The Contragredient
Joint with D. Vogan

Spherical Unitary Dual for Complex Classical Groups
Joint with D. Barbasch
The Contragredient

Problem: Compute the involution $\phi \rightarrow \phi^*$ of the space of $L$-homomorphisms corresponding to $\pi \rightarrow \pi^*$ (the contragredient)

i.e. $\phi : W'_\mathbb{F} \rightarrow ^L G$, $\pi \in \Pi_\phi \Rightarrow \pi^* \in \Pi_{\phi^*}$ what $\phi^*$?

(Assume $\phi \rightarrow \Pi_\phi$ known...)

(Well defined, same for all $\pi \in \Pi_\phi$?)

Nowhere to be found (“much needed gap in the literature”), even for $\mathbb{F} = \mathbb{R}$
Character: $\theta_{\pi^*}(g) = \theta_{\pi}(g^{-1})$

**Lemma**: There is an automorphism $C^\vee$ of $L G$ satisfying: $C^\vee(g)$ is $G^\vee$-conjugate to $g^{-1}$ for $g$ semisimple

(The Chevalley automorphism, extended to $L G$)

**Lemma**: there is an automorphism $\tau$ of $W_\mathbb{R}$ satisfying: $\tau(g)$ is $W_\mathbb{R}$ conjugate to $g^{-1}$

$(\tau(z) = z^{-1}, \tau(j) = j)$
Theorem ($\mathbb{F} = \mathbb{R}$):

(1) $\phi^* = C^\vee \circ \phi$

(2) $\phi^* = \phi \circ \tau$

Proof: Not entirely elementary; characterize $\pi^*$ by

$$\theta_{\pi^*}(g) = \theta_{\pi}(g^{-1})$$

Need a formula relating $\phi$ and $\theta_{\pi}$...

True for tori...

Key lemma: action of $C$ on the normalizer of a torus
Conjecture/Desiderata: (1) is true for all other local fields
$(\phi^* = C^\vee \circ \phi)$

Buzzard: true for unramified principal series (??)

Note: Probably nothing like $\tau$ exists in general ??
Spherical Unitary Dual for Complex Classical Groups

Joint with D. Barbasch

Hat-tip: P. Trapa, M. McGovern, E. Sommers

Barbasch 1989 (full unitary dual)

Spherical unitary dual for split real and p-adic groups:
Barbasch \sim 2005

Nice picture in terms of nilpotent orbits for $G^\vee$

**Plan:** Revisit complex case from this point of view

**Application:** Compute behavior of unitarity under the dual pair correspondence

**Application:** Organize and understand the upcoming Atlas computation of the unitary dual
Problem: How to organize the answer?

$Sp(4, \mathbb{C})$
(1) complex groups are quasisplit but not split
(2) Orbits are induced in many ways; complementary series overlap in complicated ways
(3) Organize the answer via nilpotent orbits in $G$ and/or $G^\vee$?
(4) There is no nonlinear cover of $Sp(2n, \mathbb{C})$; the oscillator representation lives on the linear group
\( G = GL(n, \mathbb{C}), SO(n, \mathbb{C}), Sp(2n, \mathbb{C}) \) (some statements hold for exceptional groups)

\( \pi(\lambda) = \) irreducible, spherical representation with infinitesimal character \( \lambda \in \mathfrak{h}^* \)

real: \( \lambda \in X^*(H) \otimes \mathbb{R} \) (\( \lambda \in \mathbb{R}^n \) in the usual coordinates)

\( \hat{\Pi}_{sph} = \{ \text{irreducible, real, spherical representations} \} \)
Definition: \( \mathcal{O} = \) nilpotent adjoint \( G \)-orbit

\[
\mathcal{C}(\mathcal{O}) = \{ \text{real, irreducible, unitary, spherical } \pi \mid AV(\pi) = \overline{\mathcal{O}} \}
\]

