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A Support Preserving Hahn-Banach Property to Determine Helson-*S* Sets

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Abstract. We imbed the space of pseudo-measures $A'(E)$ supported by a closed totally disconnected set $E \subseteq R/2\pi Z$ into a space of distributions on an "imbedding" group. The basic technique is to find a sequence of measures μ_m on E (non discrete measures generally) associated with each $T \in A'(E)$, so that, with an additional arithmetic condition, $\{\mu_m\}$ converges in a weaker than weak $*$ topology to a measure μ, and $\mu = T$. Using this framework we prove that a Helson set is a set of spectral synthesis if and only if certain of our distributions have a support preserving extension. We also introduce a uniqueness criterion, and show that the extension condition and uniqueness condition imply that $A'(E)$ is the space of measures supported by E.

$E \subseteq \mathring{\Gamma} = R/2\pi Z$ is a closed totally disconnected set. We introduce a method to associate a sequence of measures to a given pseudo-measure supported by E (§5). We employ this approach to derive necessary and sufficient conditions that Helson sets be sets of spectral synthesis (S) (§8); the first set of conditions is in terms of a Hahn-Banach extension property with "boundary" constraints (viz., (e_T) of §5). We also introduce a (Cantor) uniqueness set type of criterion (§6) on E, which, when coupled with (e_T), implies E is S (Prop. 6.1). This is interesting since we show that the Cantor set (a non-Helson set) satisfies the uniqueness condition, and are able to conclude that if a Helson set is to be an S set, then a property of Helson sets must be used to verify (e_T). Körner has recently constructed independent Helson sets which are neither S sets nor uniqueness sets.

§0 contains the necessary notation and some remarks about the general problem to associate measures with a given pseudo-measure. §1–4 are necessary for §5–6, §8, and are used in §7 to prove an analytic result on absolutely convergent Fourier series. In §1–3 we construct a large group Γ in which to imbed E, and define an LB space of functions on Γ which plays a key role in the sequel. The technique of §6 centers about the notions of §5 and the Dieudonné-Grothendieck theorem.

0. Notation and Remarks

$A(\mathring{\Gamma})$ is the space of absolutely convergent Fourier series on $\mathring{\Gamma}$, normed by $\|\phi\|_A = \sum |a_k|$, where $\phi(\gamma) = \sum a_k e^{ik\gamma}$. We define

$$k(E) = \{\phi \in A(\mathring{\Gamma}) \colon \ \phi = 0 \text{ on } E\}$$

$$j(E) = \{\phi \in k(E) \colon \ \operatorname{supp} \phi \cap E = \emptyset\};$$

and set $A(E)=A(\mathring{\Gamma})/k(E)$, $A_j(E)=A(\mathring{\Gamma})/\overline{j(E)}$ with corresponding quotient norms, $\| \ \|_{A(E)}$, $\| \ \|_j$. If $\psi\in A(\mathring{\Gamma})$ we write $\tilde{\psi}(\gamma)=\psi(\gamma)+\overline{j(E)}$. Also m is Haar measure on Γ.

$C(E)$ is the space of continuous functions on E with dual $M(E)$, the space of bounded Radon measures supported by E; the total variation norm on $M(E)$ is denoted by $\| \ \|_1$. The dual of $A(E)$ (resp., $A_j(E)$) is $A'_S(E)$ (resp., $A'(E)$). $A'(E)$ is the space of *pseudo-measures with support in* E. $A'_0(E)=\{T\in A'(E): \lim_{|n|\to\infty} \hat{T}(n)=0\}$ and $M_0(E)=M(E)\cap A'_0(E)$.

E is an *S-set* if $A'(E)=A'_S(E)$, an *Helson set* if $A'_S(E)=M(E)$, and a set of *strong spectral resolution* if $A'(E)=M(E)$. E is a *Kronecker set* if for every $\varepsilon>0$ and for every $\phi\in C(E)$, $|\phi|=1$, there is $n\in Z$ such that

$$\sup_{\gamma\in E} |\phi(\gamma)-e^{in\gamma}|<\varepsilon.$$

E is a Cantor U-set (resp., *weak* U-set) if $A'_0(E)=\{0\}$ (resp., $M_0(E)=\{0\}$).

Finally, let $\mathscr{F}$ be the family of all open (in E) and closed subsets of E, and set $\mathscr{C}E=\bigcup I_j$ where $I_j\equiv(\lambda_j,\gamma_j)$ is an open interval, $m(I_j)=\varepsilon_j$, and $E\subseteq[0,2\pi)$. For each $F\in\mathscr{F}$, $\phi_F\in C^\infty(\mathring{\Gamma})$ denotes a function equal to 1 on a neighborhood of F and equal to 0 on a neighborhood of $E-F$.

Remark 1. Let $T\in A'(E)$. If $m(E)=0$ it is relatively straightforward [2] to construct a sequence of measures μ_n with finite support such that $\mu_n\to T$ on $C^1(\mathring{\Gamma})$. The finite support is a mixed blessing since one cannot hope to conclude weak $*$ convergence of such a sequence except in the simplest cases. Naturally, for a given Helson set E, it is desirable to find a weak $*$ convergent sequence of measures for T if E is to be S. This is the motive for our approach in §5.

2. Remark 1 leads to the general problem to study those weaker than weak $*$ topologies which preserve some of the important properties of weak $*$ convergence. This is the motivation for [3], § 2, and Prop. 5.1b.

1. The Imbedding Group Γ

Let G be the additive group of real-valued elements in $A(\mathring{\Gamma})$, taken with the discrete topology; and let Γ be its compact dual group. Γ is connected since $\{0\}$ is the only compact subgroup of G, and thus Γ is also a divisible group.

