Invited Paper

Pyramidal Riesz products associated with subband coding and selfsimilarity

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ABSTRACT

A new Riesz product is formulated in terms of waveletpacket pyramidal tree structures. A path within such a structure is determined by a lacunarity criterion and a sequence $\{H_{\epsilon_j}\}$ of filters, where each H_{ϵ_j} is one or the other element of a prescribed subband coding pair $\{H_0, H_1\}$ of FIR filters. The Riesz product associated with a given path is a continuous Radon measure. Path based criteria are given to determine singular and absolutely continuous pairs of such measures. The measures have full support, and their approximants exhibit fractal behavior. These properties can be used to design a secure transmission scheme in communications theory.

Keywords: Riesz product, waveletpackets, selfsimilarity

1 Introduction

In 1918 F. Riesz introduced the infinite product $\prod_{j=1}^{\infty}(1+\cos 4^j t)$ and proved that it is a continuous measure whose Fourier coefficients do not tend to zero at $\pm\infty$.[19] This product is an example of a so-called *Classical Riesz Product*, and such products provide examples of singular measures and have been a useful device in analysis since their inception.[13–16]

Quadrature mirror filters (QMFs) originated as a speech processing tool in the 1970s and are intimately related to multiresolution analyses (MRAs) in wavelet theory.[9] In fact, wavelet orthonormal bases which arise from MRAs are constructed in terms of QMFs. QMFs are filters H_0, H_1 defined on $\mathbf{T} = \mathbf{R}/\mathbf{Z}$ for which

$$|H_0(\gamma)|^2 + |H_1(\gamma)|^2 = 2 \ \ a.e.,$$

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where $H_0(\gamma) = \sum_m h_0[m]e^{-2\pi i m\gamma}$, $H_1(\gamma) = \sum_m h_1[m]e^{-2\pi i m\gamma}$, and $h_1[m] = (-1)^m\overline{h_0}[1-m]$. The concept of ***waveletpackets*** is due to Coifman, Meyer, and Wickerhauser. In the Fourier domain, waveletpackets W_n are defined in terms of QMFs H_0, H_1 as

$$\widehat{W_n}(\gamma) = \prod_{j=1}^{\infty} H_{\epsilon_j}(\frac{\gamma}{2^j}),$$

where ϵ_j is 0 or 1, $n = \sum_{j=1}^{\infty}\epsilon_j 2^{j-1}$, and " $\hat{}$ " designates is the Fourier transform.[7] The choice of either H_0 or H_1 at each step in this product can be viewed as a path down a binary tree.

In this paper the notion of a Pyramidal Riesz Product is formulated in terms of waveletpacket pyramidal tree structures. The underlying filters are not QMFs, but must satisfy related properties. New singular measures are constructed, and selfsimilarity arises in an unexpected way.

Specifically, we define Pyramidal Riesz Products P_k in *Section 2*, as well as proving some of their elementary properties. In *Section 3*, we prove that $\lim_{k\to\infty} P_k$ is often a continuous measure, and then exhibit new singular measures. The main result of *Section 4* allows us to construct mutually singular Pyramidal Reisz Product measures along fixed paths of our pyramidal tree. The theory for the multipath case, both singular and absolutely continuous, is due to Bernstein.[3] Finally in *Section 5*, we illustrate our selfsimilarity results, which are based on our observations relating primitives of full support Pyramidal Riesz Product measures and primitives of measure zero support Cantor measures. Details for this material and applications to secure transmission are developed elsewhere.[2] Our notation is standard, and can be found in basic texts.[14]

2 Pyramidal Riesz Products

Let H be an FIR (finite impulse response) filter, written as the trigonometric polynomial,

$$H(\gamma) = \sum_{m\in E_H} h[m]e^{-2\pi i m\gamma},$$

where

$$E_H = \{m_j \in \mathbf{N} : 1 \le m_1 < m_2 < \cdots < m_L\} \tag{1}$$

and $\{h[m]\} \subseteq \mathbf{R}$. E_H is the *spectrum* of H. A *filter path* H_ϵ is a sequence

$$H_\epsilon = \{H_{\epsilon_j} : j \in \mathbf{N} \; and \; \epsilon_j \in \{0,1\}\}$$

of FIR filters H_0, H_1 having a common spectrum E_H.

