A (p, q) version of Bourgain's theorem

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Abstract. Let $1 < p, q < \infty$ satisfy $\frac{1}{p} + \frac{1}{q} = 1$. We construct an orthonormal basis $\{b_n\}$ for $L^2(\mathbb{R})$ such that $\Delta_p(b_n)$ and $\Delta_q(\widehat{b_n})$ are both uniformly bounded in n. Here $\Delta_\lambda(f) \equiv \inf_{a \in \mathbb{R}} \left(\int |x-a|^\lambda |f(x)|^2 dx\right)^{\frac{1}{2}}$. This generalizes a theorem of Bourgain and is closely related to recent results on the Balian-Low theorem.

1. Introduction

Given a square integrable function $f \in L^2(\mathbb{R})$, we formally define the Fourier transform of f by

$$\widehat{f}(\gamma) = \int f(t) e^{-2\pi i t \gamma} dt,$$

where integration is over the real line $\mathbb{R}$. The uncertainty principle in harmonic analysis is the general statement that a function and its Fourier transform can not both be "too well localized". For example, Heisenberg's inequality states that if $f \in L^2(\mathbb{R})$ is of norm one then

$$\frac{1}{4\pi} \leq \Delta(f)\Delta(\widehat{f}). \tag{1.1}$$

Here $\Delta(\cdot)$ is defined by

$$\Delta(f) = \left(\int |t - \mu(f)|^2 |f(t)|^2 dt\right)^{\frac{1}{2}} \tag{1.2}$$

where

$$\mu(f) = \int t |f(t)|^2 dt. \tag{1.3}$$

For an overview of recent mathematical work on the uncertainty principle we refer the reader to [**FS**], [**B1**], [**HJ**].

This paper deals with how the uncertainty principle constrains the time and frequency localization of the elements in an orthonormal basis for $L^2(\mathbb{R})$. We generalize a theorem of Bourgain on the construction of orthonormal bases which are uniformly well-localized with respect to the (t^2, γ^2) weights implicit in (1.1). We

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consider the more general class of non-symmetric weights given by (t^p, γ^q), where $\frac{1}{p} + \frac{1}{q} = 1$.

2. Background

2.1. The (t^2, γ^2) weight. The Balian-Low theorem [**Bal**], [**Lo**], [**G2**] is the classical example of an uncertainty principle for orthonormal bases. If, for a given $f \in L^2(\mathbb{R})$, we define the *Gabor system*, $\mathcal{G}(f, a, b) = \{f_{m,n} : m, n \in \mathbb{Z}\}$, by

$$f_{m,n}(t) = e^{2\pi i t m b} f(t - na), \tag{2.1}$$

then the Balian-Low theorem states that when $\mathcal{G}(f, 1, 1)$ is an orthonormal basis for $L^2(\mathbb{R})$ we have

$$\int |t|^2 |f(t)|^2 dt = \infty \quad \text{or} \quad \int |\gamma|^2 |\widehat{f}(\gamma)|^2 d\gamma = \infty. \tag{2.2}$$

In particular, either $\Delta(f_{m,n}) = \Delta(f) = \infty$ for all $m, n \in \mathbb{Z}$ or $\Delta(\widehat{f_{m,n}}) = \Delta(\widehat{f}) = \infty$ for all $m, n \in \mathbb{Z}$. Thus, if a Gabor system forms an orthonormal basis for $L^2(\mathbb{R})$, then its elements either have uniformly poor localization in time or uniformly poor localization in frequency. The Balian-Low theorem is true in much greater generality than above, e.g., [**DJ**], [**GHHK**], [**BCM**], [**GH**].

A recent result due to the authors together with W. Czaja and P. Gadziński shows that the (t^2, γ^2) weights used in (2.2) can not be significantly weakened. In [**BCGP**] it was shown that there exists $f \in L^2(\mathbb{R})$ such that $\mathcal{G}(f, 1, 1)$ is an orthonormal basis for $L^2(\mathbb{R})$ and such that, for each $d > 2$,

$$\int \frac{1 + |t|^2}{\log^d(|t| + 2)} |f(t)|^2 dt < \infty \tag{2.3}$$

and

$$\int \frac{1 + |\gamma|^2}{\log^d(|\gamma| + 2)} |\widehat{f}(\gamma)|^2 d\gamma < \infty. \tag{2.4}$$

In view of this result and the Balian-Low theorem, it is natural to ask what happens for general orthonormal bases, i.e., those which are not necessarily Gabor systems. Namely, can a general orthonormal basis have "uniform" localization with respect to the (t^2, γ^2) weights? This question was first posed by Balian [**Bal**] and answered by Bourgain [**Bou**] in 1986.

Bourgain showed that given any $\epsilon > 0$, there exists an orthonormal basis $\{b_n : n \in \mathbb{N}\}$ for $L^2(\mathbb{R})$ such that

$$\forall n \in \mathbb{N}, \quad \Delta(b_n) \leq \frac{1}{2\sqrt{\pi}} + \epsilon \quad \text{and} \quad \Delta(\widehat{b_n}) \leq \frac{1}{2\sqrt{\pi}} + \epsilon. \tag{2.5}$$

This basis is uniformly localized with respect to (t^2, γ^2) in the sense that the $\Delta(b_n)$ and $\Delta(\widehat{b_n})$ are uniformly bounded. To put this in perspective, note that there are $\psi \in \mathcal{S}(\mathbb{R})$, the Schwartz class, which generate wavelet orthonormal bases, $\{\psi_{m,n} : m, n \in \mathbb{Z}\}$, for $L^2(\mathbb{R})$, [**LM**]. Since $\psi \in \mathcal{S}(\mathbb{R})$, each $\Delta(\psi_{m,n})$ and $\Delta(\widehat{\psi_{m,n}})$ is finite. However, these variances are not uniformly bounded for any wavelet system, although their product may be, e.g., see [**B1**].

The constant $\frac{1}{2\sqrt{\pi}}$ in (2.5) is significant since $g(t) = 2^{1/4} e^{-\pi t^2}$ implies $\Delta(g) = \Delta(\widehat{g}) = \frac{1}{2\sqrt{\pi}}$. Moreover, it is well known that this choice of g gives equality in (1.1), i.e., the Gaussian is a minimizer for Heisenberg's inequality. Thus, each of the elements in Bourgain's basis is almost optimally localized with respect to Heisenberg's inequality.