\( AV(\pi) \) is the associated variety of \( \pi \)

\[
[AV(\pi) = WF(\pi) = AV(\text{Ann}(\pi))] \text{ via various identifications}
\]

\[
\widehat{\Pi}_{\text{sph}} = \bigcup_{\mathcal{O}} \mathcal{C}(\mathcal{O}) \quad \text{(disjoint union)}
\]
Nilpotent Orbits in Classical Groups

$GL(n)$: partitions of $n$

$Sp(2n)$: partitions of $n$, odd parts have even multiplicity

$O(n)$: partitions of $n$, even parts have even multiplicity
Induction of Orbits

\( M \subset G = \text{Levi factor} \)

\( \mathcal{O} = \text{Ind}_M^G(\mathcal{O}_M) \)

induction \( GL(n) \): combine orbits \( \text{Ind}_{GL(a) \times GL(b)}^{GL(n)}(p \otimes q) = p \oplus q \)

\((a_1, a_2, \ldots, ) \oplus (b_1, b_2, \ldots) = (a_1 + b_1, a_2 + b_2, \ldots)\)

Type \( \text{X} = \text{B,C,D} \): double the \( GL(m) \) partitions, combine, and \( X \)-collapse

Example:

\( \text{Ind}_{GL(1) \times SL(2)}^{Sp(4)}(\text{trivial}) = (2) \oplus (11) = (31) \rightarrow (22) \)

\( \text{Ind}_{GL(2)}^{Sp(4)}(\text{trivial}) := (22) \) (double \( (11) \) for \( GL(2) \) to get \( (22) \))
**Problem:** orbits can be induced in more than one way (leading to overlapping series of representations)

**Definition:** $\mathcal{O} = \text{Ind}_M^G(\mathcal{O}_M)$ is proper if no collapsing is required.

$\mathcal{O}$ is P-rigid if it is not properly induced.

**Lemma:** (type BCD) $\mathcal{O}$ is P-rigid if and only if all parts $1, 2, \ldots, k$ occur with nonzero multiplicity.

(Rigid is slightly stronger: certain rows of multiplicity 2 are not allowed)

**Lemma:** $\mathcal{O}$ is uniquely properly induced from a P-rigid orbit i.e. $(M, \mathcal{O}_M)$ unique up to $G$-conjugacy

**Conjecture:** Proper induction is equivalent to: the corresponding moment map is birational.
Program:

1. \( \mathcal{O} \) P-rigid \( \rightarrow \lambda = \lambda(\mathcal{O}) \rightarrow \pi(\lambda) \) unipotent

2. \( \mathcal{O} \) arbitrary \( \rightarrow (M, \mathcal{O}_M) \) \((\mathcal{O}_M\) rigid, \(\mathcal{O}\) properly induced from \(\mathcal{O}_M)\)

\[ M = \text{GL}(c_1) \times \cdots \times \text{GL}(c_k) \times M_0 \]

\( \mathcal{O}_M \) P-rigid \( \rightarrow \tau \) unipotent for \( M_0 \)

\[
\text{Ind}^G_{\text{GL}(c_1) \times \cdots \times \text{GL}(c_k) \times M_0} (1 \otimes \cdots \otimes 1 \otimes \tau)
\]

is irreducible and unitary
Program (continued):

Study:

\[
\text{Ind}_G^{GL(c_1) \times \cdots \times GL(c_k) \times M_0} (|det|^{x_1} \otimes \cdots \otimes |det|^{x_k} \otimes \tau)
\]

Since induction is irreducible when all \(x_i = 0\), some deformations are allowed...