DEFINITION 2.1. *a. A sequence $\{n_j : j \in \mathbf{N}\} \subseteq \mathbf{N}$ is λ-lacunary if*

$$\forall j \in \mathbf{N}, \quad \frac{n_{j+1}}{n_j} \ge 2\lambda + 1.$$

b. Let H_ϵ be a filter path, each of whose elements has spectrum E. If $\{n_j\}$ is m_L-lacunary, and if $\{a_j\} \subseteq \mathbf{R}$, then the corresponding Pyramidal Riesz Products P_k, $k \in \mathbf{N}$, are defined as

$$P_k(\gamma) = \prod_{j=1}^{k}(1 + a_j Re H_{\epsilon_j}(n_j\gamma)) = \prod_{j=1}^{k}(1 + a_j Re \sum_{m\in E} h_{\epsilon_j}[m]e^{-2\pi i m n_j\gamma}),$$

PROPOSITION 2.2. *Let H_ϵ be a filter path, let $\{n_j\}$ be m_L - lacunary, and let $\{a_j\} \subseteq \mathbf{R}$.*

a. For each $j \in \mathbf{N}$,

$$n_{j+1} - m_L(n_1 + \cdots + n_j) > m_L(n_1 + \cdots + n_j).$$

b. P_k *has the cosine series expansion,*

$$P_k(\gamma) = 1 + \sum_{j=1}^{d_k} c[j]\cos(2\pi j\gamma), \tag{2}$$

where

$$d_k = m_L(n_1 + \cdots + n_k).$$

Furthermore, noting that $P_{k+1} = P_k + (P_{k+1} - P_k)$, *we have in the case* $a_{k+1} \neq 0$ *that all the terms of the cosine series expansion of* $P_{k+1} - P_k$ *are of degree greater than* d_k.

c. If $j \neq m_{i_1}n_{j_1} \pm m_{i_2}n_{j_2} \pm \cdots$, *then* $c[j] = 0$. *If*

$$j = m_{i_1}n_{j_1} \pm m_{i_2}n_{j_2} \pm \cdots = m_{i_1}n'_{j_1} \pm m_{i_2}n'_{j_2} \pm \cdots, \tag{3}$$

then the sums in (3) are identical, i.e., no two terms of the expansion (2) of the product P_k *are of the same degree.*

Proof. a. We proceed by induction. Since $\frac{n_2}{n_1} \geq 2m_L + 1$ we have $n_2 \geq (2m_L+1)n_1 > 2m_L n_1$. Assuming $n_j > 2m_L(n_1 + \ldots + n_{j-1})$, we obtain

$$n_{j+1} \geq (2m_L+1)n_j > 2m_L n_j + 2m_L(n_1 + \ldots + n_{j-1}) = 2m_L(n_1 + \ldots + n_j),$$

and so part *a* is valid.

b. We expand the product P_k to obtain (2) by the cosine formula $\cos x \cos y = \frac{1}{2}(\cos(x+y) + \cos(x-y))$. In the formal expansion of P_k as a sum of cosines, the degree is bounded above by the degree of the cosine term with highest degree in the expansion of the product

$$\prod_{j=1}^{k} a_j h_{\epsilon_j}[m_L] \cos 2\pi m_L n_j \gamma. \tag{4}$$

Without loss of generality we shall assume that $a_j \neq 0$. Using the cosine formula, we see that the highest term of P_k is

$$\frac{1}{2^{k-1}} \prod_{j=1}^{k} a_j h_{\epsilon_j}[m_L] \cos 2\pi m_L (n_1 + \ldots + n_k)\gamma. \tag{5}$$

