1 Stirling Numbers

For $n$ a positive integer let

$$x^n = x(x - 1) \ldots (x - n + 1)$$

Also set $x^0 = 1$, so $x^n$ is a polynomial of degree $n$ in the indeterminate $x$, with top order term $x^n$. For example $x^2 = x(x - 1) = x^2 - x$, $x^3 = x^3 - 3x^2 + 2x$.

**Lemma 1.1** Suppose $f(x) = a_0 + a_1 x + \ldots a_n x^n = \sum_{i=0}^{n} a_i x^i$ is a polynomial of degree $n$ with $a_i \in \mathbb{Z}$. Then

$$f(x) = \sum_{i=0}^{n} b_i x^i$$

for some unique integers $b_j$.

Let $V$ the real vector space of polynomials of degree less than or equal to $n$. This is just a formal way of saying

$$V = \{a_0 + a_1 x a_2 x^2 + \ldots a_n x^n\} \quad (a_i \in \mathbb{R})$$

with the usual addition of polynomials, multiplication of polynomials by a scalar. This vector space is obviously $n$-dimensional: it has a basis $\{1, x, x^2, \ldots, x^n\}$. This means every polynomial is uniquely a sum of these monomials (which is obvious).

**Lemma 1.2** The polynomials $\{x^0, x^1, x^2, \ldots, x^n\}$ are also a basis of $V$. 
Proof. Every $x^k$ is a sum of the $x^\ell$. We have to show the converse: you can write $x^k$ as a sum of terms $x^\ell$. (This shows the $x^k$ span, and there are exactly $n + 1$ of them, so they are a basis.)

Consider the matrix of $\{^nk\}$:

$$
\begin{pmatrix}
\binom{0}{k} & \binom{0}{k-1} & \binom{0}{k-2} & \cdots \\
\binom{1}{k} & \binom{1}{k-1} & \binom{1}{k-2} & \cdots \\
\binom{2}{k} & \binom{2}{k-1} & \binom{2}{k-2} & \cdots \\
\binom{3}{k} & \binom{3}{k-1} & \binom{3}{k-2} & \cdots \\
\binom{4}{k} & \binom{4}{k-1} & \binom{4}{k-2} & \cdots \\
\end{pmatrix}
= \begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 \\
0 & 1 & 3 & 1 & 0 \\
0 & 1 & 7 & 6 & 1 \\
\end{pmatrix}
$$

The fact that $\{^nk\} = 0$ for $k > n$ says this is lower triangular, and $\{^nk\} = 1$ says it has ones on the diagonal. The determinant of this matrix (of whatever size $n$) is 1. Therefore the matrix is invertible. This is precisely what is necessary. \qed

In fact we’ve shown more:

Lemma 1.3 For each $n$ there is a formula

$$x^n = b_0x^0 + b_1x^1 + \ldots + b_nx^n$$

where the $b_i$ are integers.

This is because the formula for the $(i, j)$ entry of the inverse of a matrix $A$ is $(-1)^{i-j}A_{ji}\det(A)^{-1}$ where $A_{ji}$ is a determinant of a sub-matrix of $A$. If all entries of $A$ are integers, and the determinant is one, these are integers. Furthermore the inverse is lower diagonal.

Definition 1.4 The Stirling numbers of the first and second kind are defined
as follows. For \( n \geq 1, k \geq 0 \):

\[
x^n = \sum_{i=0}^{n} \binom{n}{i} x^i
\]

\[
x^m = \sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} x^i
\]

In particular \( \binom{n}{k} = \binom{n}{k} = 0 \) for \( k > n \). Also

\[
\binom{0}{0} = \binom{0}{0} = 1
\]

\[
\binom{0}{1} = \binom{0}{1} = 0 \quad (k > 0)
\]

From the Definition it is immediate that:

\[
\sum_{k=0}^{n} \binom{n}{k} \binom{n}{k} (-1)^{n-k} = \delta_{m,n}
\]

\[
\sum_{k=0}^{n} \binom{n}{k} \binom{n}{k} (-1)^{n-k} = \delta_{m,n}
\]

where \( \delta_{m,n} = 1 \) if \( m = n \), and 0 otherwise.

There are Pascal-style recurrence relations for these.