Example: if all \(x_i\) are small, deform them to 0

Example: \(G = Sp(6)\), \(\lambda = (.4, .5, .8)\). Deform to \((.4, .4, .8)\), which is induced from the unitary representation:

\[
Stein(.4) \otimes \pi(.8) \quad \text{on} \quad GL(2) \times SL(2)
\]

Basic idea (Barbasch, 1989): these operations suffice to find all the irreducible unitary ones
Program (continued):

Punch line:
Recall $\mathcal{O} \to M, \tau,$

\[
(*) \quad \text{Ind}^G_{\prod GL(c_1) \times \cdots \times GL(c_k) \times M_0} (|det|^{x_1} \otimes \cdots \otimes |det|^{x_k} \otimes \tau)
\]

Main Theorem: (rough version):

(0) The 0-complementary series $\mathcal{C}(0_p)$ can be explicitly described ($M = GL(1)^n$)

(1) The representations $(*)$ which are irreducible, and can be irreducibly deformed to a unitarily induced representation can be described in terms of $\mathcal{C}(0_p)$ for a smaller group

(2) This gives all the irreducible unitary representations $(*)$

(3) The complementary series $\mathcal{C}(\mathcal{O})$ consists of precisely these representations.

Recall $\widehat{\Pi}_{sph} = \bigcup_{\mathcal{O}} \mathcal{C}(\mathcal{O})$
Data on the Dual Group:

From now on take $G = Sp(2n)$

$A(O) = \text{Cent}_G(X)/\text{Cent}_G(X)^0$

$\overline{A}(O) = \text{Lusztig’s quotient}$

**Lemma:** $O^\vee = \text{nilpotent orbit for } SO(2n + 1)$

$O^\vee = b_0, a_1, b_1, \ldots a_r, b_r \quad b_0 \leq a_1 \leq b_1 \leq \ldots$

$O^\vee = (b_0)(a_1, b_1) \ldots (a_r, b_r)$

$\overline{A}(O^\vee) = (\mathbb{Z}/2\mathbb{Z})^k$ where $k$ is the number of $a_i < b_i$ with $b_i$ odd
d: duality of nilpotent orbits:

\[ d : \mathcal{O} \rightarrow d(\mathcal{O}) = \text{special nilpotent } G^\vee \text{-orbit} \]

**Proposition** (Barbasch/Vogan, Sommers) If \( \mathcal{O}^\vee \) is even, there is a canonical bijection

\[ \overline{A}(\mathcal{O}^\vee) \leftrightarrow \{ \mathcal{O} \mid d(\mathcal{O}) = \mathcal{O}^\vee \} \]

\( (\mathcal{O}^\vee, s) \rightarrow \mathcal{O} \)
Lemma: If $\mathcal{O}$ is P-rigid then $d(\mathcal{O})$ is even

Definition
A P-rigid symbol for $G^\vee$ is:

$$\Sigma = (b_0)(a_1, b_1)_{\epsilon_1} \cdots (a_r, b_r)_{\epsilon_r}$$

with $a_i, b_i$ odd, $a_i < b_i$, $\epsilon_i = \pm 1$

Assume:

(1) $\epsilon_i = 1 \rightarrow b_i - a_i > 2$

(2) $\epsilon_i = \epsilon_{i+1} = -1 \Rightarrow b_i < a_{i+1}$

These are certain pairs $(\mathcal{O}^\vee, s)$

Lemma The P-rigid symbols parametrize P-rigid orbits
**Definition:** \( \Sigma = (b_0) (a_1, b_1) \epsilon_1 \cdots (a_r, b_r) \epsilon_r \)

\((a, b)_1 \rightarrow \frac{1}{2} (b - 1, b - 3, \ldots, -a + 1) \ (a + b)/2 \) terms

\((a, b)_{-1} \rightarrow \frac{1}{2} (b - 1, b - 3, \ldots, -a + 1) + \frac{1}{2} (1, \ldots, 1) \ (a + b)/2 \) terms

\((b_0) \rightarrow \frac{1}{2} (b - 1, b - 3, \ldots, 1) \ (b - 1)/2 \) terms

Do this for each \( i \), concatenate \( \rightarrow \lambda = \lambda(\Sigma) = \lambda(\mathcal{O}) \)