2.2. The (t^p, γ^q) weights. The three results in the previous subsection give insight into the limits of the uncertainty principle for the t^2 and γ^2 weights. Our investigation in this paper deals with the more general weights t^p and γ^q, where $\frac{1}{p}+\frac{1}{q}=1$ and $1<p,q<\infty$. In this setting, one has the following analogue to the Balian-Low theorem which follows from work of Feichtinger and Gröchenig. Suppose $\epsilon>0$, $f\in L^2(\mathbb{R})$, and $\mathcal{G}(f,1,1)$ is an orthonormal basis for $L^2(\mathbb{R})$. Then

$$\int |t|^{p+\epsilon}|g(t)|^2dt=\infty \quad \text{or} \quad \int |\gamma|^{q+\epsilon}|\widehat{g}(\gamma)|^2d\gamma=\infty. \tag{2.6}$$

This result is proven by combining Theorem 4.4 of [**FG1**] and Theorem 1 of [**G1**].

As in the case $(p,q)=(2,2)$, we proved that Gabor bases *are* possible if the t^p and γ^q weights are weakened slightly. In particular, it was shown in [**BCGP**] that there exists $f\in L^2(\mathbb{R})$ such that $\mathcal{G}(f,1,1)$ is an orthonormal basis for $L^2(\mathbb{R})$ and such that, for every $d>2$,

$$\int \frac{1+|t|^p}{\log^d(|t|+2)}|f(t)|^2dt<\infty \tag{2.7}$$

and

$$\int \frac{1+|\gamma|^q}{\log^d(|\gamma|+2)}|\widehat{f}(\gamma)|^2d\gamma<\infty. \tag{2.8}$$

2.3. Statement of the main result. In view of the results in the previous subsection, we now consider the question of whether or not there is an analogue of Bourgain's theorem for the weights (t^p,γ^q). The following definition provides an appropriate generalization of $\Delta(\cdot)$.

DEFINITION 2.1. *Given $f\in L^2(\mathbb{R})$ and $\lambda>0$. We define*

$$\Delta_\lambda(f)=\inf_{a\in\mathbb{R}}\left(\int |t-a|^\lambda|f(t)|^2dt\right)^{\frac{1}{2}}.$$

It is easy to verify that when $\lambda=2$, and $||f||_{L^2(\mathbb{R})}=1$, this definition agrees with the one given by (1.2) and (1.3). Let $C_c^\infty(\mathbb{R})$ be the space of compactly supported, infinitely differentiable functions on $\mathbb{R}$. We now state our main result.

THEOREM 2.2. *Assume $1<p,q<\infty$, where $\frac{1}{p}+\frac{1}{q}=1$. Fix $\epsilon>0$ and $\varphi\in C_c^\infty(\mathbb{R})$ with $||\varphi||_{L^2(\mathbb{R})}=1$. There exists an orthonormal basis, $\{b_n : n\in\mathbb{N}\}\subseteq C_c^\infty(\mathbb{R})$, for $L^2(\mathbb{R})$ such that*

$$\forall n\in\mathbb{N},\quad \Delta_p(b_n)\le\left(\int |t|^p|\varphi(t)|^2dt\right)^{\frac{1}{2}}+\epsilon\equiv C_{p,\varphi}+\epsilon \tag{2.9}$$

and

$$\forall n\in\mathbb{N},\quad \Delta_q(\widehat{b_n})\le\left(\int |\gamma|^q|\widehat{\varphi}(\gamma)|^2d\gamma\right)^{\frac{1}{2}}+\epsilon\equiv C_{q,\varphi}+\epsilon. \tag{2.10}$$

For perspective, let us mention that Cowling and Price [**CP**] proved analogues of Heisenberg's inequality for the (t^p,γ^q) weights. Their results are quite general, but as a special case one has that if $1 < p, q < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$, then there exists a constant $0 < K_{p,q}$ such that for all $f \in L^2(\mathbb{R})$ of norm one there holds

$$K_{p,q} \leq \ [\Delta_p(f)]^{\frac{2}{p}} \ [\Delta_q(\widehat{f}\ )]^{\frac{2}{q}}. \tag{2.11}$$

Our main result, Theorem 2.2, allows one to construct orthonormal bases whose elements are almost optimally localized with respect to the Cowling-Price uncertainty principle, (2.11).

3. Preliminary lemmas

In this section we shall state several lemmas which will be needed to prove Theorem 2.2.

3.1. Decay rates of inverses of matrices. Theorem 3.2 relates the off-diagonal decay of an invertible matrix to the off-diagonal decay of its inverse. The results are due to Jaffard, [**J**], and have been further studied and simplified by Strohmer in [**S**]. We also note that Bourgain implicitly made use of similar implications in [**Bou**]. For example, see the transition between equations (2.11) and (2.12) in [**Bou**].

The following definition appears in [**S**].

Definition 3.1. *Let $A = (A_{m,n})_{m,n\in\mathcal{I}}$ be a matrix, where the index set is $\mathcal{I} = \mathbb{Z}, \mathbb{N}$, or $\{0, \cdots, N-1\}$. Fix $s > 1$. We say that A belongs to $\mathcal{Q}_s$ if the coefficients $A_{m,n}$ satisfy*

$$\exists C > 0 \text{ such that } \forall m, n \in \mathcal{I}, \quad |A_{m,n}| < \frac{C}{(1 + |m-n|)^s}.$$

We say that A belongs to $\mathcal{E}_s$ if

$$\exists C > 0 \text{ such that } \forall m, n \in \mathcal{I}, \quad |A_{m,n}| < Ce^{-s|m-n|}.$$

Theorem 3.2 (Jaffard). *Let $A : l^2(\mathcal{I}) \to l^2(\mathcal{I})$ be an invertible matrix, where $\mathcal{I} = \mathbb{Z}, \mathbb{N}$, or $\{0, \cdots, N-1\}$. Then*

$$A \in \mathcal{Q}_s \implies A^{-1} \in \mathcal{Q}_s$$

and

$$A \in \mathcal{E}_s \implies A^{-1} \in \mathcal{E}_{s'},$$

for some $0 < s' \leq s$.

The case $\mathcal{I} = \{0, 1, 2, \cdots, N-1\}$ should be interpreted as follows. We quote from [**S**]: "View the $n \times n$ matrix A_n as a finite section of an infinite dimensional matrix A. If we increase the dimension of A_n (and thus consequently the dimension of $(A_n)^{-1}$) we can find uniform constants independent of n such that the corresponding decay properties hold."

Let us next comment on the constants which arise in Jaffard's theorem. We restrict ourselves to the case $\mathcal{I} = \{0, 1, \cdots, N-1\}$. Suppose that the A_N are sections of the infinite matrix A, and that

$$\exists C > 0 \text{ such that } \forall N \geq 1 \text{ and } \forall\ j, k \in \mathcal{I}, \ \ |A_N(j,k)| \leq \frac{C}{1 + |j-k|^s}.$$

Further, suppose for simplicity that there is a fixed $0 < r < 1$ such that

$$\forall N \geq 1, \ B_N \equiv I_N - A_N \text{ satisfies } ||B_N|| \leq r < 1,$$

where I_N is the $N \times N$ identity matrix. Jaffard's theorem then says that there exists C' such that

$$\forall N \geq 1 \text{ and } \forall\ j,k \in \mathcal{I}, \quad |A_N^{-1}(j,k)| \leq \frac{C'}{1+|j-k|^s}.$$

The constant C' depends only on r, s, and C. The proof of this assertion can be obtained by examining Jaffard's proofs, [**J**].