For each $\gamma\in\mathring{\Gamma}$ we identify the element $f_\gamma\in\Gamma$ by

$$\forall\phi\in G, \quad (f_\gamma,\phi)=e^{i\phi(\gamma)}. \tag{1.1}$$

Thus we have the injection

$$u\colon \mathring{\Gamma}\to\Gamma$$
$$\gamma\rightsquigarrow f_\gamma,$$

and by (1.1) and the definition of the topology on Γ, u is continuous. In particular uE is compact in Γ.

We could take G to have the further property that each $\phi \in G$ equal zero at a fixed $\lambda_0 \notin E$. This has the advantage of yielding an easy proof of the fact that

$$\{f \sim r_j, \gamma_j \in \Gamma\colon\ (f, \phi) = \prod e^{i r_j \phi(\gamma_j)}, \sum |r_j| < \infty, \gamma_j \in \mathring{\Gamma}\}$$

is dense in Γ (e.g. [3]).

Define $\mathscr{T}$ to be the set of trigonometric polynomials Ψ on Γ, i.e.

$$\Psi(f) = \sum_1^n a_j (f, \phi_j), \tag{1.2}$$

some $\phi_j \in G$. Then $A(\Gamma)$, the space of absolutely convergent Fourier series on Γ, is the completion of $\mathscr{T}$ where Ψ (of (1.2)) is normed by

$$\|\Psi\|_{A(\Gamma)} = \sum_1^n |a_j|.$$

Finally, define $k(uE)$ and $j(uE)$ analogous to the way we defined $k(E)$ and $j(E)$.

2. An *LB* Function Space on Γ

For each $\Psi \in A(\Gamma)$ define

$$G_\Psi = \{\phi \in G\colon\ \hat{\Psi}(\phi) \neq 0\},$$

and for each integer $k > 0$ let

$$\mathscr{A}_k = \{\Psi \in A(\Gamma)\colon\ \forall \phi \in G_\Psi, \|\phi\|_A \leqq k\}.$$

Clearly,

Proposition 2.1. *$\mathscr{A}_k$ is a closed vector subspace of $A(\Gamma)$.*

Thus we define the *LB* space

$$\mathscr{A} = \bigcup \mathscr{A}_k$$

noting that $\mathscr{A}$ is bornological, barrelled, and non-metrizable, and that the dual $\mathscr{A}'$ is Fréchet. Obviously, $\bar{\mathscr{A}} = A(\Gamma)$ since $\mathscr{T} \subseteq \mathscr{A}$, the imbedding $\mathscr{A} \to A(\Gamma)$ is continuous since $\mathscr{A}$ is bornological, and consequently we have the natural imbedding $A'(\Gamma) \subseteq \mathscr{A}'$. It is easy to check that $A'(\Gamma)$ is $\beta(\mathscr{A}', \mathscr{A})$ dense in $\mathscr{A}'$.

3. Canonical Maps

From the definition of $\mathscr{A}_k$,

Proposition 3.1. *Let $\Psi \in \mathscr{A}$. Then*

$$u'\Psi(\lambda) = \sum \hat{\Psi}(\phi)\, e^{i\phi(\lambda)} \tag{3.1}$$

is an element of $A(\mathring{\Gamma})$.

u' is thus a well-defined linear map $\mathscr{A} \to A(\Gamma)$ and so we can define the associated canonical linear map

$$u'_j\colon\ \mathscr{A} \to A_j(E).$$

Proposition 3.2. $u'_j \mathscr{A} = A_j(E)$.

Proof. Take $\psi(\gamma) = \sum a_k e^{ik\gamma} A(\mathring{\Gamma})$ and $\psi_n(\gamma) = \sum_{|k| \leqq n} a_k e^{ik\gamma}$.

Let N be a neighborhood of E with $m(N) < 2\pi$ and choose $\phi_k \in G$ such that $\phi_k(\gamma) = k\gamma$ on N.

Define $\Psi_n(f) = \sum_{|k| \leqq n} a_k(f, \phi_k) \in \mathscr{A}$ noting that $u'\Psi_n = \psi_n$ on N and $u'\Psi_n \in A(\Gamma)$.

Since $u'\Psi_n - \psi_n \in j(E)$ we have $u'_j \Psi_n = \tilde{\psi}_n$.

Therefore

$$\|\tilde{\psi} - u'_j \Psi_n\|_j = \|\tilde{\psi} - \tilde{\psi}_n\|_j \leqq \sum_{|k| > n} |a_k|. \qquad qed.$$

Since $\mathscr{A}$ is bornological and by properties of bounded sets in $\mathscr{A}$,

Proposition 3.3. *u' and u'_j are continuous.*

Example 3.1. Clearly, there are $\Psi \in A(\Gamma) - \mathscr{A}$ such that $u'_j \Psi \in A_j(E)$. Take $\|\phi_k\|_A \to \infty$, $a_k = (\alpha/k^2) \exp\{-\|\phi_k\|_A\}$, and

$$\Psi(f) = \sum_1^\infty a_k(f, \phi_k).$$

Thus $\|\Psi\|_{A(\Gamma)}$ can be made as small as we like by choice of α, and $\Psi \notin \mathscr{A}$ by choice of $\{\phi_k\}$. Now

$$\Big\|\sum_1 a_k e^{i\phi_k}\Big\|_A \leqq |\alpha| \sum_1 \frac{1}{k^2}$$

so that $u'_j \Psi \in A_j(E)$.

