

Proof. For the first one, note that

$$(x-n)x^{\underline{n}} = x(x-1)\ldots(x-n+1)(x-n) = x^{\underline{n+1}}$$

So

$$(x-n)\sum \left[{n \atop k} \right] (-1)^{n-k} x^k = \sum \left[{n+1 \atop k} \right] (-1)^{n+1-k} x^k$$

and multiplying in the first term gives

$$\sum {n \brack k} (-1)^{n-k} x^{k+1} - \sum n {n \brack k} (-1)^{n-k} x^k = \sum {n+1 \brack k} (-1)^{n+1-k} x^k$$

Look at the coefficient of x^m on both sides; in the first term take $k+1=m$, and take $k=m$ in the others.

$${n \brack m-1} (-1)^{n-(m-1)} - {n \brack m} (-1)^{(n-m)} = {n+1 \brack m} (-1)^{n+1-m)}$$

and cancelling signs gives

$${n \brack m-1} + {n \brack m} = {n+1 \brack m}$$

For the second identity, use $x * x^n = x^{n+1}$, so

$$x \sum {n \brace k} x^{\underline{k}} = \sum {n+1 \brace k} x^{\underline{k}}$$

Now write $x x^{\underline{k}} = (x-k+k) x^{\underline{k}} = x^{\underline{k+1}} + k x^{\underline{k}}$, so

$$\sum {n \brace k} x^{\underline{k+1}} + \sum {n \brace k} k x^{\underline{k}} = \sum {n+1 \brace k} x^{\underline{k}}$$

and equating the coefficient of $x^{\underline{m}}$ gives

$${n \brace m-1} + m {n \brace m} = {n+1 \brace m}$$

$\square$

Here is a combinatorial interpretation of the Stirling numbers of the second kind.

Consider the ways of distributing n distinct balls into k *identical* boxes, with at least one ball in each box. We haven't considered identical boxes before; this means you can permute the boxes at will. See the 12-fold way notes on the class web site.

Example 1: For 4 balls a, b, c, d into 4 boxes: only 1, you put one ball in each box, and the order of the boxes doesn't matter.

Example 2: 4 balls into 3 boxes: $([a],[b],[c,d])$ or $([a],[c],[b,d])$ or $([b],[c],[a,d])$, a total of 3.

Example 3: 4 balls into 2 boxes: $([a],[b,c,d])$,$([b],[a,c,d])$, $([c],[a,b,d])$, $([d],[a,b,c])$. Also $([a,b],[c,d])$, $([a,c],[b,d])$, or $([a,d],[b,c])$. The total is 7.

Let $B(n,k)$ be the number of ways of doing this.

Obvioiusly $B(n,0)=0$, $B(n,1)=1$, and $B(n,n)=1$. Also $B(n,k)=0$ if $k>n$.

I claim these numbers satisfy the same recurrence relation as the $\left\{ {n \atop k} \right\}$:

$$B(n+1,m) = mB(n,m) + B(n,m-1)$$

Why? How many ways are there of putting $n+1$ balls into m boxes? Well, you could put ball 1 in a box by itself. There are $B(n,m-1)$ ways of doing the rest. On the other hand, suppose ball 1 is not in a box by itself. Then, you can remove ball 1 from its box, and still have at least one ball in each box. There are $B(n,m)$ ways of putting the n balls in the m boxes. But you also have to choose which box ball 1 came out of. This gives $mB(n,m)$. Voila!

Since these numbers satisfy the same recurrence relation as the $\left\{ {n \atop k} \right\}$, and agree for $k=0,1$, we conclude:

Lemma 1.6 $\left\{ {n \atop k} \right\}$ *is the number of ways of distributing* n *distinct balls into* k *identical boxes, with at least* 1 *ball in each box.*

Equivalently: $\left\{ {n \atop k} \right\}$ is the number of ways of partitioning an n-set into k disjoint (non-empty) subsets.