3.2. A simple lemma.

LEMMA 3.3. *Fix $\varphi \in L^2(\mathbb{R})$ and $1 < q < \infty$. If $\int |\gamma|^q |\varphi(\gamma)|^2 d\gamma < \infty$ then*

$$\int |\gamma + N|^q |\varphi(\gamma)|^2 d\gamma \leq 3^q |N|^q ||\varphi||_2^2 + \left(\frac{3}{2}\right)^q \int_{\mathbb{R}\setminus[-2N,2N]} |\gamma|^q |\varphi(\gamma)|^2 d\gamma \tag{3.1}$$

holds for all $N \geq 0$.

PROOF. First note that

$$\int_{-2N}^{2N} |\gamma + N|^q |\varphi(\gamma)|^2 d\gamma \leq 3^q |N|^q ||\varphi||^2_{L^2(\mathbb{R})}.$$

Next note that $2N \leq \gamma$ implies $|\gamma + N|^q \leq \left(\frac{3}{2}\right)^q |\gamma|^q$. Thus

$$\int_{2N}^{\infty} |\gamma + N|^q |\varphi(\gamma)|^2 d\gamma \leq \left(\frac{3}{2}\right)^q \int_{2N}^{\infty} |\gamma|^q |\varphi(\gamma)|^2 d\gamma.$$

Likewise,

$$\int_{-\infty}^{-2N} |\gamma + N|^q |\varphi(\gamma)|^2 d\gamma \leq \left(\frac{3}{2}\right)^q \int_{-\infty}^{-2N} |\gamma|^q |\varphi(\gamma)|^2 d\gamma.$$

This completes the proof. □

4. Finite, orthonormal, well localized systems

LEMMA 4.1. *Assume $\frac{1}{p} + \frac{1}{q} = 1$, where $1 < p, q < \infty$. Fix $\epsilon > 0$ and $\varphi \in C_c^\infty(\mathbb{R})$ with $||\varphi||_{L^2(\mathbb{R})} = 1$. There is $K((p,q),\epsilon,\varphi)$ such that for each $K > K((p,q),\epsilon,\varphi)$, there exists an infinite orthonormal set $S_0 = S_0(K,\varphi) = \{s_n\}_{n=0}^\infty \subseteq C_c^\infty(\mathbb{R})$ satisfying*

$$\text{supp } s_n = \text{supp } \varphi \subseteq [-K/2, K/2], \tag{4.1}$$

$$\left(\int |t|^p |s_n(t)|^2 dt\right)^{\frac{1}{2}} \leq \left(\int |t|^p |\varphi(t)|^2 dt\right)^{\frac{1}{2}} + \epsilon \equiv C_{p,\varphi} + \epsilon, \tag{4.2}$$

and

$$\left(\int |\gamma - nK|^q |\widehat{s_n}(\gamma)|^2 d\gamma\right)^{\frac{1}{2}} \leq \left(\int |\gamma|^q |\widehat{\varphi}(\gamma)|^2 d\gamma\right)^{\frac{1}{2}} + \epsilon \equiv C_{q,\varphi} + \epsilon \tag{4.3}$$

for $n = 0, 1, 2, \cdots$.

PROOF. Throughout the proof, C will denote various constants which are independent of K. C may depend on $(p,q), \varphi$, and N, all of which are fixed throughout the proof.

i. Let $\epsilon > 0$ and let $\varphi \in C_c^\infty(\mathbb{R})$ have $L^2(\mathbb{R})$ norm one. We may assume φ satisfies

$$|\widehat{\varphi}(\gamma)| \leq \frac{C}{|\gamma|^N + 1}, \tag{4.4}$$

where $N > 4q, N \in \mathbb{N}$. Now define

$$\varphi_j(t) = e^{2\pi ijKt}\varphi(t), \quad j = 0, 1, 2, \cdots,$$

where K is a sufficiently large integer which will depend on $(p, q), \varphi, N$, and ϵ. We shall specify how large to take K during the proof. Next, define

$$h_0(t) = \varphi_0(t) \tag{4.5}$$

and

$$h_n(t) = \varphi_n(t) - \sum_{j=0}^{n-1} a_{n,j}\varphi_j(t), \quad n = 1, 2, \cdots, \tag{4.6}$$

where the $a_{n,j}$ are chosen to make h_n orthogonal to $\{\varphi_j\}_{j=0}^{n-1}$. This choice of $a_{n,j}$ implies that, for all $0 \le l \le n-1$,

$$\langle \varphi_n, \varphi_l \rangle = \sum_{j=0}^{n-1} a_{n,j} \langle \varphi_j, \varphi_l \rangle.$$

Rewriting this in matrix form, we have

$$Ga = g,$$

where $G = \begin{pmatrix} \langle \varphi_{n-1}, \varphi_{n-1} \rangle & \langle \varphi_{n-2}, \varphi_{n-1} \rangle & \cdots & \langle \varphi_0, \varphi_{n-1} \rangle \\ \langle \varphi_{n-1}, \varphi_{n-2} \rangle & \langle \varphi_{n-2}, \varphi_{n-2} \rangle & \cdots & \langle \varphi_0, \varphi_{n-2} \rangle \\ \cdots & \cdots & \cdots & \cdots \\ \langle \varphi_{n-1}, \varphi_0 \rangle & \langle \varphi_{n-2}, \varphi_0 \rangle & \cdots & \langle \varphi_0, \varphi_0 \rangle \end{pmatrix}$

$$a = \begin{pmatrix} a_{n,n-1} \\ a_{n,n-2} \\ \cdots \\ a_{n,0} \end{pmatrix} \quad \text{and} \quad g = \begin{pmatrix} \langle \varphi_n, \varphi_{n-1} \rangle \\ \langle \varphi_n, \varphi_{n-2} \rangle \\ \cdots \\ \langle \varphi_n, \varphi_0 \rangle \end{pmatrix}.$$

Note that these matrices all depend on n, but we shall usually suppress this for economy of notation. When we wish to emphasize the dependence on n, we shall write $G = G_n$.

ii. First of all, observe that G is an invertible matrix since the finite set $\{\varphi_j\}_{j=0}^{n-1}$ is linearly independent by Proposition 1 of [**HRT**]. In particular, one also has that $\{a_{n,j}\}_{j=0}^{n-1}$ is uniquely determined.