Using this example, the Baire category theorem yields

Proposition 3.4. a) *$u'_j \mathscr{A}_k$ in nowhere dense in $A_j(E)$.*

b) $A_j(E) \neq \bigcup \overline{u'_j \mathscr{A}_k}$.

c) *$\mathscr{A}$ and $\mathscr{C}\mathscr{A}$ are dense in $A_j(E)$.*

We now define the canonical transpose

$$u_j\colon\ A'(E)\to\mathscr{A}',$$

noting that u_j is a continuous 1-1 linear map since u_j' is continuous and $\overline{u_j'\mathscr{A}}=A_j(E)$. By definition, if $T\in A'(E)$, $t=u_j T$ is given by

$$\forall\,\Psi\in\mathscr{A},\qquad \langle t,\Psi\rangle=\langle T,u_j'\Psi\rangle.$$

It follows that

Proposition 3.5. *Let* $T\in A'(E)$, $u_j T=t$, *and* $\Psi\in\mathscr{A}$. *If* $\operatorname{supp}\Psi\cap uE=\emptyset$ *then* $\langle t,\Psi\rangle=0$.

4. Kronecker Sets in Γ

Let $F\in C(E)$, $|F|=1$. Because F is uniformly continuous and E is totally disconnected there is $\phi\in C(E)$, real-valued, such that

$$\forall\,\gamma\in E,\qquad F(\gamma)=e^{i\phi(\gamma)}.$$

Obviously, this statement can be strengthened considerably. The point for us is that by using the Stone-Weierstrass theorem we can show

Proposition 4.1. uE *is Kronecker (and therefore independent).*

Varopoulos [5] showed that Kronecker sets are S-sets in $\mathring{\Gamma}$, and the proof is readily extended to arbitrary compact abelian groups in which the Kronecker set is 0-dimensional (e.g. [2, Chapter 2]).

Let us show that $\dim uE=0$, i.e., that uE is totally disconnected. Take $\lambda\in E$ and let $\bigcap F_\alpha=\{\lambda\}$, $F_\alpha\in\mathscr{F}$, noting that each F_α is open and compact. u continuous implies uF_α compact, so that we will have the total disconnectedness of uE by Šura-Bura's theorem once we prove uF is open in $uE\,(F\in\mathscr{F})$. Let $\phi\in G$ be equal to 1 on a neighborhood of $F\in\mathscr{F}$ and equal to 0 on a neighborhood of $E-F$. Take $0<\varepsilon<|e^i-1|$ and define the following open sets of uE:

$$\forall\lambda\in F,\qquad V_\phi(\lambda,\varepsilon)\cap uE=\{f_\gamma\colon\ |1-e^{i(\phi(\lambda)-\phi(\gamma))}|<\varepsilon\}.$$

Clearly, if $f_\gamma\in V_\phi(\lambda,\varepsilon)\cap uE$ then $\gamma\in F$ for otherwise we get $|e^i-1|<\varepsilon$ by our choice of ϕ. Thus

$$uF=\bigcup_{\lambda\in F}[V_\phi(\lambda,\varepsilon)\cap uE]$$

and so uF is open. Consequently, from Prop. 4.1, the fact that Kronecker sets are Helson, and the Varopoulos theorem generalized to Γ –

Proposition 4.2. $A'(uE)=M(uE)$.

Actually, Saeki has extended Varopoulos' result by dropping the 0-dimensionality hypothesis. For our purposes of dealing with uE it is easier to proceed as we did than to invoke Saeki's technique.

5. Pseudo-Measures and Associated Measures

$T \in A'(E)$ is *measure approximable* (resp., *synthesis measure approximable*) if there is a sequence of subspaces $X_k \subseteq A_j(E)$ and measures $\mu_k \in M(E)$ such that

a) $\overline{\bigcup X_k} = A_j(E)$ (resp., $\bigcup \overline{X}_k = A_j(E)$)

b) $\mu_k = T$ on X_k.

We shall observe (Prop. 5.3) that under a certain extension condition ((e_T), below) on $T \in A'(E)$, T is measure approximable. Clearly, if $T \in A'(E)$ and E is synthesis measure approximable then $T \in M(E)$ for E Helson; however, in light of Prop. 3.4, this approach is not promising.

Given $t \in \mathscr{A}'$ we know that $t \in \mathscr{A}'_k$ for each k and so there is $s_k \in A'(\Gamma)$ such that $s_k = t$ on $\mathscr{A}_k$. Thus, because of the annihilating properties of $t = u_j T$, $T \in A'(E)$ (e.g., Prop. 3.5, Example 5.2 below), the following condition is meaningful:

Given $t = u_j T$, $T \in A'(E)$, $\exists$ infinitely many k for which (e_T)

$$\exists\, s_k \in A'(uE), \quad s_k = t \quad \text{on } \mathscr{A}_k .$$

We can not expect (e_T) to hold generally for all $T \in A'(E)$ (e.g. Example 6.2).

Example 5.1. Note that *if* $u_j T = t \in A'(\Gamma)$ *then* $T \in M(E)$. In fact, if there is $K > 0$ such that for all $\phi_F \in G$, $F \in \mathscr{F}$,

$$|\langle T, e^{i\phi_F}\rangle| = |\hat{t}(\phi_F)| < K$$

then $T \in M(E)$ (e.g. [3]). Further, from Prop. 5.2, $t \in M(uE)$.

Implicit in (e_T) is the hypothesis that $\mathscr{A}$ contains a large number of elements from $j(uE)$ (modifications of this observation will appear in forthcoming work).

On the other hand, there are many elements $\Psi \in \mathscr{A}$ which vanish on uE and such that $\langle t, \Psi \rangle = 0$ for $t = u_j T$; and so it is reasonable to consider "supp t" $\subseteq uE$. In fact –

Example 5.2. Take $U \subseteq V$, two compact neighborhoods of E, and let $\phi \in G$ be 2π on U and 0 on $\mathscr{C}V$. Thus, $\Psi(f) = 1 - (f, \phi) \in \mathscr{A}$ and if $t = u_j T$ then $\langle t, \Psi \rangle = 0$ since supp $T \subseteq E$. Assume without loss of generality that $U - N + N = V$, N a neighborhood of origin. Then $\Psi \in \mathscr{A}_k$ where

$$m(U - N)/m(N) < k^2$$

by a standard approximate identity argument.