**Example:** \( \Sigma = (5)(5, 7)_-(7, 11)_+, \mathcal{O} = 5555433211,RRR \)

\[
\left( \begin{array}{c}
(5) \\
(2, 1), \frac{7}{2}, \frac{5}{2}, \frac{3}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{3}{2} \\
\end{array} \right) \left( \begin{array}{c}
(5, 7)_- \\
\frac{5}{2}, \frac{3}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{3}{2} \\
\end{array} \right) \left( \begin{array}{c}
(7, 11)_+ \\
5, 4, 3, 2, 1, 0, -1, -2, -3 \\
\end{array} \right).
\]
Conjecture: $\lambda(O) = \lambda_{BV}(O)$

(certainly true for rigid orbits)

Theorem (Barbasch 1989): $O$ P-rigid $\Rightarrow \lambda = \lambda(O) \rightarrow \pi(\lambda)$ is unitary

This completes the first part of the program
Symbol:

\[ \Sigma = \{c_1, c_1\} \ldots \{c_k, c_k\} : (b_0)(a_1, b_1)_{\epsilon_1} \ldots (a_r, b_r)_{\epsilon_r} \]

Assume:
(0) \((b_0)(a_1, b_1)_{\epsilon_1} \ldots\) is a P-rigid symbol;
(1) if \(a_i < c_j < b_i, \epsilon_i = 1\) then \(c_j\) is even;
(2) if \(a_i < c_j < b_i, \epsilon_i = -1\) then \(c_j\) is odd;
(3) if \(c_i < b_0\) then \(c_i\) is even

\[ M(\Sigma) = GL(c_1) \times \cdots \times GL(c_k) \times Sp(2m) \]

**Lemma**: Symbols parametrize pairs \((M, O_M)\) where \(O_M\) is P-rigid and \(\text{Ind}_M^G(O_M)\) is proper

(Conditions 1-3 give the proper induction)