To apply Jaffard's theorem, we also need to know that the spectrum of $G = G_n$ stays uniformly bounded away from 0 independent of n. Note that the matrix G is a Toeplitz matrix, and by (4.4) it has polynomial decay of order N off the main diagonal; in fact,

$$|G(j,k)| \le \frac{C}{1 + K^N|j-k|^N} \le \frac{C}{1 + |j-k|^N}. \tag{4.7}$$

For K large enough, the first inequality of (4.7) implies $G = G_n$ is diagonally dominant and has spectrum uniformly bounded away from 0.

iii. By Jaffard's theorem, G^{-1} has the same type of decay off its main diagonal as G, namely,

$$|G^{-1}(j,k)| \le \frac{C}{1 + |j-k|^N}.$$

Also, the comments after the statement of Jaffard's theorem ensure that C is independent of K.

Therefore, noting that $a_{n,n-j}$ is the j-th element of the vector a,

$$\begin{aligned}
|a_{n,n-j}| &\leq \sum_{l=0}^{n-1} |G^{-1}(j,l)||g_l| = \sum_{l=0}^{n-1} |G^{-1}(j,l)||\langle \varphi_n, \varphi_{n-l-1}\rangle| \\
&\leq \sum_{l=0}^{n-1} \left(\frac{C}{1+|j-l|^N}\right)\left(\frac{C}{1+K^N|l+1|^N}\right) \\
&\leq \sum_{l=0}^{n-1} \frac{C}{1+|j-l|^N}\left(\frac{C}{K^N(l+1)^N}\right) \\
&\leq \frac{C}{K^N}\sum_{l=0}^{n-1} \frac{1}{(1+|j-l|^N)}\frac{1}{|l+1|^N} \\
&\leq \frac{C}{K^N}\sum_{l=1}^{\infty} \frac{1}{(1+|(j+1)-l|^N)}\frac{1}{|l|^N} \\
&\leq \left(\frac{1}{K^N}\right)\frac{C}{|j+1|^N}.
\end{aligned}$$

To see the last step, first note that

$$\sum_{1\leq l\leq \frac{j+1}{2}} \frac{1}{|l|^N(1+|j+1-l|^N)} \leq \frac{1}{(1+|\frac{j+1}{2}|^N)}\sum_{l=1}^{\infty}\frac{1}{l^N}.$$

Combining this with a similar estimate for the remaining range of summation gives the desired inequality.

Thus, we have

$$|a_{n,j}| = |a_{n,n-(n-j)}| \leq \frac{C}{K^N|n-j+1|^N}. \tag{4.8}$$

iv. Observe that

$$\sum_{j=0}^{n-1}|a_{n,j}| \leq \frac{C}{K^N}\sum_{j=0}^{n-1}\frac{1}{|n-j+1|^N} \leq \frac{C}{K^N}\sum_{j=2}^{n+1}\frac{1}{j^N} \leq \frac{C}{K^N}. \tag{4.9}$$

Combining this and (4.6), we can estimate the localization of the $h_n(t)$.

$$\begin{aligned}
\left(\int |t|^p|h_n(t)|^2dt\right)^{\frac{1}{2}} &\leq \left(\int |t|^p|\varphi_n(t)|^2dt\right)^{\frac{1}{2}} + \sum_{j=0}^{n-1}|a_{n,j}|\left(\int |t|^p|\varphi_j(t)|^2dt\right)^{\frac{1}{2}} \\
&= \left(\int |t|^p|\varphi(t)|^2dt\right)^{\frac{1}{2}} + \left(\sum_{j=0}^{n-1}|a_{n,j}|\right)\left(\int |t|^p|\varphi(t)|^2dt\right)^{\frac{1}{2}} \\
&\leq \left(\int |t|^p|\varphi(t)|^2dt\right)^{\frac{1}{2}} + \frac{C}{K^N}\left(\int |t|^p|\varphi(t)|^2dt\right)^{\frac{1}{2}}
\end{aligned}$$

Thus for K large enough,

$$\int |t|^p|h_n(t)|^2dt \leq C_{p,\varphi} + \frac{\epsilon}{2} \tag{4.10}$$

holds for all $n = 0, 1, 2, \cdots$.

v. We now estimate the localization of the $\widehat{h_n}(t)$.

$$\begin{aligned}
&\left(\int |\gamma - nK|^q |\widehat{h_n}(\gamma)|^2 d\gamma\right)^{\frac{1}{2}} \\
&\leq \left(\int |\gamma - nK|^q |\widehat{\varphi}(\gamma - nK)|^2 d\gamma\right)^{\frac{1}{2}} + \left(\int |\gamma - nK|^q \left|\sum_{j=0}^{n-1} a_{n,j}\widehat{\varphi}(\gamma - jK)\right|^2 d\gamma\right)^{\frac{1}{2}} \\
&\leq \left(\int |\gamma|^q |\widehat{\varphi}(\gamma)|^2 d\gamma\right)^{\frac{1}{2}} + \sum_{j=0}^{n-1} |a_{n,j}| \left(\int |\gamma - K(n-j)|^q |\widehat{\varphi}(\gamma)|^2 d\gamma\right)^{\frac{1}{2}}.
\end{aligned}$$

Using (4.8) and Lemma 3.3 we have

$$\begin{aligned}
&\sum_{j=0}^{n-1} |a_{n,j}| \left(\int |\gamma - K(n-j)|^q |\widehat{\varphi}(\gamma)|^2 d\gamma\right)^{\frac{1}{2}} \\
&\leq \sum_{j=0}^{n-1} |a_{n,j}| \left[|K(n-j)|^q ||\widehat{\varphi}||^2_{L^2(\mathbb{R})} + (3/2)^q \int |\gamma||\widehat{\varphi}(\gamma)|^2 d\gamma\right]^{\frac{1}{2}} \\
&\leq CK^{q/2} \sum_{j=0}^{n-1} |a_{n,j}||n-j|^{q/2} \leq CK^{q/2} \sum_{j=0}^{n-1} \frac{|n-j|^{q/2}}{K^N|n-j+1|^N} \\
&\leq \frac{C}{K^{N-q/2}}.
\end{aligned}$$

Thus, combining the above with K large enough gives

$$\left(\int |\gamma - nK|^q |\widehat{h_n}(\gamma)|^2 d\gamma\right)^{\frac{1}{2}} \leq C_{q,\varphi} + \frac{\epsilon}{2} \tag{4.11}$$

for all $n = 0, 1, 2, \cdots$.