Example 5.3. There are other extension conditions to guarantee that $T \in P'(E)$ be measure approximable and hence to deduce that if E is Helson and satisfies one of these conditions then $T \in M(E)$. For example, noting that for all k there is $s_k \in A'(\Gamma)$ such that $s_k = t$ on $\mathscr{A}_k$, we assume

$s_k = u_j S_k$, $S_k \in A'(E)$. Thus $S_k \in M(E)$ and $s_k \in M(uE)$ by Example 5.1, and so T is measure approximable.

The following is clear from Prop. 4.2 and the definition of inductive limit topologies.

Proposition 5.1. *Assume that* (e_T) *is satisfied. Then there is* $\{v_{k_n}\} \subseteq M(uE)$ *such that*

a) $\forall k_n > k_m$, $t = v_{k_n} = v_{k_m}$ *on* $\mathscr{A}_{k_m}$.

b) $v_{k_n} \to t$ *in* $\beta(\mathscr{A}', \mathscr{A})$ *where* $\beta(\mathscr{A}', \mathscr{A})$ *is the strong topology on* $\mathscr{A}'$.

By standard measure theory –

Proposition 5.2. u_j: $M(E) \to M(uE)$ *is a bijective continuous linear map where* $M(E)$ *has the induced topology from* $A'(E)$ *and* $M(uE)$ *has the induced topology from* $\mathscr{A}'$.

Prop. 5.2 is true for $M(X)$ and $M(uX)$, with any closed $X \subseteq \mathring{\Gamma}$.

Summing up the previous observations and letting $X_k = u_j' \mathscr{A}_k$ we have –

Proposition 5.3. *Assume* (e_T) *is satisfied. Then* $T \in A'(E)$ *is measure approximable; further, the subspaces* X_k *can be taken to be increasing and* $\mu_m = \mu_k$ *on* X_k *if* $m \geqq k$.

6. Conditions for Strong Spectral Resolution

We begin with the following definition. E has the weak uniqueness property $\underline{U}_N$ if there is N satisfying the following condition:

$$\forall I_a, I_b \subseteq [0, 2\pi), \qquad \text{for which } I_k \subseteq (\lambda_a, \gamma_b)$$

for infinitely many k, $\exists\, n I_{k_j} \subseteq (\lambda_a, \gamma_b)$ such that

$$d(I_{k_{j-1}}, I_{k_j}) \leqq N \min(\varepsilon_{k_{j-1}}, \varepsilon_{k_j}) \qquad \text{for } j = 1, \ldots, n,$$
$$\text{where} \quad k_0 = a \quad \text{and} \quad k_{n+1} = b$$

(and where $d(I, J)$ is the distance between I and J).

Remark 1. We refer to U_N as a weak uniqueness property because the classical necessary conditions for weak uniqueness due to Bary and Civin-Chrestenson [1, Chapter 14, § 13]; their result is only stated for U sets but it is shown that $M_0(E) \neq \{0\}$ in the proof. In this regard, consider the following condition, $\tilde{U}_1$, which is weaker than U_1:

For every interval J, which contains infinitely many I_k, $\exists\, I_{k_1}, I_{k_2} \subseteq J$ such that

$$d(I_{k_1}, I_{k_2}) \leqq \min(\varepsilon_{k_1}, \varepsilon_{k_2}).$$

Thus if E is not $\tilde{U}_1$ and if we build E by throwing out the largest I_j between any given I_p, I_q (which have already been thrown out), we have

$$\varepsilon_j/d(I_p, I_q) < 1/3 \tag{6.1}$$

since $\varepsilon_j \leqq \min(\varepsilon_p, \varepsilon_q)$. One of Bary's necessary conditions for weak uniqueness is that ratios such as (6.1) tend to 0, and for a long time it was thought that such convergence was all that was necessary.

2. Since we are interested in whether certain Helson sets are S sets, and since Helson sets are weak U-sets [2, Chapter 7], it is not unexpected to have uniqueness conditions for strong spectral resolution (e.g. Prop. 6.1 below).

Example 6.1. Clearly every countable E is $\tilde{U}_1$. On the other hand $E=\{0, 1/n: n=1, \ldots\}$ is not U_N for any N. In fact, given $I_a=[-\pi, 0)$ and $I_b=I_n=(1/(n+1), 1/n)$, if $m>n$ with $I_m=(1/(m+1), 1/m)$

$$1/m = d(I_a, I_m) > 1/m(m+1) = \min(\varepsilon_a, \varepsilon_m);$$

and if n is chosen large enough, $1/m > N/m(m+1)$ for $m>n$. Note that $\{0, 1/n: n=1, \ldots\}$ is not Helson. Now take $E=\{0, 1/3^n: n=1, \ldots\}$ (resp., $\{0, 1/2^n: n=1, \ldots\}$) which is Helson. Let I_a be as above and let $I_b=I_n=(1/3^{n+1}, 1/3^n)$ (resp., $I_b=I_n=(1/2^{n+1}, 1/2^n)$). Then for $m>n$

$$1/3^{m+1}\ (\text{resp., } 1/2^{m+1}) = d(I_a, I_m) \leqq 2/3^{m+1}\ (\text{resp., } 1/2^{m+1}) = \min(\varepsilon_a, \varepsilon_m),$$

and hence these Helson sets are U_1. We mention both the $1/2^n$ and $1/3^n$ cases since $\{0, 1/2^n\}$ is close to the "boundary" of being a Helson set, a fact which is illustrated by observing that the above inequality is not as strong for the $1/2^n$ case as for the $1/3^n$ case.