Thus the degree of the highest term is d_k. By definition,

$$\begin{aligned} P_{k+1}(\gamma) - P_k(\gamma) &= P_k(\gamma) a_{k+1} Re H_{\epsilon_{k+1}}(n_{k+1}\gamma) \\ &= P_k(\gamma) a_{k+1} (h_{\epsilon_{k+1}}[m_1] \cos 2\pi m_1 n_{k+1}\gamma + \ldots \\ &\quad + h_{\epsilon_{k+1}}[m_L] \cos 2\pi m_L n_{k+1}\gamma). \end{aligned} \tag{6}$$

Using the degree property proved before (5) and the cosine formula, we se that the degree of the lowest (first) term of $P_{k+1} - P_k$ is $m_1 n_{k+1} - d_k$. Thus part *b* is verified once we show that $m_1 n_{k+1} - d_k > d_k$. However, this inequality is clear from the m_L*-lacunarity* and part*a*.

c. In the expansion of P_k in the form (2), we see from the cosine formula that if $c[j]$ is not 0 then j must be of the form $j = m_{i_1}n_{j_1} \pm m_{i_2}n_{j_2} \pm \ldots$, a finite sum. This is the first claim of part*c*.

For the second part we proceed by induction and assume the result for P_k. The highest (last) term in P_{k+1} which involves $m_j n_{k+1}$ is of degree (bounded above by) $m_j n_{k+1} + d_k$ This is a consequence of expanding the right side of (6). The lowest term (first) term in P_{k+1} which involves $m_{j+1}n_{k+1}$ is of degree (bounded below by) $m_j n_{k+1} - d_k$. In fact, by the cosine formula, d_k is the largest amount we subtract. Thus the uniqueness is obtained once we show that

$$m_{j+1}n_{k+1} - d_k > m_j n_{k+1} + d_k.$$

Recalling that $d_k = m_L(n_1 + \ldots + n_k)$, this inequality is a consequence of part a and the fact the $(m_{j+1} - m_j)n_k + 1 \geq n_{k+1}$. □

REMARK 2.3. *a.* F. Riesz[19] introduced Riesz products in 1918 for the filter path H_ϵ, where each $H_{\epsilon_j} = e^{-2\pi i\gamma}$, for the $\frac{3}{2}$-*lacunary* sequence $\{n_j : j \in \mathbf{N}\}$ defined by $n_j = 4^{j-1}$, and for the sequence $\{a_j\}$, where each $a_j = 1$. The *Classical Riesz products*, which generalize Riesz's example, are defined as

$$P_k(\gamma) = \prod_{j=1}^{k}(1 + a_j \cos 2\pi n_j \gamma), \tag{7}$$

where $-1 \leq a_j \leq 1$ and $\{n_j\}$ is *1- lacunary*, i.e., $\frac{n_{j+1}}{n_j} \geq 3$.

b. Besides our present development, there is a recent interesting generalization of Riesz products due to Brown and Dooley.[5] Their generalization includes products of the form

$$\prod_{j=1}^{k}(1 + a_j \cos 2\pi n_1 n_2 \cdots n_j \gamma),$$

where $-1 \leq a_j \leq 1$ and $n_j \in \mathbf{Z}$.[4]

c. The *spectrum* of P_k is the subset S of $\{1, \cdots, d_k\}$ for which $c[j] \neq 0$ for all $j \in S$.

EXAMPLE 2.4. Let $\lambda \in \mathbf{N}$ and let $n_1 = 1$. We say that $\{n_j\}$ is a *minimal* λ - *lacunary* sequence if

$$\forall j > 1,\ n_{j+1} = (2\lambda + 1)n_j.$$

For any such λ, $n_j = (2\lambda + 1)^{j-1}$. Hence, if $\lambda = 1$ then $n_j = 3^{j-1}$, if $\lambda = 2$ then $n_j = 5^{j-1}$, etc.

Suppose that $\{a_j\} \subseteq \mathbf{R} \setminus \{0\}, \lambda \in \mathbf{N}$, and H_ϵ is a filter path where each filter H_{ϵ_j} has the spectrum $E = \{1, 2, \cdots, \lambda\}$. Then the union of the spectra of the P_k is $\mathbf{N} \cup \{0\}$.