vi. It remains to normalize the h_n. First note that

$$||\varphi_n - h_n||_{L^2(\mathbb{R})} \leq \sum_{j=0}^{n-1} |a_{n,j}| \leq \frac{C}{K^N} \sum_{j=0}^{n-1} \frac{1}{|n-j+1|^N} \leq \frac{C}{K^N} \sum_{j=2}^{\infty} \frac{1}{j^N} = \frac{C}{K^N},$$

so that

$$1 = ||\varphi_n||_{L^2(\mathbb{R})} \leq ||h_n||_{L^2(\mathbb{R})} + ||h_n - \varphi_n||_{L^2(\mathbb{R})} \leq ||h_n||_{L^2(\mathbb{R})} + \frac{C}{K^N}$$

and

$$||h_n||_{L^2(\mathbb{R})} \leq ||\varphi||_{L^2(\mathbb{R})} + ||h_n - \varphi_n||_{L^2(\mathbb{R})} \leq 1 + \frac{C}{K^N}.$$

Thus we have

$$1 - \frac{C}{K^N} \leq ||h_n||_{L^2(\mathbb{R})} \leq 1 + \frac{C}{K^N}. \tag{4.12}$$

Finally, let $s_n(t) = h_n(t)/||h_n||_{L^2(\mathbb{R})}$. Taking K large enough and combining (4.10), (4.11), and (4.12) now shows that that (4.2) and (4.3) hold. □

Lemma 4.2. *Assume* $\frac{1}{p}+\frac{1}{q}=1$, *where* $1<p,q<\infty$. *Fix* $\epsilon>0$ *and* $\varphi\in C_c^\infty(\mathbb{R})$ *with* $||\varphi||_{L^2(\mathbb{R})}=1$. *Fix* $K\in\mathbb{N}$ *sufficiently large. For each* $T>1$ *there exists a finite orthonormal set,*

$$S=S(T,K)=S(T,K,\varphi)=\{s_{m,n}:0\le m<\lfloor T^{2/q}\rfloor, 0\le n<\lfloor T^{2/p}\rfloor\}\subseteq C_c^\infty(\mathbb{R}),$$

of cardinality $\lfloor T^{2/p}\rfloor\lfloor T^{2/q}\rfloor\le T^2$ *satisfying*

$$\text{supp } s_{m,n}\subseteq\left[\frac{1}{2}T^{2/p},2T^{2/p}K\right], \tag{4.13}$$

$$\left(\int|t-Kn-T^{(2/p)}K|^p|s_{m,n}(t)|^2dt\right)^{\frac{1}{2}}\le C_{p,\varphi}+\epsilon, \tag{4.14}$$

and

$$\left(\int|\gamma-Km|^q|\widehat{s_{m,n}}(\gamma)|^2d\gamma\right)^{\frac{1}{2}}\le C_{q,\varphi}+\epsilon, \tag{4.15}$$

for all $0\le m<\lfloor T^{2/q}\rfloor, 0\le n<\lfloor T^{2/p}\rfloor$. *Here* $C_{p,\varphi}$ *and* $C_{q,\varphi}$ *are defined as in the previous lemma.*

Proof. Let $\{s_m\}_{m=0}^{\lfloor T^{2/q}\rfloor-1}$ be defined using the system from the previous lemma. Define

$$s_{m,n}(t)=s_m(t-nK-T^{(2/p)}K)\quad\text{for}\quad 0\le m<T^{2/q}\quad\text{and}\quad 0\le n<T^{2/p}.$$

Now, (4.14) and (4.15) hold by the previous lemma. Since K was chosen large enough so that supp $\varphi\subseteq[-K/2,K/2]$, it follows that

$$\text{supp } s_{m,n}\subseteq\left[nK+T^{2/p}K-K/2,nK+T^{2/p}K+K/2\right],$$

so that all the $s_{m,n}$ are supported in

$$\left[T^{2/p}K-K/2,(T^{2/p}-1)K+T^{2/p}K+K/2\right]\subseteq\left[\frac{1}{2}T^{2/p},2KT^{2/p}\right].$$

□

Lemma 4.3. *Fix* $\epsilon>0$ *and* $\varphi\in C_c^\infty(\mathbb{R})$. *There exists a constant* C *such that for each* $S(T,K)$ *as in the previous lemma and for every* $\Phi\in$ span $S(T,K)$ *and each* $0\le y\le T^{2/q}K$ *one has*

$$\int|\gamma-y|^q|\widehat{\Phi}(\gamma)|^2d\gamma\le CK^qT^2||\Phi||^2_{L^2(\mathbb{R})}. \tag{4.16}$$

Proof. *i.* First note that

$$\int_{-2T^{2/q}K}^{2T^{2/q}K}|\gamma-y|^q|\widehat{\Phi}(\gamma)|^2d\gamma\le 3^qK^qT^2||\Phi||^2_{L^2(\mathbb{R})}.$$

ii. Recall that

$$\text{span } S(T,K)=\text{span}\{\varphi_{m,n}:0\le m<\lfloor T^{2/q}\rfloor, 0\le n<\lfloor T^{2/p}\rfloor\},$$

where

$$\varphi_{m,n}(t)=e^{2\pi iKmt}\varphi(t-nK-T^{2/p}K).$$

Next, note that for K large enough, $\{\varphi_{m,n}:m,n\in\mathbb{Z}\}$ is a Riesz basis for its closed linear span. To see this, it suffices to show that $\{g_{m,n}:m,n\in\mathbb{Z}\}=\mathcal{G}(\varphi,K,K)$ is a Riesz basis for its closed linear span. Using Theorem 9 of Chapter

1, Section 8 in [**Y**] this is equivalent to proving that the Gram matrix $G_{(j,k),(l,m)} = \langle g_{j,k}, g_{l,m}\rangle$ defines a bounded positive operator on $l^2(\mathbb{Z}\times\mathbb{Z})$. Since $\varphi \in \mathcal{S}(\mathbb{R})$ one may directly verify that the Gram matrix is positive and bounded for all large enough K. In particular, for K large enough one can use Schur's test (see Lemma 6.2.1 in [**G2**]) to show that $M \equiv G - I$ satisfies $||M|| < \frac{1}{2}$, where I is the identity matrix, and $||\cdot||$ denotes the operator norm induced by the $l^2(\mathbb{Z}\times\mathbb{Z})$ norm. Hence there exists K_0 such that for all $K > K_0$, $\{\varphi_{m,n}\}$ is a Riesz basis for its closed linear span.

By the definition of Riesz basis there exist $0 < A \leq B < \infty$, such that for each finite sum $\Phi(t) = \sum d_{m,n}\varphi_{m,n}(t)$

$$A\sum |d_{m,n}|^2 \leq ||\Phi||^2_{L^2(\mathbb{R})} \leq B\sum |d_{m,n}|^2. \tag{4.17}$$

For us the constants A, B can be chosen independent of T and K. To see this, first note that the proof of Theorem 9 in Chapter 1, Section 8 of [**Y**] shows that one may take $B = ||G||$ and $A = ||G^{-1}||^{-1}$, where G is the Gram matrix defined above. A direct calculation with the Gram matrix shows that since $\varphi \in \mathcal{S}(\mathbb{R})$, the norm of the Gram matrix and its inverse approach 1 as $K \to \infty$.