Example 6.2. The Cantor ternary set E is U_1 (and, of course, not Helson). Again, this is an easy calculation. In light of Prop. 6.1 and the fact that $A'_S(E) \neq M(E)$, (e_T) is not satisfied for some $T \in A'(E)$.

Example 6.3. There is a standard technique to construct perfect Helson sets E due to Carleson, Kahane-Salem, and Rudin [2, Chapter 5]. We shall show that such an E is U_N.

Take $\lambda_1^1 \in (0, \pi/2)$ such that $\{\pi, \lambda_1^1\}$ is independent (over the rationals), choose $\gamma_2^1 \in (3\pi/2, 2\pi)$, and form the interval $I_0=(\gamma_2^1, \lambda_1^1) \subseteq [-\pi, \pi)$ of length ε_0. We next take $\lambda_1^1 < \gamma_1^1 < \lambda_2^1 < \gamma_2^1$ such that $\varepsilon_0 \geqq \gamma_1^1 - \lambda_1^1 = \gamma_2^1 - \lambda_2^1 = L_1$, $\{\pi, \lambda_1^1, \lambda_2^1\}$ is independent, and $\varepsilon_1^0 = \lambda_2^1 - \gamma_1^1 \geqq L_1$; and write $[\lambda_j^1, \gamma_j^1] = S_j^1$, $E^1 = S_1^1 \cup S_2^1$, $I_1^0 = (\gamma_1^1, \lambda_2^1)$. For the inductive step we first recall from Kronecker's theorem [2, Chapter 5] that if $\{\pi, \lambda_1^n, \ldots, \lambda_{2^n}^n\}$ is independent and a measure μ is supported by $\{\lambda_1^n, \ldots, \lambda_{2^n}^n\}$ then there is N_n for which

$$\sup_{0<m<N_n} |\hat{\mu}(m)| > \tfrac{1}{2} \|\mu\|_1;$$

next we note that if $\{\pi, \lambda_1^n, \ldots, \lambda_{2^n}^n\}$ is independent and $V_1, \ldots, V_{2^n}$ are intervals disjoint from the λ_j^n, π then there are $\lambda_j \in V_j$ such that

$$\{\pi, \lambda_1^n, \ldots, \lambda_{2^n}^n, \lambda_1, \ldots, \lambda_{2^n}\}$$

is independent [2, Chapter 4]. We form $E = \bigcap E^n$ where $E^n = S_1^n \cup \cdots \cup S_{2^n}^n$, $m(S_j^n) = L^n$, $S_{2j-1}^n, S_{2j}^n \subseteq S_j^{n-1}$, and $\{\pi, \lambda_1^n, \ldots, \lambda_{2^n}^n\}$ is independent. Thus at the n-th stage we throw out the open intervals I_j^{n-1} of length ε_j^{n-1}, $j = 1, \ldots, 2^{n-1}$, subject to the conditions

$$L_n \leqq \varepsilon_j^{n-1}, \quad j = 1, \ldots, 2^{n-1},$$

and

$$\lim_n N_n 2^n L_n^{\frac{1}{2}} = 0.$$

We need these two conditions to show $A'(E) = M(E)$, and the inequality obviously tells us that E is U_1.

Proposition 6.1. *Let E be a U_N set. If (e_T) is satisfied for $T \in A'(E)$ then $T \in M(E)$.*

Proof. Take $X_k = u_j' \mathscr{A}_k$ and μ_k as in the definition of measure approximable and Prop. 5.3.

Without loss of generality assume $N = 1$.

Let $V \subseteq E$ be open (in the relative topology obviously).

We shall show that $\lim \mu_k(V)$ exists so that by the Grothendieck-Dieudonné theorem, μ_k converges in the weak topology on $M(E)$ to a measure μ; consequently, $\mu_k \to \mu$ in the weak$*$ topology.

For $\tilde{\phi} \in \bigcup X_k$, with $\tilde{\phi} \in X_m$, say, we have

$$\langle \mu, \tilde{\phi} \rangle = \lim \langle \mu_k, \tilde{\phi} \rangle = \langle \mu_m, \tilde{\phi} \rangle = \langle T, \tilde{\phi} \rangle;$$

thus $T = \mu$ on a dense subset of $A_j(E)$ and so $T = \mu$.

Now, since V is open, we write

$$V = \bigcup F_n,$$

where $F_n \subseteq V$, $F_n \in \mathscr{F}$, and $F_n \subseteq F_{n+1}$.

Recall that the de la Vallée-Poussin kernel $\phi_{\varepsilon,\gamma}$ about $\gamma \in \mathring{\Gamma}$ is non-negative, equals 1 on an ε-neighborhood of γ, equals 0 outside a 2ε-neighborhood of γ, and has the property that $\|\phi_{\varepsilon,\gamma}\|_A \leqq 3$ [2, Chapter 1].

For each F_n we observe that there is a finite number of de la Vallée-Poisson kernels $\phi_{n,1}, \ldots, \phi_{n,k_n}$ (for various ε's and γ's in F_n), such that

$$\begin{gathered} \operatorname{supp} \phi_{n,j} \cap \operatorname{supp} \phi_{n,k} = \emptyset, \\ \forall \gamma \in F_n, \exists j \quad \text{such that } \phi_{n,j}(\gamma) = 1, \\ \forall \gamma \in E - F_n, \quad \forall j, \quad \phi_{n,j}(\gamma) = 0. \end{gathered} \tag{6.2}$$

To see this we first note that since V is open. $V=E\cap(\cup J_m)$ where J_m is an open interval; and a straightforward compactness argument shows that $F\in\mathscr{F}$ is the intersection of E with a finite number of intervals each having their endpoints in the I's.