Further, for the Classical Riesz products, the example $n_j = 3^{j-1}$ is the only case where it is possible for the union of the spectra of the P_k to be $\mathbf{N} \cup \{0\}$.

For the Classical Riesz products defined in (7) it is well known that

$$\lim_{k \to \infty} P_k = \mu \geq 0,\ \sigma(\mathbf{M}(\mathbf{T}), \mathbf{C}(\mathbf{T})),$$

where μ is a continuous measure. Also if $\sum a_j^2 = \infty$ then μ is a singular measure. Early contributions to this subject include those made by Riesz[19] and Zygmund.[20] There have been many other contributions.[5,4,6,11,13,15,16,18] The ideas of Peyrière are explicitly used in *Section* 4.

We shall address the problem of computing $\lim P_k = \mu$ for the Pyramidal Riesz Products defined above. In particular, we shall think of a filter path H_ϵ as a " path through a tree." Thus at the *kth* step we view the Pyramidal Riesz Product P_k as information at a node on the $k\,th$ level of the tree. The information in P_{k+1} will depend on which filter H_0 or H_1 is used as $H_{\epsilon_{k+1}}$. We shall investigate properties of μ, as well as ultimately comparing μ and $\tilde{\mu}$ for different filter paths H_ϵ and $H_{\tilde{\epsilon}}$.

3 Continuous Pyramidal Riesz Product Measures

If H_ϵ is a filter path, we shall suppose that the impulse responses h_0 and h_1 of H_0 and H_1 satisfy the conditions

$$\sum |h_0[m]| = \sum |h_1[m]| = C_1$$

$$\sum |h_0[m]|^2 \le C_2, \quad \sum |h_1[m]|^2 \le C_2.$$

THEOREM 3.1. *Let H_ϵ be a filter path, let $\{n_j\}$ be m_L-lacunary, and let $\{P_l\}$ be the corresponding sequence of Pyramidal Riesz Products. If $|a_j| \le 1/C_1$ then there exists a subsequence $\{P_{l_k}\}$ that converges to a continuous measure μ_a in $\sigma(\mathbf{M(T)}, \mathbf{C(T)})$.*

Proof. The existence of μ_a follows from the fact that

$$\forall l, \quad \|P_l\|_1 = \int_{\mathbf{T}} |P_l(\gamma)|\, d\gamma = \int_{\mathbf{T}} P_l(\gamma)\, d\gamma = 1,$$

and from Alaoglu's theorem.

The continuity of μ_a is established using Wiener's condition[1]:

$$\sum_{x \in \mathbf{T}} |\mu_a\{x\}|^2 = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{-N}^{N} |\hat{\mu}_a[n]|^2. \tag{8}$$

If the limit on the right hand side equals zero then μ_a is continuous. Thus, we need to estimate $\sum_n |\hat{\mu}_a[n]|^2$. We let $d_N =$ *degree of* $P_N(\gamma)$. A counting argument shows

$$d_N \ge \frac{(2m_L+1)^N - 1}{2},$$

and that

$$\begin{aligned} \sum_{-d_N}^{d_N} |\hat{\mu}_a[n]|^2 &\le \sum_{n=1}^{N} \binom{N}{n} \left(\frac{C_2}{2C_1^2}\right)^n \\ &= \left(\frac{C_2 + 2C_1^2}{2C_1^2}\right)^N. \end{aligned}$$

Thus,

$$\frac{1}{2d_N+1} \sum_{-d_N}^{d_N} |\hat{\mu}_a[n]|^2 \le \frac{1}{(2m_L+1)^N} \left(\frac{C_2 + 2C_1^2}{2C_1^2}\right)^N. \tag{9}$$

Since $C_2 \le C_1^2$ and $m_L \ge 1$, $C_2 + 2C_1^2 < (2m_L+1)2C_1^2$; hence, the right side of (9) goes to 0 as $N \longrightarrow \infty$, and this establishes the continuity of μ_a. □

