Math 744, Fall 2014
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Homework I SOLUTIONS

(1) Consider the action of $SO(n+1)$ acting on $S^n \subset \mathbb{R}^{n+1}$.
(a) Show this action is transitive.
(b) Compute $\mathrm{Stab}_G(v)$ where $v = (1, 0, \dots, 0)$.
(c) Show there is an isomorphism $SO(n+1)/SO(n) \simeq S^n$ (it is enough to give the bijection).

Solution

(a) We need to show that if $||\vec{v}|| = ||\vec{w}|| = 1$ then there is an element of $SO(n+1)$ taking $\vec{v}$ to $\vec{w}$. This is Witt's theorem.

It is enough to take $\vec{v} = (1, 0, \dots, 0)$. Suppose $\vec{w} = (a_1, \dots, a_n)$. Take the first column of g to be the vector w. Then take the remaining columns to be an orthonormal basis of $w^\perp$.

(b) It is easy to see $g\vec{v} = \vec{v}$ if and only if the first column of g is $(1, 0, \dots, 0)$. The condition for such a matrix to be orthgonal is that the first row is also $(1, 0, \dots, 0)$, and the matrix in the lower right hand corner is orthgonal. The determinant one condition gives $\mathrm{Stab}_{SO(n+1)}(\vec{v})$ is the matrices of the form $\mathrm{diag}(1, h)$ with $h \in SO(n)$. This is isomorphic to $SO(n)$.

(c) Take $H = SO(n)$ embedded as in (b). The map takes the coset gH to $g\vec{v}$. This is well defined (since H stabilizes $\vec{v}$), surjective (by (a)), and injective (since H is the full stabilizer of $\vec{v}$).

(2)
(a) Show that $\{(z, w) \in \mathbb{C}^2 \mid z^2 + w^2 = 1\} \simeq \mathbb{C}^*$
(b) Show that $SO(2, \mathbb{C}) \simeq \mathbb{C}^*$
(c) Show that $SO(2, \mathbb{R}) \simeq S^1$
(d) Show that $SO(1, 1) \simeq \mathbb{R}^*$. Recall $SO(1, 1)$ is the group preserving a symmetric bilinear form on $\mathbb{R}^2$ of signature $(1, 1)$.
(e) Show that $O(2)$ contains $SO(2)$ as a subgroup of index 2, that $O(2)$ is no abelian, and the elements of $O(2) - SO(2)$ constitute a single conjugacy class.

Solution

It is easiest to start with (b). It is easy to see that $SO(2, \mathbb{C})$ is the set of matrices

$$\begin{pmatrix} z & w \\ -w & z \end{pmatrix} \tag{1)(a}$$

such that $z^2 + w^2 = 1$. To see this, the condition is that the rows must be orthonormal. So the first row is (z, w) as indicated. The second is $(-w, z)$ or $(z, -w)$; the determinant one condition gives $(-w, z)$. This matrix is diagonalizable, it diagonalizes to

$$\begin{pmatrix} z + iw & 0 \\ 0 & z - iw \end{pmatrix} \tag{1)(b}$$

using

(1)(c) $$J = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}$$

The inverse map takes

(1)(d) $$\begin{pmatrix} a & 0 \\ 0 & \frac{1}{a} \end{pmatrix}$$

to

(1)(e) $$\begin{pmatrix} a + \frac{1}{a} & i(a - \frac{1}{a}) \\ -i(a - \frac{1}{a}) & a - \frac{1}{a} \end{pmatrix}$$

Then (a) follows from looking at the first row of (1a) and the first entry of (1b).

Also (c) follows by taking $z, w \in \mathbb{R}$.

(d) It is standard to take the form $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. But you can take any symmetrix matrix equivalent to this, and it is best to take $J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. This is the same up to change of basis, i.e. up to $J \to AJA^t$. It is easy to see that g satisfies $\{g \mid gJg^t = J\}$ if and only if $g = \text{diag}(a, \pm\frac{1}{a})$, so the determinant one condition gives $\text{diag}(a, 1/a)$, which is isomorphic to $\mathbb{R}^*$.