By the previous observation $F\in\mathscr{F}$, $F_n\cap J_m$ determines a finite number of I_{a_j}, I_{b_j} depending on n and m, and for each of these we use the U_1 property. Let us label all the closed intervals in J_m contiguous to the I's (obtained from U_1) by $H_1, \ldots, H_p$.

Consequently we choose $\phi_{n,j}$ about $\gamma\in H_k$ so that it is equal to 1 on H_k, and we have (6.2).

Now, for a given F_n, define $\tilde{\phi}_n\in A_j(E)$ by

$$\phi_n(\gamma)=\frac{1}{1-e^i}\left[k_n-\sum_{j=1}^{k_n} e^{i\phi_{n,j}(\gamma)}\right].$$

Clearly, $\tilde{\phi}_n\in X_3$ since $\|\phi_{n,j}\|_A\leqq 3$.

Because of (6.2) we note that if $\gamma\in F_n$,

$$\phi_n(\gamma)=\frac{1}{1-e^i}\left[k_n-(k_n-1)-e^i\right]=1,$$

since $\phi_{n,j}(\gamma)=1$ for precisely one j and $\phi_{n,j}(\gamma)=0$ for the remaining j; similarly, if $\gamma\in E-F_n$, $\phi_{n,j}(\gamma)=0$ for each j and so $\phi_n(\gamma)=0$.

Thus $\lim\phi_n=\chi_V$ pointwise; and from our definitions

$$\sup\{|\phi_n(\gamma)|:\gamma\in E, n\}<\infty.$$

Therefore, we can use the Riesz representation theorem and see that for each $k\geqq 3$, $\mu_k(V)$ is well-defined by

$$\lim_n\langle\mu_k,\tilde{\phi}_n\rangle=\mu_k(V).$$

Now,

$$\lim_{k\geqq 3}\mu_k(V)=\lim_k\lim_n\langle\mu_k,\tilde{\phi}_n\rangle=\lim_k\lim_n\langle T,\tilde{\phi}_n\rangle=\lim_n\langle T,\tilde{\phi}_n\rangle,$$

where the right hand side exists since $\lim_n\langle T,\tilde{\phi}_n\rangle=\lim_n\langle\mu_k,\tilde{\phi}_n\rangle$.

Thus, $\lim_k\mu_k(V)$ exists. *qed.*

We can't replace U_1 by $\tilde{U}_1$ in the proof since there is no guarantee that V would be covered by the above procedure.

Example 6.4. Let us look at the relation between U_N and Helson sets. For convenience of explication suppose that E is $\tilde{U}_1$ (the point of this example being the same for U_N). Assume E is a perfect Helson set. If E is not $\tilde{U}_1$ then there is an admissible J (as in the definition of $\tilde{U}_1$) such that for all I_{k_1}, $I_{k_2}\subseteq J$, $d(I_{k_1}, I_{k_2})>\min(\varepsilon_{k_1},\varepsilon_{k_2})$. Without loss of generality we suppose that J has endpoints in I_p, I_q, each with length greater than or equal to the length of any I between them. We build $J\cap E$ by throwing out I_1^1, the largest I between I_p and I_q; then at the second step we throw out I_2^1, I_2^2 the largest I's between I_p and I_1^1, and I_1^1 and I_q, respectively; etc. In this way because of estimates like (6.1), it is reasonable to try to find situations where there are perfect P, Q for which

$$P+Q\subseteq J\cap E\subseteq E \tag{6.3}$$

(since at each stage of building $J\cap E$ there is a certain symmetry and we have much of J left over).

We can prove that E Helson and (6.3) give a contradiction so that in this setting we'd have "Helson implies $\tilde{U}_1$". To get the contradiction we construct the continuous Cantor-Lebesgue positive measures μ_P, μ_Q supported by P, Q, respectively, and have

$$\mu_P * \mu_Q(E)>0$$

since $P+Q\subseteq E$. This contradicts the recent Salinger-Varopoulos theorem [2, Chapter 7]: if E is Helson and μ, ν are positive continuous measures in $R/2\pi Z$ then $\mu*\nu(E)=0$. This latter result, by the way, is based on the Kahane-Salem theorem which estimates the number points of "general" arithmetic progressions that lie in a Helson set.

Remark. In light of the Bary theorem mentioned in Remark 1 of this section and the importance of the non-existence of inclusions like (6.3) for Helson sets, we would now like to define a more general notion than U_N. The point is to find such a notion so that Helson sets are included and the analogue of Prop. 6.1 is true; and to investigate the cases where (6.3) holds for perfect non-U_N sets.

7. A Property of $A(\mathring{\Gamma})$

Consider the following property on $A(\mathring{\Gamma})$ and E: $\exists K$ such that $\forall n\in Z$, $\exists \phi_n\in G$ and $\exists N_n$, a finite disjoint union of closed intervals covering E, for which

$$\begin{aligned} &\|\phi_n\|_A<K,\\ \forall \gamma\in N_n,\quad &e^{in\gamma}=e^{i\phi_n(\gamma)}. \end{aligned} \tag{P}$$

Note that for Helson sets there is $K>0$ such that $\|\phi_n\|_{A(E)}<K$ where $\phi_n(\gamma)=e^{in\gamma}$ on E. Also observe that our requirements for N_n are weaker than stipulating that N_n be a neighborhood of E.

The following result (and proof) is amusing for one-point sets, E.

Proposition 7.1. *Property P never holds.*

Proof. Let $\psi(\gamma)=\sum a_k e^{ik\gamma}\in A(\mathring{\Gamma})$ and define

$$\Psi_N(f)=\sum_{|n|\leq N} a_n(f,\phi_n)$$

where the ϕ_n are chosen by property P. Thus there is a k such that for all N, $\Psi_N \in \mathscr{A}_k$.