In other words there are canonical bijections:
(1) Orbits \(O\)
(2) Pairs \((M, O_M)\) \((O_M\ P\text{-rigid, the induction is proper})\)
(3) P-rigid symbols \(\Sigma\)
| \(|\Sigma\) | \(|(L, \mathcal{O}_L)\) | \(|\mathcal{O}\) | \(|\lambda\) |
|---|---|---|---|
| \((11)\) | \((Sp(10), \text{triv})\) | \(^{10}\) | \((5, 4, 3, 2, 1)\) |
| \((1)(1, 9)_{+}\) | \((Sp(10), 2^2 1^6)\) | \(2^2 1^6\) | \((4, 3, 2, 1, 0)\) |
| \((1)(1, 9)_{-}\) | \((Sp(10), 21^8)\) | \(21^8\) | \((\frac{9}{2}, \frac{7}{2}, \frac{5}{2}, \frac{3}{2}, \frac{1}{2})\) |
| \((1)(3, 7)_{+}\) | \((Sp(10), 2^4 1^2)\) | \(2^4 1^2\) | \((3, 2, 1, 0, -1)\) |
| \((1)(3, 7)_{-}\) | \((Sp(10), 2^3 1^4)\) | \(2^3 1^4\) | \((\frac{9}{2}, \frac{7}{2}, \frac{5}{2}, \frac{3}{2}, \frac{1}{2})\) |
| \([2, 2]: (7)\) | \((GL(2) \times Sp(6), \text{triv})\) | \(3^2 1^4\) | \((3, 2, 1, \frac{1}{2}, -\frac{1}{2})\) |
| \([1, 1]: (1)(1, 7)_{+}\) | \((GL(1) \times Sp(8), \text{triv} \times 2^2 1^4)\) | \(421^4\) | \((3, 2, 1, 0, 0)\) |
| \([1, 1]: (1)(1, 7)_{-}\) | \((GL(1) \times Sp(8), \text{triv} \times 21^6)\) | \(41^6\) | \((\frac{7}{2}, \frac{5}{2}, \frac{3}{2}, \frac{1}{2}, 0)\) |
| \([5, 5]: (1)\) | \((Sp(10), \text{triv})\) | \(2^5\) | \((2, 1, 0, -1, -2)\) |
| \((3)(3, 5)_{-}\) | \((Sp(10), 33211)\) | \(33211\) | \((\frac{5}{2}, \frac{3}{2}, \frac{1}{2}, -\frac{1}{2}, 1)\) |
| \([1, 1]: (1)(3, 5)_{-}\) | \((GL(1) \times Sp(8), \text{triv} \times 2^3 1^2)\) | \(42211\) | \((\frac{5}{2}, \frac{3}{2}, \frac{1}{2}, -\frac{1}{2}, 0)\) |
| \([2, 2]: (1)(1, 5)_{+}\) | \((GL(2) \times Sp(6), \text{triv} \times 2211)\) | \(4411\) | \((2, 1, 0, \frac{1}{2}, -\frac{1}{2})\) |
| \([1, 1]^2: (1)(1, 5)_{+}\) | \((GL(1)^2 \times Sp(6), \text{triv} \times 2211)\) | \(6211\) | \((2, 1, 0, 0, 0)\) |
| \([1, 1]^2: (1)(1, 5)_{-}\) | \((GL(1)^2 \times Sp(6), \text{triv} \times 21^4)\) | \(61^4\) | \((\frac{5}{2}, \frac{3}{2}, \frac{1}{2}, 0, 0)\) |
| \([4, 4]: (3)\) | \((GL(4) \times Sp(2), \text{triv})\) | \(3322\) | \((\frac{3}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{3}{2}, 1)\) |
| \([1, 1]: (4, 4): (1)\) | \((GL(1) \times GL(4), \text{triv})\) | \(42^3\) | \((\frac{3}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{3}{2}, 0)\) |
| \([3, 3]: (1)(1, 3)_{-}\) | \((GL(3) \times Sp(4), \text{triv} \times 211)\) | \(433\) | \((\frac{5}{2}, \frac{1}{2}, 1, 0, -1)\) |
| \([2, 2]: (3, 3)_{-}\) | \((GL(2) \times GL(3), \text{triv})\) | \(442\) | \((1, 0, -1, \frac{1}{2}, -\frac{1}{2})\) |
| \([1, 1]^2: (3, 3)_{-}\) | \((GL(1)^2 \times GL(3), \text{triv})\) | \(622\) | \((1, 0, -1, 0, 0)\) |
| \([2, 2]^2: (3)\) | \((GL(2)^2 \times Sp(4), \text{triv})\) | \(55\) | \((\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, 0)\) |
| \([1, 1]^3: (1)(1, 3)_{-}\) | \((GL(1)^3 \times Sp(4), \text{triv} \times 211)\) | \(811\) | \((\frac{3}{2}, \frac{1}{2}, 0, 0, 0)\) |
| \([1, 1] \{2, 2\}: (1)\) | \((GL(1) \times GL(2)^2, \text{triv})\) | \(64\) | \((\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, 0)\) |
| \([1, 1]^3 \{2, 2\}: (1)\) | \((GL(1)^3 \times GL(2), \text{triv})\) | \(82\) | \((\frac{1}{2}, -\frac{1}{2}, 0, 0, 0)\) |
| \([1, 1] \{4\}: (1)\) | \((GL(1)^4, \text{triv})\) | \(10\) | \((0, 0, 0, 0, 0)\) |
\[ \Sigma = \{c_1, c_1\} \ldots \{c_k, c_k\} : (b_0)(a_1, b_1)_{\epsilon_1} \ldots (a_r, b_r)_{\epsilon_r} \]

\[ M = M(\sigma), \ (M, O_M) \rightarrow \tau \text{ on } Sp(2m) \]

\[ \mathcal{O} = \text{Ind}_{M}^{G}(O_M) \]

\[ \lambda(\Sigma) \text{ as before, using: } \{c_i, c_i\} \rightarrow (c_i, c_i)_1 \rightarrow \frac{1}{2}(c_i - 1, \ldots, -c_i + 1) \]

\[ \lambda(\Sigma) = \lambda(\mathcal{O}) \]

\[ I(\Sigma) = \text{Ind}_{GL(c_1) \times \cdots \times GL(c_k) \times Sp(2m)}^{Sp(2n)}(1 \otimes \cdots \otimes 1 \otimes \tau) \]