EXAMPLE 3.2. We just proved that $\lim P_k = \mu$ is a continuous measure for Pyramidal Riesz Products as long as $\|a_j\|_\infty$ is sufficiently small. If the condition of λ- *lacunarity* is weakened then we can lose the continuity of μ. For perspective recall that the minimal *1-lacunary* sequence is $n_j = 3^{j-1}, j = 1, 2, \ldots$. In the $\frac{1}{2}$ *− lacunary* case, which does not allow a spectrum of the form E_H in (1), we have[17]

$$P_k(\gamma) = \prod_{j=1}^{k} (1 + \cos 2\pi(2^{j-1}\gamma))$$

and

$$\lim_{k\to\infty} P_k = \delta,\ \sigma(\mathbf{M}(\mathbf{T}), \mathbf{C}(\mathbf{T})). \tag{10}$$

To verify (10) we first note that

$$P_k(\gamma) = \frac{\sin^2(2\pi 2^{k-1}\gamma)}{2^k \sin^2 \pi\gamma} \tag{11}$$

which follows from an induction argument. Since the right side of (11) is the Fejér kernel, which is an approximate identity for $L^1(\mathbf{T})$, we obtain (10) by a standard calculation.[1]

EXAMPLE 3.3. Let $H(\gamma) = e^{-2\pi i\gamma} + e^{-2\pi i 2\gamma}$ so that $E_H = \{1, 2\}$, and let H_ϵ be the filter path each of whose elements is H. Consider the (minimal) *2-lacunary* sequence $\{5^{j-1}\}$ and the associated Pyramidal Riesz Products P_k for the constant sequence $a_j = \frac{1}{2}$ for each j. It is easy to see that the spectrum of each P_k is $\{1, \cdots, d_k\}$. This observation, combined with *Theorem* 3.1 and the remark in *Example* 2.4 about the Classical Riesz products for $n_j = 3^{j-1}$, gives a measure $\mu = \lim_{k\to\infty} P_k$ which is *not* a Classical Riesz product.

4 Singular Pyramidal Riesz Product Measures

The proof of *Theorem 4.2* that follows is based on a proof by Peyrière.[18] Our proof requires the following orthogonality lemma.

LEMMA 4.1. *Let H_ϵ be a filter path for which $\sum_m |h_0[m]| = \sum_m |h_1[m]| = C_1$ and*

$$\sum |h_0[m]|^2 = \sum |h_1[m]|^2 = C_2.$$

Then $\{H_{\epsilon_j}(n_j\gamma) - C_2\frac{a_j}{2}\}$ and $\{H_{\epsilon_j}(n_j\gamma) - C_2\frac{b_j}{2}\}$ form orthogonal sets of functions in $L^2_{\mu_a}(\mathbf{T})$ and $L^2_{\mu_b}(\mathbf{T})$, respectively.

THEOREM 4.2. *Let H_ϵ be a filter path, satisfying the additional conditions of the orthogonality lemma, let $\{n_j\}$ be $m_L -$ lacunary, and let $\{a_j\}$ and $\{b_j\}$ be two sequences such that $\|a\|_\infty \le \frac{1}{C_1}$ and $\|b\|_\infty \le \frac{1}{C_1}$. If μ_a and μ_b are the measures associated with the Pyramidal Riesz Products for the path H_ϵ and the sequences $\{a_j\}$ and $\{b_j\}$, respectively, and if $\sum_j |b_j - a_j|^2 = \infty$, then $\mu_a \perp \mu_b$.*