**Lemma 1.5**

\[
\binom{n+1}{m} = n \binom{n}{m} + \binom{n}{m-1}
\]

\[
\binom{n+1}{m} = m \binom{n}{m} + \binom{n}{m-1}
\]

**Proof.** For the first one, note that

\[
(x - n)x^m = x(x - 1) \ldots (x - n + 1)(x - n) = x^{n+1}
\]

So

\[
(x - n) \sum \binom{n}{k} (-1)^{n-k} x^k = \sum \binom{n+1}{k} (-1)^{n+1-k} x^k
\]
and multiplying in the first term gives

\[ \sum \binom{n}{k}(-1)^{n-k}x^{k+1} - \sum \binom{n}{k}(-1)^{n-k}x^{k} = \sum \binom{n+1}{k}(-1)^{n+1-k}x^{k} \]

Look at the coefficient of \(x^{m}\) on both sides; in the first term take \(k+1 = m\), and take \(k = m\) in the others.

\[ \binom{n}{m-1}(-1)^{n-(m-1)} - \binom{n}{m}(-1)^{(n-m)} = \binom{n+1}{m}(-1)^{n+1-m} \]

and cancelling signs gives

\[ \binom{n}{m-1} + \binom{n}{m} = \binom{n+1}{m} \]

For the second identity, use \(x \ast x^n = x^{n+1}\), so

\[ x \sum \binom{n}{k}x^{k} = \sum \binom{n+1}{k}x^{k} \]

Now write \(xx^{k} = (x - k + k)x^{k} = x^{k+1} + kx^{k}\), so

\[ \sum \binom{n}{k}x^{k+1} + \sum \binom{n}{k}kx^{k} = \sum \binom{n+1}{k}x^{k} \]

and equating the coefficient of \(x^{m}\) gives

\[ \binom{n}{m-1} + m \binom{n}{m} = \binom{n+1}{m} \]

\[ \Box \]

Here is a combinatorial interpretation of the Stirling numbers of the second kind.

Consider the ways of distributing \(n\) distinct balls into \(k\) identical boxes, with at least one ball in each box. We haven’t considered identical boxes before; this means you can permute the boxes at will. See the 12-fold way notes on the class web site.

Example 1: For 4 balls \(a, b, c, d\) into 4 boxes: only 1, you put one ball in each box, and the order of the boxes doesn’t matter.
Example 2: 4 balls into 3 boxes: ([a], [b], [c, d]) or ([a], [c], [b, d]) or ([b], [c], [a, d]), a total of 3.

Example 3: 4 balls into 2 boxes: ([a], [b, c, d]), ([b], [a, c, d]), ([c], [a, b, d]), ([d], [a, b, c]). Also ([a], [b, c, d]), ([a], [b, d]), or ([a], [b, c]). The total is 7.

Let $B(n, k)$ be the number of ways of doing this. Obviously $B(n, 0) = 0$, $B(n, 1) = 1$, and $B(n, n) = 1$. Also $B(n, k) = 0$ if $k > n$.

I claim these numbers satisfy the same recurrence relation as the $\{n\choose k\}$:

$$B(n + 1, m) = mB(n, m) + B(n, m - 1)$$

Why? How many ways are there of putting $n + 1$ balls into $m$ boxes? Well, you could put ball 1 in a box by itself. There are $B(n, m - 1)$ ways of doing the rest. On the other hand, suppose ball 1 is not in a box by itself. Then, you can remove ball 1 from its box, and still have at least one ball in each box. There are $B(n, m)$ ways of putting the $n$ balls in the $m$ boxes. But you also have to choose which box ball 1 came out of. This gives $mB(n, m)$. Voila!

Since these numbers satisfy the same recurrence relation as the $\{n\choose k\}$, and agree for $k = 0, 1$, we conclude:

**Lemma 1.6** $\{n\choose k\}$ is the number of ways of distributing $n$ distinct balls into $k$ identical boxes, with at least 1 ball in each box.

Equivalently: $\{n\choose k\}$ is the number of ways of partitioning an $n$-set into $k$ disjoint (non-empty) subsets.

Here is a combinatorial interpretation of the Stirling numbers $\left[n\atop k\right]$ of the first kind.

First, here is a standard way to write a permutation of $n$ in terms of cycles. For example, $(1, 2, 3)$, in cycle notation, denotes the permutation
1 → 2 → 3 → 1. More examples:

(1, 2, 3)(4) : 1 → 2 → 3 → 4; 4 goes to itself
(1, 2)(3, 4) : 1 ↔ 2; 3 ↔ 4
(1)(2)...(n) : the identity (trivial) permutation

Note that there are \((n−1)\) \(n\)-cycles: \((1, 2, \ldots, n) = (2, 3, \ldots, n, 1) = (3, 4, \ldots, n, 1, 2)\), so assume 1 is first, and there are \((n−1)\) distinct other ones.

For example \(\left[3\atop 2\right] = 3\): (1, 2)(3); (1, 3)(2) or (2, 3)(1)
\(\left[4\atop 2\right] = 11\): (1, 2)(3, 4); (1, 3)(2, 4); (1, 4)(2, 3); also (1)(2, 3, 4); (1)(2, 4, 3),
\ldots (8 of these), for a total of 11.

**Lemma 1.7** \(\left[\begin{array}{c} n \\ k \end{array}\right]\) is the number of permutations of \(n\) with \(k\) cycles.