**Proposition**: \( I(\Sigma) \) is irreducible and unitary

(unitarity is obvious)
Inducing Data:

\[(\Sigma, \nu) = \{c_1, c_1\} x_1 \ldots \{c_k, c_k\} x_k : (b_0)(a_1, b_1)\epsilon_1 \ldots (a_r, b_r)\epsilon_r\]

\[I(\Sigma, \nu) = \text{Ind}^{Sp(2n)}_{GL(c_1) \times \ldots \times GL(c_k) \times Sp(2m)}(|\text{det}|^{x_1} \otimes \ldots \otimes |\text{det}|^{x_k} \otimes \tau)\]

For each \(1 \leq j \leq k\) define:

\[X_j = \begin{cases} 
B & a_i \leq c_j \leq b_i \text{ for some } 1 \leq i \leq r; \\
B & c_j \leq b_0; \\
C & \text{otherwise}. 
\end{cases}\]
Relabel thing by grouping $c_i$s which are equal:

\[
\{c_1, c_1\}_{x_1^1} \cdots \{c_1, c_1\}_{x_{d_1}^1} \cdots \{c_\ell, c_\ell\}_{x_1^\ell} \cdots \{c_\ell, c_\ell\}_{x_{d_\ell}^\ell}
\]

with $0 < c_1 < c_2 < \cdots < c_\ell$.

\[x_1^j \leq x_2^j \leq \cdots \leq x_{d_j}^j \quad (1 \leq j \leq \ell). \quad (1)\]
Theorem:
(1) $I(\Sigma, \nu)$ is irreducible and unitary if $(x^j_1, \ldots, x^j_{d_j})$ is in the 0-complementary of type $X_j$ for all $j$

and if $c_{j+1} = c_j + 1$ then $x^j_s + x^j_{s+1} < \frac{3}{2}$ for all $s$ and $t$.

Call these $(\Sigma, \nu)$ or $\nu$ admissible

$\lambda = \lambda(O) = \lambda(\Sigma, \nu), \pi(\Sigma, \nu) = I(\Sigma, \nu) = \pi(\lambda)$

(2) $I(\Sigma, \nu)$ (admissible) satisfies $AV(I(\sigma, \nu)) = \overline{O}$

(3) $\mathcal{C}(O) = \{I(\Sigma, \nu) \mid \nu \text{ is admissible}\}$

Recall $\hat{\Pi}_{sph} = \bigcup_{\mathcal{O}} \mathcal{C}(O)$
0-Complementary Series:

**type B:** \( \lambda = (x_1, \ldots, x_n) \): \( |x_i| < \frac{1}{2} \) for all \( i \)

**type C:** \( \lambda = (x_1, x_1, \ldots, x_r, y_1, \ldots, y_s) \)

\( 0 \leq x_1 \leq \cdots \leq x_r \leq \frac{1}{2} < y_1 < y_2 < \cdots < y_s < 1. \)

(1) \( x_i + x_j, x_i + y_j \neq 1 \) for all \( i, j \);

(2) If \( s \geq 1 \), \( |\{1 \leq i \leq r | 1 - x_i < y_1\}| \) is even

(3) For all \( 1 \leq j \leq s - 1 \), \( |\{i | y_j < 1 - x_i < y_{j+1}\}| \) is odd

(In short: all irreducible deformations, removing Stein complementary series)
Appendix: Dual Pair Correspondence for $Sp(4)/SO(5)$

$Sp(4)$
\[ SO(5) \]
$Sp(4)/SO(5)$ Dual Pair Correspondence