Setting $\psi_N(\gamma) = \sum_{|n| \leqq N} a_n e^{in\gamma}$,

$$\|\tilde{\psi} - u'_j \Psi_N\|_j \leqq \|\tilde{\psi} - \tilde{\psi}_N\|_j + \|\tilde{\psi}_N - u'_j \Psi_N\|_j.$$

Clearly, $\|\tilde{\psi} - \tilde{\psi}_N\|_j \to 0$ as $N \to \infty$.

Also, since $u' \Psi_N = \psi_N$ on an N_n we have

$$\|\tilde{\psi}_N - u'_j \Psi_N\|_j = 0,$$

because N_n is an S-set.

Consequently, $\tilde{\psi} \in \overline{u'_j \mathscr{A}_k}$, and hence $\bigcup_k \overline{u'_j \mathscr{A}_k} = A_j(E)$, contradicting Prop. 3.4. *qed.*

Remark. 1. E Helson implies $m(E)=0$ [2, Chapter 7].

2. If $m(E)=0$ then

$$\forall D>0, \quad \forall M>0, \quad \forall n, \quad \text{and} \quad \forall 1 \leqq p < \infty$$

there is a neighborhood $N_{n,p}$ of E of the form $\bigcup_1^k I_j$, I_j open disjoint intervals, and a function ϕ, $\sup\{|\phi(\gamma)|: \gamma \in N_{n,p}\} < D$ such that

$$\phi(\gamma) = n\gamma + 2\pi k_j \quad \text{on} \quad I_j$$

and

$$\Big(\int_{N_{n,p}} |\phi'|^2\Big)^{\frac{1}{2}} < M.$$

8. Extension and Helson Set Conditions for Spectral Synthesis

Let $\mathscr{A}(uE)$ be the quotient space of restrictions of the elements of $\mathscr{A}$ to uE; and recall from the proof of Prop. 4.1 that $u' \Psi \in C(E)$ when $\Psi \in C(uE)$.

Proposition 8.1. *E is Helson if and only if $\mathscr{A}(uE) = C(uE)$.*

Proof. Take $\Psi \in C(uE)$. Since E is Helson there is

$$\phi(\gamma) = \sum a_n e^{in\gamma} \in A(\mathring{\Gamma})$$

such that $\phi = u' \Psi$ on E.

For each n, consider a finite decomposition $\{F_1, \ldots, F_{m_n}\} \subseteq \mathscr{F}$ of E and $\psi_n \in G$ such that $\|\psi_n\|_\infty \leqq 31$ and

$$\psi_n(\gamma) = n\gamma + 2\pi k_j$$

on F_j, $j = 1, \ldots, m_n$.

Let $K_E > 0$ be the Helson constant: $\|\ \|_{A(E)} \leqq K_E \|\ \|_{\infty, E}$.

Thus, there is $k > 31\, K_E$ and $\{\phi_n\} \subseteq G$ such that $\|\phi_n\|_A \leqq k$ and $\phi_n = \psi_n$ on E; the fact that we can take ϕ_n real follows since $(\phi_n + \bar{\phi}_n)/2 = \psi_n$ on E and $\|(\phi_n + \bar{\phi}_n)/2\|_A \leqq \|\phi_n\|_A$.

Consequently, $\Phi(f) = \sum a_n(f, \phi_n) \in \mathscr{A}_k$ and $\Phi = \Psi$ on uE. *qed.*

Proposition 8.2. *Let E be Helson.*

a) *If (e_T) is satisfied then $T \in M(E)$.*

b) *E is S if and only if (e_T) is satisfied for all $T \in A'(E)$.*

Proof. a) Because of Prop. 5.1 b, $\lim \langle v_k, \Psi \rangle$ exists for all $\Psi \in \mathscr{A}$ (where we preserve the notation of § 5).

Since $v_k \in M(uE)$, if $\Psi \in \mathscr{A}(uE)$ and $\Psi_e \in \mathscr{A}$ is any extension, then $\langle v_k, \Psi \rangle = \langle v_k, \Psi_e \rangle$.

Thus, $\lim \langle v_k, \Psi \rangle$ exists for all $\Psi \in \mathscr{A}(uE)$; and so, from Prop. 8.1, $\{v_k\}$ converges in the weak $*$ topology to a measure $v \in M(uE)$.

Consequently $v = t$ on $\mathscr{A}$ and so $t \in M(uE)$ since $\bar{\mathscr{A}} = C(\Gamma)$ (because $\mathscr{T} \subseteq \mathscr{A}$, Γ is compact, G separates points on Γ, and by the Stone-Weierstrass theorem).

We are done by Example 5.1 or Prop. 5.2.

b) We need only show that $M(E) = A'(E)$ implies (e_T). This is obvious since if $T \in M(E)$ then $t = u_j T \in M(uE)$. *qed.*

In the same way,

Proposition 8.3. *Let E be a Helson set. E is an S set if and only if $t = u_j' T$ is well-defined on $\mathscr{A}(uE)$ for each $T \in A'(E)$.*

In fact, for E Helson and $T \in A'(E)$ we need only show: For each $F \in \mathscr{F}$, there is $\psi = \phi_F$ with $\|\psi\|_A \leqq K_E$ (since E is Helson such ψ exist) such that $\hat{t}(\psi) = \hat{t}(\phi_F)$; then $T \in M(E)$.