We may conclude that there exists C, independent of T and K, such that for each $\Phi = \sum d_{m,n}\varphi_{m,n} \in \operatorname{span} S(T,K)$

$$\left(\sum |d_{m,n}|^2\right)^{\frac{1}{2}} \leq C||\Phi||_{L^2(\mathbb{R})}. \tag{4.18}$$

Here, and below, sums are over $0 \leq m < \lfloor T^{2/q}\rfloor, 0 \leq n < \lfloor T^{2/p}\rfloor$.

iii. We need to show that

$$\int_{2T^{2/q}K}^{\infty} |\gamma - y|^q |\widehat{\Phi}(\gamma)|^2 d\gamma \leq CK^q T^2 ||\Phi||^2_{L^2(\mathbb{R})}. \tag{4.19}$$

iii.a. First note that since $0 \leq y \leq T^{2/q}K$, we have

$$\left(\int_{2T^{2/q}K}^{\infty} |\gamma - y|^q |\widehat{\Phi}(\gamma)|^2 d\gamma\right)^{\frac{1}{2}} \leq \left(\int_{2T^{2/q}K}^{\infty} |\gamma|^q |\widehat{\Phi}(\gamma)|^2 d\gamma\right)^{\frac{1}{2}}. \tag{4.20}$$

iii.b. To estimate the right side of (4.20) we begin as follows:

$$\begin{aligned}\left(\int_{2T^{2/q}K}^{\infty} |\gamma|^q |\widehat{\Phi}(\gamma)|^2 d\gamma\right)^{\frac{1}{2}} &= \left(\int_{2T^{2/q}K}^{\infty} |\gamma|^q \left|\sum_{m,n} d_{m,n}\widehat{\varphi_{m,n}}(\gamma)\right|^2 d\gamma\right)^{\frac{1}{2}}\\ &\leq \sum_{m,n} |d_{m,n}| \left(\int_{2T^{2/q}K}^{\infty} |\gamma|^q \left|\widehat{\varphi_{m,n}}(\gamma)\right|^2 d\gamma\right)^{\frac{1}{2}}\\ &\leq ||d_{m,n}||_{l^2(\mathbb{Z}^2)} \sum_{m,n} \left(\int_{2T^{2/q}K}^{\infty} |\gamma|^q \left|\widehat{\varphi_{m,n}}(\gamma)\right|^2 d\gamma\right)^{\frac{1}{2}}\\ &\leq C||\Phi||_{L^2(\mathbb{R})} \sum_{m,n} \left(\int_{2T^{2/q}K}^{\infty} |\gamma|^q \left|\widehat{\varphi_{m,n}}(\gamma)\right|^2 d\gamma\right)^{\frac{1}{2}}. \qquad (4.21)\end{aligned}$$

Next note that

$$|\widehat{\varphi_{m,n}}(\gamma)| = |\widehat{\varphi}(\gamma - Km)|. \tag{4.22}$$

Thus,

$$\sum_{m,n}\left(\int_{2T^{2/q}K}^{\infty}|\gamma|^q\left|\widehat{\varphi_{m,n}}(\gamma)\right|^2d\gamma\right)^{\frac{1}{2}}=\sum_{m,n}\left(\int_{2T^{2/q}K}^{\infty}|\gamma|^q\left|\widehat{\varphi}(\gamma-mK)\right|^2d\gamma\right)^{\frac{1}{2}}$$

$$=\lfloor T^{2/p}\rfloor\sum_{m=0}^{\lfloor T^{2/q}\rfloor-1}\left(\int_{2T^{2/q}K}^{\infty}|\gamma|^q|\widehat{\varphi}(\gamma-mK)|^2d\gamma\right)^{\frac{1}{2}}$$

$$\le T^2\left(\int_{2T^{2/q}K}^{\infty}|\gamma|^q|\widehat{\varphi}(\gamma-T^{2/q}K)|^2d\gamma\right)^{\frac{1}{2}}=T^2\left(\int_{T^{2/q}K}^{\infty}|\gamma+T^{2/q}K|^q|\widehat{\varphi}(\gamma)|^2d\gamma\right)^{\frac{1}{2}}$$

$$\le T^2 2^{q/2}\left(\int_{T^{2/q}K}^{\infty}|\gamma|^q|\widehat{\varphi}(\gamma)|^2d\gamma\right)^{\frac{1}{2}}\le\frac{CT^2}{|T^{2/q}K|^{2q}}.$$

The final inequality holds since $\varphi\in\mathcal{S}(\mathbb{R})$. Together with (4.21), this gives

$$\left(\int_{2T^{2/q}K}^{\infty}|\gamma|^q|\widehat{\Phi}(\gamma)|^2d\gamma\right)^{\frac{1}{2}}\le\frac{C||\Phi||_{L^2(\mathbb{R})}}{K^{2q}T^2}. \tag{4.23}$$

Combining (4.20) and (4.23) yields (4.19), as desired.
iv. It remains to show that

$$\int_{-\infty}^{-2T^{2/q}K}|\gamma-y|^q|\widehat{\Phi}(\gamma)|^2d\gamma\le CK^qT^2||\Phi||^2_{L^2(\mathbb{R})}.$$

This follows by calculations similar to those in part *iii.* The proof is complete. □

5. A (p,q) version of Bourgain's theorem

We are now ready to prove our main result, Theorem 2.2. The proof follows that of Bourgain, [**Bou**], which, in turn, depends on an idea of W. Rudin, [**R**], used to construct certain orthonormal bases for $H^2(B)$, where B is the unit ball of $\mathbb{C}^n$.

Proof of Theorem 2.2. Throughout the proof C will denote various constants which are independent of n, T_n, K, Θ, and any indices.

Let $\{f_n\}_{n\in\mathbb{N}}\subseteq C_c^\infty(\mathbb{R})$ be sequence which is dense in the unit sphere of $L^2(\mathbb{R})$. Let $\epsilon>0$ and $\varphi\in C_c^\infty(\mathbb{R})$ be given. Let K be sufficiently large to ensure we may use Lemma 4.2 applied to $\frac{\epsilon}{2}$. The orthonormal basis we construct will be of the form $\bigcup_{n=1}^\infty B_n$ where B_n is a finite orthonormal subset of $C_c^\infty(\mathbb{R})$. We shall construct the B_n inductively.

