Math 744, Fall 2014  
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Homework I

(1) Consider the action of $SO(n + 1)$ acting on $S^n \subset \mathbb{R}^{n+1}$.
   (a) Show this action is transitive.
   (b) Compute $\text{Stab}_G(v)$ where $v = (1, 0, \ldots, 0)$.
   (c) Show there is an isomorphism $SO(n + 1)/SO(n) \simeq S^n$ (it is enough to give the bijection).

(2)
   (a) Show that $\{(z, w) \in \mathbb{C}^2 \mid z^2 + w^2 = 1\} \simeq \mathbb{C}^*$
   (b) Show that $SO(2, \mathbb{C}) \simeq \mathbb{C}^*$
   (c) Show that $SO(2, \mathbb{R}) \simeq S^1$
   (d) Show that $SO(1, 1) \simeq \mathbb{R}^*$. Recall $SO(1, 1)$ is the group preserving a symmetric bilinear form on $\mathbb{R}^2$ of signature $(1, 1)$.
   (e) Show that $O(2)$ contains $SO(2)$ as a subgroup of index 2, that $O(2)$ is no abelian, and the elements of $O(2) - SO(2)$ constitute a single conjugacy class.

(3) Show that the proper algebraic subsets of the one dimensional vector space $\mathbb{C}$ are the finite sets.

(4) Show that the Euclidean topology on $\mathbb{C}^n$ is finer than the Zariski topology.

(5) Show that $\text{Hom}_{\text{alg}}(G_m, G_m) \simeq \mathbb{Z}$; the left hand side is the set of morphisms from $G_m$ to $G_m$ (as algebraic varieties) which are also group homomorphisms.

(6) Recall an action of an algebraic group $G$ on an algebraic variety $X$ is a morphism of varieties $G \times X \to X$, $(g, x) \to g \cdot x$, satisfying $g \cdot (h \cdot x) = (gh) \cdot x$, and $e \cdot x = x$.
   (a) Consider the action of $GL(n, K)$ on $K^n$ ($K$ is any field). Determine the orbits of $GL(n, K)$ and $SL(n, K)$ on $K^n$.
   (b) Show that $GL(2, K)$ acts transitively on $\mathbb{P}^1$, the set of lines through the origin in $K^2$. Compute the stabilizer of a point. Compute the orbits of $GL(2, K)$ on $\mathbb{P}^1 \times \mathbb{P}^1$. 