Proof. Since $\sum_j |b_j - a_j|^2 = \infty$, the uniform boundedness principle can be used to show there exists a sequence $\{\alpha_j\} \in \ell^2$ such that $\sum \alpha_j(b_j - a_j) = \infty$ and $\alpha_j(b_j - a_j) \ge 0$. Now, letting

$$S_N(\gamma) = \sum_{j=1}^{N} \alpha_j (H_{\epsilon_j}(n_j\gamma) - \frac{C_2 a_j}{2})$$

and

$$T_N(\gamma) = \sum_{j=1}^{N} \alpha_j (H_{\epsilon_j}(n_j\gamma) - \frac{C_2 b_j}{2}),$$

it can be shown, using the orthogonality lemma, and the existence of $\{\alpha_j\} \in \ell^2$, that subsequences $\{S_{m_k}\}$ and $\{T_{m_k}\}$ converge μ_a a.e. and μ_b a.e., respectively. If we assume $\mu_a \not\perp \mu_b$, then there exists γ such that

$$\exists \lim_{k\to\infty} S_{m_k}(\gamma) = \lim_{k\to\infty} \sum_{1\le j\le m_k} \alpha_j (H_{\epsilon_j}(n_j\gamma) - \frac{C_2 a_j}{2})$$

and

$$\exists \lim_{k\to\infty} T_{m_k}(\gamma) = \lim_{k\to\infty} \sum_{1\le j\le m_k} \alpha_j (H_{\epsilon_j}(n_j\gamma) - \frac{C_2 b_j}{2}).$$

Subracting these terms, we obtain

$$0 \leq \sum_{j \geq 1} \alpha_j (b_j - a_j) < \infty,$$

a contradiction. □

REMARK 4.3. Let H_ϵ and $H_{\tilde{\epsilon}}$ be filter paths where all but finitely many "coordinates" are different, and let $\sum |a_j|^2 = \infty$. With the additional restriction,

$$\sum h_0[m] h_1[m]| = 0,$$

it can be shown that $\mu \perp \tilde{\mu}$.[3]

5 Selfsimilarity and Riesz products

This section motivates the relationship between Pyramidal Riesz Products and the modern analysis of selfsimilarity in certain signals. Such signals arise in topics such as turbulence, DNA studies, and dynamical systems. Jaffard's theory of selfsimilar functions[10] is the mathematical model compatible with the Riesz product data we have found. There are analytic results establishing the relationship between Jaffard's selfsimilarity and Classical and Pyramidal Riesz Products.[2]

In order to formulate our idea, let us first recall that if μ_C is the Cantor measure supported by the $\frac{1}{3}$-Cantor set, then the Lebesgue measure of supp μ_C is 0. On the other hand, if μ is the Classical Riesz Product corresponding to $n_j = 3^{j-1}$ then the Lebesgue measure of supp μ is 1. (We have a new proof of this result, which is well known, as well as a proof of the corresoponding result for Pyramidal Riesz Products.[2]) We would *not* be surprised if *Figure* 1 represented a normalized primitive of μ_C. What *is* interesting, however, is that *Figure* 1 represents the function

$$F_4(\gamma) = \frac{1}{2\pi} \int_0^\gamma P_4\left(\frac{\lambda}{2\pi}\right) d\lambda \quad \text{on} \quad [0, 2\pi]$$

where $\lim P_k = \mu$ in $\sigma(\mathbf{M(T)}, \mathbf{C(T)})$.