i. Suppose $B_1,\dots,B_{n-1}$ are already defined such that B_j is a finite orthonormal subset of $C_c^\infty(\mathbb{R})$ and the elements of $\bigcup_{j=1}^{n-1}B_j$ are mutually orthonormal. Define $F_n=f_n-P_{[B_1,\dots,B_{n-1}]}f_n$, where $P_{[B_1,\dots,B_{n-1}]}$ is the orthogonal projection of $L^2(\mathbb{R})$ onto

$$[B_1,\dots,B_{n-1}]\equiv\operatorname{span}\bigcup_{l=1}^{n-1}B_l.$$

For the base case of the induction we simply let $F_1=f_1$. Using F_n, we now prepare to construct B_n.
i.a. Note that

$$||F_n||^2_{L^2(\mathbb{R})}\le 1 \tag{5.1}$$

because $F_n \perp P_{[B_1,\cdots,B_n-1]}f_n$ and $||f_n||_{L^2(\mathbb{R})} = 1$.
i.b. Since f_n and all elements of the B_j are in $C_c^\infty(\mathbb{R})$, it follows that $F_n \in C_c^\infty(\mathbb{R})$.
Choose $T_n > 2$ large enough so that

$$\frac{\lfloor T_n^{2/p} \rfloor \lfloor T_n^{2/q} \rfloor}{T_n^2} > \frac{1}{4}, \tag{5.2}$$

$$\text{supp } F_n \subseteq \left[-\frac{1}{2} T_n^{2/p}, \frac{1}{2} T_n^{2/p} \right], \tag{5.3}$$

$$\text{supp } b \subseteq [-\frac{1}{2} T_n^{2/p}, \frac{1}{2} T_n^{2/p}] \quad \text{for all} \quad b \in \bigcup_{j=1}^{n-1} B_j, \tag{5.4}$$

and

$$\forall y \geq 1, \ \int_{\mathbb{R}\setminus[-2y,2y]} |\gamma|^q |\widehat{F_n}(\gamma)|^2 d\gamma \leq \frac{T_n}{1+|y|^N}, \tag{5.5}$$

where we have used the fact that $\widehat{F_n} \in \mathcal{S}(\mathbb{R})$ $(F_n \in C_c^\infty(\mathbb{R}))$.
ii. Let

$$S = S(T_n, K) = \{s_{j,k}^n : 0 \leq j < \lfloor T_n^{(2/p)} \rfloor \text{ and } 0 \leq k < \lfloor T_n^{(2/q)} \rfloor\}$$

be the system from Lemma 4.2 applied with $\frac{\epsilon}{2}$ instead of ϵ. We shall switch from the double indexing (j,k) to single indexing, and enumerate the elements of the system as $\{s_l^n\}_{l=1}^{\lfloor T_n^{2/p} \rfloor \lfloor T_n^{2/q} \rfloor}$. If l_1, l_2 are the indices for which $s_l^n = s_{l_1,l_2}^n$, let

$$x(s_l^n) = Kl_1 + T_n^{(2/p)}K \quad \text{and} \quad y(\widehat{s_l^n}) = Kl_2,$$

so that by Lemma 4.2

$$\left(\int |t - x(s_j^n)|^p |s_j^n(t)|^2 dt \right)^{\frac{1}{2}} \leq C_{p,\varphi} + \frac{\epsilon}{2} \tag{5.6}$$

and

$$\left(\int |\gamma - y(\widehat{s_j^n})|^q |\widehat{s_j^n}(\gamma)|^2 d\gamma \right)^{\frac{1}{2}} \leq C_{q,\varphi} + \frac{\epsilon}{2}. \tag{5.7}$$

Note that

$$T_n^{(2/p)}K \leq x(s_j^n) \leq 2KT_n^{2/p} \quad \text{and} \quad 0 \leq y(\widehat{s_j^n}) \leq KT_n^{2/q}. \tag{5.8}$$

Let $0 < \Theta < \frac{1}{4}$ be sufficiently small and be fixed throughout the proof. We shall be more precise later about how small to take Θ. For now, note that K is fixed throughout the proof, so that Θ may depend on K (but not T_n). Let $\nu_n = \lfloor T_n^{2/p} \rfloor \lfloor T_n^{2/q} \rfloor$. Now define

$$\begin{aligned}
b_1^n(t) &= \frac{\Theta}{T_n} F_n(t) + \alpha_{n,1} s_1^n(t) \\
b_2^n(t) &= \frac{\Theta}{T_n} F_n(t) + \sigma_{n,1} s_1^n(t) + \alpha_{n,2} s_2^n(t) \\
&\vdots \\
b_{\nu_n}^n(t) &= \frac{\Theta}{T_n} F_n(t) + \sigma_{n,1} s_1^n(t) + \cdots + \sigma_{n,\nu_n-1} s_{\nu_n-1}^n(t) + \alpha_{n,\nu_n} s_{\nu_n}^n(t),
\end{aligned}$$

where the $\sigma_{n,j}$ and $\alpha_{n,j}$ are chosen to ensure that $\{b_j^n\}_{j=1}^{\nu_n}$ is orthonormal.
ii.a. The choice of $\sigma_{n,j}$ and $\alpha_{n,j}$ implies that

$$|1-\alpha_{n,j}| \leq \frac{\Theta}{T_n} \quad \text{for} \quad j=1,2,\cdots,T_n^2, \tag{5.9}$$

and

$$|\sigma_{n,j}| \leq \frac{\Theta}{T_n^2} \quad \text{for} \quad j=1,2,\cdots,T_n^2-1. \tag{5.10}$$

To see this, first note that $\{F_n\}\bigcup S(T_n,K)$ is an orthogonal set. Therefore, $\{b_j^n\}_{j=1}^{\nu_n}$ being orthonormal implies that for $l=1,2,\cdots,T_n^2$ we have

$$0=\frac{\Theta^2}{T_n^2}||F_n||^2_{L^2(\mathbb{R})}+\sigma_{n,1}^2+\cdots+\sigma_{n,l-1}^2+\sigma_{n,l}\alpha_{n,l} \tag{5.11}$$

and for $l=1,2,\cdots,T_n^2-1$

$$\alpha_{n,l}^2=1-\frac{\Theta^2}{T_n^2}||F_n||^2_{L^2(\mathbb{R})}-\sigma_{n,1}^2-\cdots-\sigma_{n,l-1}^2. \tag{5.12}$$

ii.b. Using (5.11) and (5.12) we shall now prove (5.9) and (5.10) by induction. The case $j=1$ of (5.9) holds since (5.12) implies

$$1=\frac{\Theta^2}{T_n^2}||F_n||^2_{L^2(\mathbb{R})}+\alpha_{n,1}^2.$$

Since $2<T_n$ and $\Theta<\frac{1}{4}$, we may choose $0<\alpha_{n,1}\leq 1$. Therefore,

$$|1-\alpha_{n,1}|\leq|1-\alpha_{n,1}^2|\leq\frac{\Theta^2}{T_n^2}\leq\frac{\Theta}{T_n}.$$