Here is a combinatorial interpretation of the Stirling numbers $\left[{n \atop k} \right]$ of the first kind.

First, here is a standard way to write a permutation of n in terms of *cycles.* For example, $(1,2,3)$, in cycle notation, denotes the permutation

$1 \to 2 \to 3 \to 1$. More examples:

$$\begin{aligned}(1,2,3)(4) &: 1 \to 2 \to 3 \to 4; 4 \text{ goes to itself}\\ (1,2)(3,4) &: 1 \leftrightarrow 2; 3 \leftrightarrow 4\\ (1)(2)...(n) &: \text{the identity (trivial) permutation}\\ (1,2,...,n) &: \text{ the } n\text{-cycle } 1 \to 2 \to \cdots \to n \to 1\end{aligned}$$

Note that there are $(n-1)!$ n-cycles: $(1,2,\ldots,n) = (2,3,\ldots n,1) = (3,4,\ldots,n,1,2)$, so assume 1 is first, and there are $(n-1)!$ distinct other ones.

For example $\begin{bmatrix}3\\2\end{bmatrix} = 3$: $(1,2)(3); (1,3)(2)$ or $(2,3)(1)$

$\begin{bmatrix}4\\2\end{bmatrix} = 11$: $(1,2)(3,4); (1,3)(2,4); (1,4)(2,3)$; also $(1)(2,3,4); (1)(2,4,3)$, $\ldots$ (8 of these), for a total of 11.

Lemma 1.7 $\begin{bmatrix}n\\k\end{bmatrix}$ *is the number of permutations of n with k cycles.*

Proof. let $B(n,k)$ be the number of permutations of an n-set with k cycles. Then $B(n,0) = 0$ and $B(n,1) = (n-1)!$.

By induction. To compute $B(n,k)$ consider the position of 1. If it is in a cycle by itself, (1), this leaves $B(n-1,k-1)$ others. Otherwise, there are $B(n-1,k)$ permutations of $n-1$ with k cycles. Now you can put in the 1 to the left of any j: there are $n-1$ ways to do this. So

$$B(n,k) = B(n-1,k-1) + (n-1)B(n-1,k)$$

This is the same as the recurrence in Lemma 1.5. $\square$

2 Generalized Stirling Numbers

Recall the Stirling numbers of the second kind $\left\{\begin{matrix}n\\k\end{matrix}\right\}$ satisfy

(2.1)(a) $$\left\{\begin{matrix}n+1\\k\end{matrix}\right\} = k\left\{\begin{matrix}n\\k\end{matrix}\right\} + \left\{\begin{matrix}n\\k-1\end{matrix}\right\}$$

provided $n,k \geq 0$. If we assume

(2.1)(b) $$\left\{\begin{matrix}n\\0\end{matrix}\right\} = \delta_{n,0}, \left\{\begin{matrix}0\\k\end{matrix}\right\} = \delta_{0,k}$$

Then (a) and (b) together determine $\left\{ {n \atop k} \right\}$ for all $k, n \geq 0$.

Now: use (a) and (b) to define $\left\{ {n \atop k} \right\}$ for *all* integers n and k. It is easy to see they uniquely determine $\left\{ {n \atop k} \right\}$.

Do the same thing to define $\left[{n \atop k} \right]$ for all n, k.

Then something remarkable happens. If you fill out the upper left hand corner of the triangle of the $\left\{ {n \atop k} \right\}$, for $n, k < 0$, you see you get the $\left[{m \atop \ell} \right]$ for $m, \ell > 0$.