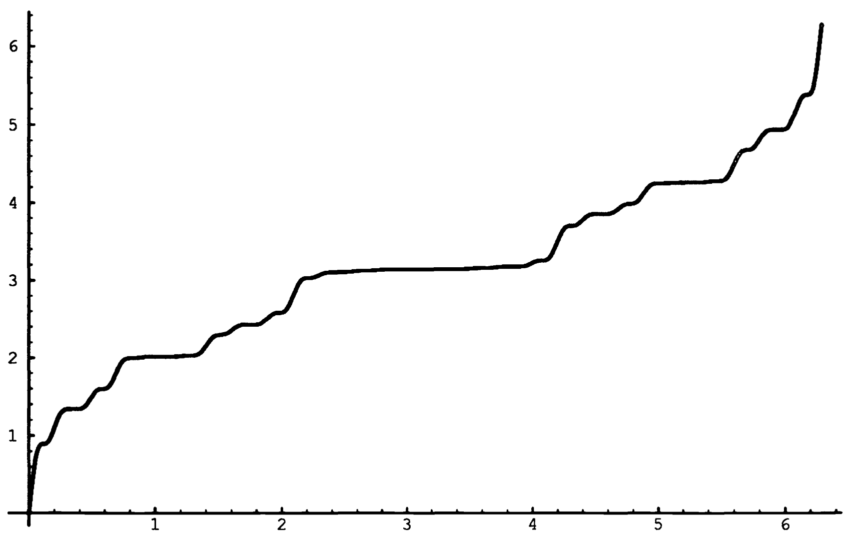

Figure 1: $F_4(\gamma)$

This means that the flat regions of the graph of F_4 (as well as of the other F_k) not only give rise to a measure with ***full support***, but, in fact, are locally not flat and are inherently selfsimilar, e.g., ***Figures*** 2 and 3. We should point out that our observations on this matter have recently led to a related connection between μ_C and μ.[12]

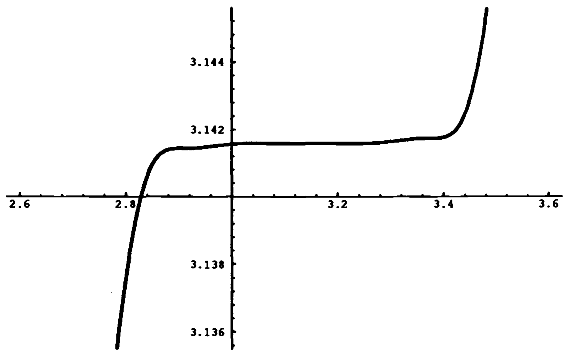

Figure 2: behavior of $F_4(\gamma)$ around π

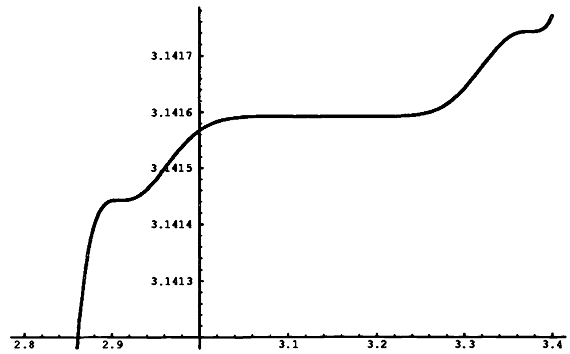

Figure 3: behavior of $F_4(\gamma)$ around π

Figure 4 represents the function

$$F_5(\gamma) = \frac{1}{2\pi}\int_0^{\gamma} P_5(\frac{\lambda}{2\pi})\,d\lambda \quad \text{on} \quad [0, 2\pi].$$

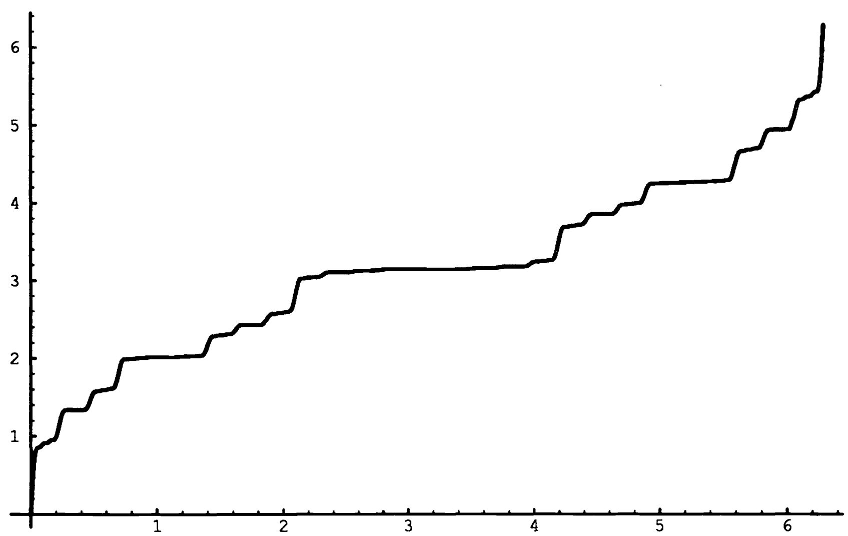

Figure 4: $F_5(\gamma)$

In *Figures* 5-7 we examine the Pyramidal Riesz Product ν defined in *Example* 3.3 by

$$a_j = \frac{1}{2},\ H(\gamma) = e^{-2\pi i\gamma} + e^{-2\pi i(2\gamma)}, \text{ and } \ n_j = 5^{j-1}.$$