Example 8.1. Since Helson sets E have $m(E) = 0$ a natural technique to prove that $t = u_j T$ is well-defined on $\mathscr{A}(uE)$ (i.e., Prop. 8.3) is to utilize the representation

$$T = \sum k_j(\delta_{\lambda_j} - \delta_{\gamma_j}) \quad \text{on} \quad C^1(\mathring{\Gamma}),$$
$$\sum e^{r|k_j|} \varepsilon_j < \infty, \qquad \text{some } r$$

for $\hat{T}(0) = 0$ [2, Chapter 2]. In fact, if we chose G to be the real continuously differentiable elements of $A(\mathring{\Gamma})$ and defined $\mathscr{A}_k$ by the restriction that $\|\phi'\|_\infty \leqq k$ for $\phi \in G_\Psi$ then $t = u_j T$ is well-defined. On the other hand we do not get (8.1) with these spaces and so cannot conclude that $T \in M(E)$ as in Prop. 8.3.

Example 8.2. Along the same lines of Prop. 8.3 and Example 8.1 we see that if E is Helson and $G = G_E \subseteq A(\mathring{\Gamma})$ exists such that

$$\forall F \in \mathscr{F} \quad \exists \phi \in G \quad \text{for which } \phi = 1 \text{ on a neighborhood of } F$$
$$\text{and } \phi = 0 \text{ on a neighborhood of } E - F$$

and

$$\exists M>0 \quad \text{for which } \|\phi'\|_\infty \leqq M\,\|\phi\|_A$$

then E is an S-set. Verifying such conditions is of course related to Remark 1, § 6.

We say uE in $\underline{\mathscr{A}-S}$ if for all $t\sim T\in A'(E)$ and for all $\Psi\in\mathscr{A}$ vanishing on in E, $\langle t, \Psi\rangle=0$. The above results show that if E is Helson then E is S if and only if uE if $\mathscr{A}-S$. It would be interesting to weaken the Helson set hypothesis here. Neither Malliavin's non-S criteria or Körner's example provide any help. On the other hand we can easily check that –

Proposition 8.4. *Assume that for each* $t\sim T\in A'(E)$, $\{\phi_n\}\subseteq G$ *with* $\sup\|\phi_n\|_A<\infty$, *and* $\{s_n\}\subseteq R$ *increasing to infinity, there is* $\mu\in M(uE)$ *such that for all* n, $|\hat\mu(\phi_n)-\hat t(\phi_n)|<s_n$. *Then* uE *is* $\mathscr{A}-S$.

The hypothesis of Prop. 8.4 characterizes S sets when we deal only with pseudo-measures (as a simple Hahn-Banach argument shows).

Using the technique of Prop. 8.1 we find

Proposition 8.5. *Given real* $\phi\in C^\infty(\mathring{\Gamma})$ *and* E *Helson. There is real* $\psi_n\in C^\infty(\mathring{\Gamma})$ *such that*

$$\|\psi_n\|_A\leqq 31\,K_E+1$$

and for each $\gamma\in S_n=\{\lambda_1,\gamma_1,\dots,\lambda_n,\gamma_n\}$,

$$e^{i\psi_n}(\gamma)=e^{i\phi(\gamma)}.$$

This leads us to define the following set of numbers $\{\phi(n,F)\colon F\in\mathscr{F}, n=1,2,\dots\}$ which we call the *Helson indicator* for a given Helson set E. From Prop. 8.5 and for a given n and $\phi=\phi_F$ choose $q_n\in k(S_n)$ and $g_n\in j(S_n)$ such that $\|\phi+q_n\|_A\leqq 31\,K_E+1$ and $\|q_n-g_n\|_A<1$; let $\varepsilon=\varepsilon(n,F)$ have the property that

$$\|g_n*p_\varepsilon-g_n\|_A<1$$

where p_ε is the Friedrich mollifier of support $[-\varepsilon,\varepsilon]$.

For convenience in Prop. 8.6 we consider the bounded pseudo-measures $A_b'(E)$ (e.g. [3]); this amounts to considering those $T\in A'(E)$ with $\{k_j\}$ (of Example 8.1) bounded. The result is easily reformulated for $A'(E)$.

Proposition 8.6. *Given* E *Helson and* $T\in A_b'(E)$. *If there is* K *such that for all* $F\in\mathscr{F}$ *we can choose* n *for which*

$$\sum_{j=n+1}\varepsilon_j<K\,\varepsilon(n,F),$$

then $T\in M(E)$.

Proof. Set $\phi=\phi_F$, $F\in\mathscr{F}$, and $f_n=e^{i\phi}-e^{i\psi_n}$ from Prop. 8.5. From the structure of ϕ there is N_F such that $\phi'=0$ on (λ_j,γ_j) if $j\geqq N_F$.

Taking $n > N_F$ we have

$$|\langle T, f_n \rangle| = \left| \sum_{n+1}^{\infty} k_j \int_{\lambda_j}^{\gamma_j} \psi_n'(\gamma)\, e^{i\psi_n(\gamma)}\, d\gamma \right|;$$

and since $\psi_n' = \phi' + g_n * p_\varepsilon'$ we compute

$$|\langle T, f_n \rangle| \leqq C' \sum_{n+1}^{\infty} k_j \int_0^{2\pi} |p_\varepsilon'| = \frac{C}{\varepsilon(n, F)} \sum_{n+1} \varepsilon_j.$$

Given the hypothesis and the norm bound on $\|\psi_n\|_A$ we have $|\langle T, e^{i\phi_F} \rangle|$ uniformly bounded. Hence $T \in M(E)$. *qed.*

Because of Prop. 8.6 we consider E with the property: there is $\alpha_E > 0$ such that for each N we can find $n > N$ for which

$$\min d(t_j, t_k) \geqq \alpha_E \sum_{n+1} \varepsilon_j, \tag{P}$$

where $j, k \leqq n$ and t_j is λ_j or γ_j. The Cantor set is not (P), which is encouraging, whereas some calculations show that there is no reason to expect a pseudo-measure on an Helson-(P) set to be a measure.

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