Lemma 2.2 *For all integers n, k:*

$$\left\{ {n \atop k} \right\} = \left[{-k \atop -n} \right]$$

Proof. It is enough to show that the numbers $\left[{-k \atop -n} \right]$ satisfy the correct recurrence relation. That is, $\left\{ {n \atop k} \right\}$ are determined by

$$\left\{ {n+1 \atop k} \right\} = k\left\{ {n \atop k} \right\} + \left\{ {n \atop k-1} \right\} \quad (n, k \geq 0) \tag{2.4}$$

Replace each $\left\{ {a \atop b} \right\}$ with $\left[{-b \atop -a} \right]$:

$$\left[{-k \atop -n-1} \right] - k\left[{-k \atop -n} \right] + \left[{-k+1 \atop -n} \right] \tag{2.5}$$

We don't know this is true, but if we can prove it, then $\left[{n \atop k} \right]$ satisfy the same recurrence relation as $\left\{ {-k \atop -n} \right\}$, proving the result. Set $\ell = -k, m = -n-1$, so (b) gives

$$\left[{\ell \atop m} \right] = -\ell\left[{\ell \atop m+1} \right] + \left[{\ell+1 \atop m+1} \right] \tag{2.6}$$

or

$$\begin{bmatrix} \ell \\ m \end{bmatrix} + \ell \begin{bmatrix} \ell \\ m+1 \end{bmatrix} = \begin{bmatrix} \ell+1 \\ m+1 \end{bmatrix} \tag{2.7}$$

which is (1.5). □

There is also a connection with polynomials.

Consider the recurrence relation

$$(x-n)x^{\underline{n}} = x^{\underline{n+1}} \quad (n \geq 0). \tag{2.8}$$

Write this in the form

$$x^{\underline{n}} = \frac{x^{\underline{n+1}}}{x-n} \quad (n \geq 0) \tag{2.9}$$

Even though the right hand side has $x - n$ in the denominator, this is an equality of polynomials (of degree n) in x.

Use (2.9), inductively, to *define* $x^{\underline{n}}$ for $n \leq -1$. That is

$$\begin{aligned} x^{\underline{-1}} &= \frac{x^{\underline{0}}}{x-(-1)} = \frac{1}{x+1} \\ x^{\underline{-2}} &= \frac{x^{\underline{-1}}}{x-(-2)} = \frac{1}{(x+1)(x+2)} \dots \\ x^{\underline{-n}} &= \frac{1}{(x+1)(x+2)\dots(x+n)} \quad (n \geq 1). \end{aligned}$$

Alternatively,

$$x^{\underline{n}} = \frac{1}{(x-n)(x-n-1)\dots(x+1)} \quad (n \leq -1)$$

Note that $x^{\underline{n}}$ is a polynomial of degree n if $n \geq 0$, or 1 over a polynomial of degree $-n$ if $n \leq -1$.

Then the *exact same recurrence relation* (2.8) holds for all n.

Furthermore, the defining relations now hold for all n, k, as *formal power series*:

$$\begin{aligned} x^n &= \sum_{i=0}^{n} \left\{ n \atop i \right\} x^{\underline{i}} \\ x^{\underline{n}} &= \sum_{i=0}^{n} (-1)^{n-i} \begin{bmatrix} n \\ i \end{bmatrix} x^i \end{aligned}$$

Stirling numbers $S(n,k) = \left\{ {n \atop k} \right\}$ of the second kind

n/k	-5	-4	-3	-2	-1	0	1	2	3	4	5	
-5	1											
-4	10	1										
-3	35	6	1									
-2	50	11	3	1								
-1	24	6	2	1	1							
0						1						
1							1					
2							1	1				
3							1	3	1			
4							1	7	6	1		
5							1	15	25	10	1	
6							1	31	90	65	15	1

Stirling numbers $s(n,k) = \left[{n \atop k} \right]$ of the second kind

n/k	-5	-4	-3	-2	-1	0	1	2	3	4	5	
-5	1											
-4	10	1										
-3	25	6	1									
-2	15	7	3	1								
-1	1	1	1	1	1							
0						1						
1							1					
2							1	1				
3							2	3	1			
4							6	11	6	1		
5							24	50	35	10	1	
6							120	274	225	85	15	1