If the Pyramidal Riesz Products Q_k are approximants of ν, then *Figure* 5 represents the function

$$G_2(\gamma) = \frac{1}{2\pi}\int_0^{\gamma} Q_2(\frac{\lambda}{2\pi})\,d\lambda \quad \text{on} \quad [0, 2\pi]$$

and *Figure* 6 represents the function

$$G_3(\gamma) = \frac{1}{2\pi}\int_0^{\gamma} Q_3(\frac{\lambda}{2\pi})\,d\lambda \quad \text{on} \quad [0, 2\pi]. \tag{12}$$

These graphs are compatible with the fact that ν has full support, as opposed to *Figures* 1 and 4. The selfsimilar nature of these graphs is apparent from *Figure* 7, which is a local version of (12).

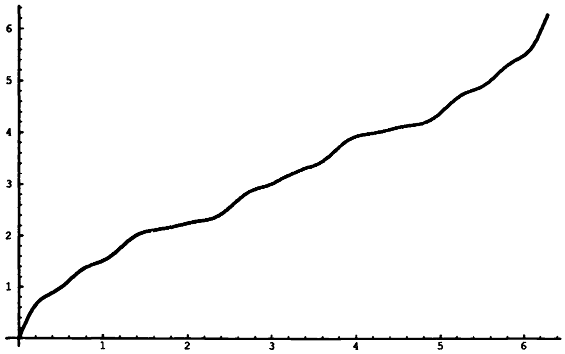

Figure 5: $G_2(\gamma)$

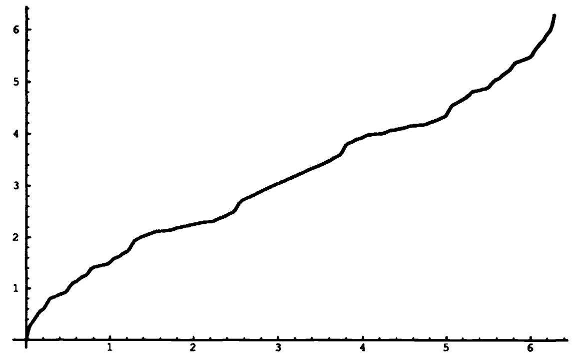

Figure 6: $G_3(\gamma)$

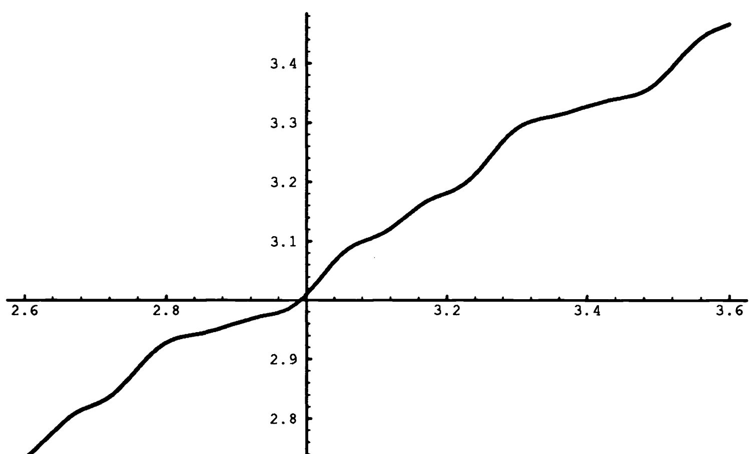

Figure 7: behavior of $G_3(\gamma)$ around π

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