**Proof.** Let \(B(n, k)\) be the number of permutations of an \(n\)-set with \(k\) cycles. Then \(B(n, 0) = 0\) and \(B(n, 1) = (n−1)!!\).

By induction. To compute \(B(n, k)\) consider the position of 1. If it is in a cycle by itself, (1), this leaves \(B(n−1, k−1)\) others. Otherwise, there are \(B(n−1, k)\) permutations of \(n−1\) with \(k\) cycles. Now you can put in the 1 to the left of any \(j\): there are \(n−1\) ways to do this. So
\[
B(n, k) = B(n−1, k−1) + (n−1)B(n−1, k)
\]
This is the same as the recurrence in Lemma 1.5. \(\square\)

## 2 Generalized Stirling Numbers

Recall the Stirling numbers of the second kind \(\{\begin{array}{c} n \\ k \end{array}\}\) satisfy

\[
\{n + 1\} = k\{\begin{array}{c} n \\ k \end{array}\} + \{\begin{array}{c} n \\ k−1 \end{array}\}
\]
provided \(n, k \geq 0\). If we assume

\[
\{\begin{array}{c} n \\ 0 \end{array}\} = 0, \{\begin{array}{c} 0 \\ k \end{array}\} = \delta_{0,k}
\]
Then (a) and (b) together determine \( \{\frac{n}{k}\} \) for all \( k, n \geq 0 \).

Now: use (a) and (b) to define \( \{\frac{n}{k}\} \) for all integers \( n \) and \( k \). It is easy to see they uniquely determine \( \{\frac{n}{k}\} \).

Do the same thing to define \( [\frac{n}{k}] \) for all \( n, k \).

Then something remarkable happens. If you fill out the upper left hand corner of the triangle of the \( \{\frac{n}{k}\} \), for \( n, k < 0 \), you see you get the \( [\frac{m}{\ell}] \) for \( m, \ell > 0 \).

**Lemma 2.2** For all integers \( n, k \):

\[
\{\frac{n}{k}\} = [\frac{-k}{-n}]
\]

**Proof.** It is enough to show that the numbers \( [\frac{-k}{-n}] \) satisfy the correct recurrence relation. That is, \( \{\frac{n}{k}\} \) are determined by

\[
\{\frac{n+1}{k}\} = k\{\frac{n}{k}\} + \{\frac{n}{k-1}\} \quad (n, k \geq 0)
\]

Replace each \( \{\frac{a}{b}\} \) with \( [\frac{-b}{-a}] \):

\[
[\frac{-k}{-n-1}] - k[\frac{-k}{-n}] + [\frac{-k+1}{-n}]
\]

We don’t know this is true, but if we can prove it, then \( [\frac{n}{k}] \) satisfy the same recurrence relation as \( [\frac{-k}{-n}] \), proving the result. Set \( \ell = -k, m = -n - 1 \), so (b) gives

\[
[\frac{\ell}{m}] = -\ell[\frac{\ell}{m+1}] + [\frac{\ell+1}{m+1}]
\]
or
\[(2.7) \quad \ell \binom{\ell}{m} + \ell \binom{\ell}{m+1} = \ell \binom{\ell+1}{m+1}\]
which is (1.5).

There is also a connection with polynomials. Consider the recurrence relation

\[(2.8) \quad (x - n)x^n = x^{n+1} \quad (n \geq 0).\]

Write this in the form
\[(2.9) \quad x^n = \frac{x^{n+1}}{x - n} \quad (n \geq 0)\]

Even though the right hand side has \(x - n\) in the denominator, this is an equality of polynomials (of degree \(n\)) in \(x\).

Use (2.9), inductively, to define \(x^n\) for \(n \leq -1\). That is
\[
x^{-1} = \frac{x^0}{x - (-1)} = \frac{1}{x + 1}
\]
\[
x^{-2} = \frac{x^{-1}}{x - (-2)} = \frac{1}{(x + 1)(x + 2)} \ldots
\]
\[
x^{-n} = \frac{1}{(x + 1)(x + 2) \ldots (x + n)} \quad (n \geq 1).
\]

Alternatively,
\[
x^n = \frac{1}{(x - n)(x - n - 1) \ldots (x + 1)} \quad (n \leq -1)
\]

Note that \(x^n\) is a polynomial of degree \(n\) if \(n \geq 0\), or 1 over a polynomial of degree \(-n\) if \(n \leq -1\).

Then the exact same recurrence relation (2.8) holds for all \(n\).

Furthermore, the defining relations now hold for all \(n, k\), as formal power series:
\[
x^n = \sum_{i=0}^{n} \binom{n}{i} x^i
\]
\[
x^n = \sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} x^i
\]
Stirling numbers $S(n, k) = \binom{n}{k}$ of the second kind

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Stirling numbers $s(n, k) = \left[\frac{n}{k}\right]$ of the second kind

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