(e) Since $\det(g) = \pm 1$ for $g \in O(2)$ there is an exact sequence $1 \to SO(2) \to O(2) \to \mathbb{Z}/2\mathbb{Z} \to 1$. In fact $O(2) = \langle SO(2), \text{diag}(1, -1)\rangle$. The element $\epsilon = \text{diag}(1, -1)$ acts by inverse on S^1, i.e. $\epsilon g \epsilon = g^{-1}$. This gives $g\epsilon g = \epsilon$, or $g\epsilon g^{-1} = \epsilon g^2$. Since the square map is surjective for $SO(2)$ this proves the result.

You can think of $O(2)$ as the "infinite dihedral group".

(3) Show that the proper algebraic subsets of the one dimensional vector space $\mathbb{C}$ are the finite sets.

Solution

This amounts to the fact that a polynomial in one variable only has finitely many roots.

(4) Show that the Euclidean topology on $\mathbb{C}^n$ is finer than the Zariski topology.

Solution

It is enough to show Zariski-closed implies Euclidean closed. A Zariski closed set is the intersection of the zeros of a set of polynomials. Since polynomials are continuous in the Euclidean topology, their zeros are closed in the Euclidean topology.

(5) Show that $\text{Hom}_{\text{alg}}(G_m, G_m) \simeq \mathbb{Z}$; the left hand side is the set of morphisms from G_m to G_m (as algebraic varieties) which are also group homomorphisms.

Solution

The key point is to use $\mathrm{Hom}_{alg}(G_m, G_m) \simeq \mathrm{Hom}(k[x, x^{-1}], k[x, x^{-1}])$. A homomorphism is given by $f(x) = \sum_m^n a_k x^k$. To be an algebra homomorphism it must satisfy $f(xy) = f(x)f(y)$, or $\sum_{i,j} a_i a_j x^i y^j = \sum_k a_k (xy)^k$. On the right hand side, the only terms which appear are $x^k y^k$. It follows that there can only be one term on the left: $f(x) = ax^k$ for some k, and $a = 1$.

(6) Recall an action of an algebraic group G on an algebraic variety X is a morphism of varieties $G \times X \to X$, $(g, x) \to g \cdot x$, satisfying $g \cdot (h \cdot x) = (gh) \cdot x$, and $e \cdot x = x$.
(a) Consider the action of $GL(n, K)$ on K^n (K is any field). Determine the orbits of $GL(n, K)$ and $SL(n, K)$ on K^n.

Solution

$GL(n)$ acts transitively on $K^n - \{0\}$: $g * (1, 0, ..., 0) = (a_1, \ldots, a_n)$ if the first column of g is $(a_1, \ldots, a_n)$, and if this isn't the 0 vector this can be filled out to an element of $GL(n)$. There are two orbits.

If $n \geq 2$, the same holds for $SL(n)$. After permuting we may assume $a_1 \neq 0$; take

$$g = \begin{pmatrix} a_1 & 0 & 0 & \ldots & 0 \\ a_2 & 1 & 0 & \ldots & 0 \\ a_3 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ a_n & 0 & 0 & \ldots & a_1^{-1} \end{pmatrix}$$

If $n = 1$ of course $SL(1) = 1$ and the orbits are points.
(b) Show that $GL(2, K)$ acts transitively on P^1, the set of lines through the origin in K^2. Compute the stabilizer of a point. Compute the orbits of $GL(2, K)$ on $P^1 \times P^1$.

The transitivity is clear from (a). The stabilizer of the line through $(1, 0)$ is the Borel subgroup $\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$. All lines are related by the action of $GL(2)$; correspondingly all Borel subgroups are conjugate.

Compute the orbits of $GL(2)$ on $P^1 \times P^1$.

We know that $P^1 = G/B$, so we're computing $G\backslash G/B \times G/B$. There is a bijection:

$$G\backslash(G/B \times G/B) \longleftrightarrow B\backslash G/B$$

given by $G(xB, yB) \to Bx^{-1}yB$.

It is not hard to see that $B\backslash G/B = B \cup BwB$ where $w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. We have to show any element not in B is in BwB:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \frac{bc-ad}{c} & -a \\ 0 & -c \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & \frac{d}{c} \\ 0 & 1 \end{pmatrix}$$

if $c \neq 0$.

This is a general fact: $B\backslash G/B \simeq W$ (the Bruhat decomposition).