Using this, the case $j=1$ of (5.10) now follows since, by (5.11),

$$0=\frac{\Theta^2}{T_n^2}||F_n||^2_{L^2(\mathbb{R})}+\alpha_{n,1}\sigma_{n,1},$$

which implies

$$|\sigma_{n,1}|\leq\frac{\Theta^2}{T_n^2}\frac{1}{|\alpha_{n,1}|}\leq\frac{\Theta^2}{T_n^2}\frac{1}{(1-\Theta/T_n)}\leq\frac{\Theta}{T_n^2}.$$

The last inequality holds because $\Theta<\frac{1}{4}$ and $T_n>2$.
ii.c. Next, assume $|\sigma_{n,j}|\leq\frac{\Theta}{T_n^2}$ holds for $j<l$. We may once again choose $0<\alpha_{n,l}\leq 1$. Since the cardinality of $S(T_n,K)$ is at most T_n^2,

$$|1-\alpha_{n,l}|\leq|1-\alpha_{n,l}^2|\leq\frac{\Theta^2}{T_n^2}+\sum_{j=1}^{l-1}\sigma_{n,j}^2\leq\frac{\Theta^2}{T_n^2}+T_n^2\frac{\Theta^2}{T_n^4}\leq 2\frac{\Theta^2}{T_n^2}\leq\frac{\Theta}{T_n},$$

and (5.9) follows by induction. For (5.10), assume that $|\sigma_{n,j}|\leq\frac{\Theta}{T_n^2}$ for $j<l$ and $|1-\alpha_{n,l}|\leq\frac{\Theta}{T_n}$. Thus,

$$|\sigma_{n,l}|\leq\frac{1}{|\alpha_{n,l}|}\left(\frac{\Theta^2}{T_n^2}||F_n||^2_{L^2(\mathbb{R})}+\sum_{j=1}^{l-1}\sigma_{n,j}^2\right)\leq\frac{1}{(1-\Theta/T_n)}\left(2\frac{\Theta^2}{T_n^2}\right)\leq\frac{\Theta}{T_n^2},$$

and (5.10) holds by induction.

iii. By (5.9) and (5.10), we know that $\sigma_{n,j}$ is close to zero and $\alpha_{n,j}$ is close to one. Thus, we expect to have b_j^n close to s_j^n. In fact,

$$||b_j^n - s_j^n||_{L^2(\mathbb{R})} \leq 3\frac{\Theta}{T_n}. \tag{5.13}$$

To see this, note that by (5.9) and (5.10)

$$\begin{aligned}
||b_j^n - s_j^n||_{L^2(\mathbb{R})} &\leq ||b_j^n - \alpha_{n,j} s_j^n||_{L^2(\mathbb{R})} + |1 - \alpha_{n,j}| \\
&\leq ||b_j^n - \alpha_{n,j} s_j^n||_{L^2(\mathbb{R})} + \frac{\Theta}{T_n} \\
&= \left(\frac{\Theta^2}{T_n^2} ||F_n||^2_{L^2(\mathbb{R})} + \sum_{k=1}^{j-1} |\sigma_{n,k}|^2 \right)^{\frac{1}{2}} + \frac{\Theta}{T_n} \\
&\leq \left(\frac{\Theta^2}{T_n^2} + \left(T_n^2 \frac{\Theta^2}{T_n^4} \right) \right)^{\frac{1}{2}} + \frac{\Theta}{T_n} \leq 3\frac{\Theta}{T_n}.
\end{aligned}$$

iv. Let us now prove that

$$\Delta_p(b_j^n) \leq \left(\int |t - x(s_j^n)|^p |s_j^n(t)|^2 dt \right)^{\frac{1}{2}} + CK^{p/2}\Theta \leq C_{p,\varphi} + \epsilon. \tag{5.14}$$

Using (5.8), (5.13), and the fact that the b_j^n are supported in $[-\frac{1}{2}T_n^{2/p}, 2T_n^{2/p}K]$ (since F_n and $s_{n,j}$ are), we have

$$\begin{aligned}
&\left(\int |t - x(s_j^n)|^p |b_j^n(t)|^2 dt \right)^{\frac{1}{2}} \\
&\leq \left(\int |t - x(s_j^n)|^p |b_j^n(t) - s_j^n(t)|^2 dt \right)^{\frac{1}{2}} + \left(\int |t - x(s_j^n)|^p |s_j^n(t)|^2 dt \right)^{\frac{1}{2}} \\
&\leq |2T_n^{2/p}K + 2KT_n^{2/p}|^{p/2} ||b_j^n - s_j^n||_{L^2(\mathbb{R})} + \left(\int |t - x(s_j^n)|^p |s_j^n(t)|^2 dt \right)^{\frac{1}{2}} \\
&\leq CK^{p/2} T_n ||b_j^n - s_j^n||_{L^2(\mathbb{R})} + \left(\int |t - x(s_j^n)|^p |s_j^n(t)|^2 dt \right)^{\frac{1}{2}} \\
&\leq CK^{p/2}\Theta + \left(\int |t - x(s_j^n)|^p |s_j^n(t)|^2 dt \right)^{\frac{1}{2}}.
\end{aligned}$$

Assume Θ was chosen small enough to ensure $C\Theta K^{p/2} < \frac{\epsilon}{2}$. Thus, by (5.6) we have

$$\Delta_p(b_j^n) \leq C_{p,\varphi} + CK^{(p/2)}\Theta < C_{p,\varphi} + \epsilon. \tag{5.15}$$

v. Here we shall prove that

$$\Delta_q(\widehat{b_j^n}) \leq \left(\int |\gamma - y(\widehat{s_j^n})|^q |\widehat{s_j^n}(\gamma)|^2 d\gamma \right)^{\frac{1}{2}} + C\Theta K^{(q/2)} < C_{q,\varphi} + \epsilon.$$

v.a. First we show that

$$\left(\int |\gamma - y(\widehat{s_j^n})|^q |\frac{\Theta}{T_n} \widehat{F_n}(\gamma)|^2 d\gamma \right)^{\frac{1}{2}} \leq C\Theta K^{(q/2)}. \tag{5.16}$$

This follows from (5.5), (5.8), and Lemma 3.3:

$$\int |\gamma - y(\widehat{s_j^n})|^q |\widehat{F_n}(\gamma)|^2 d\gamma \le 3^q |y(\widehat{s_j^n})|^q ||\widehat{F_n}||^2_{L^2(\mathbb{R})} + \frac{(3/2)^q T_n}{1+|y(\widehat{s_j^n})|^M}$$
$$\le CT_n^2 K^q + CT_n \le CT_n^2 K^q.$$

v.b. Next, we show that

$$\left(\int |\gamma - y(\widehat{s_j^n})|^q |\widehat{b_j^n}(\gamma) - \widehat{s_j^n}(\gamma) - \frac{\Theta}{T_n}\widehat{F_n}(\gamma)|^2 d\gamma\right)^{\frac{1}{2}} \le C\Theta K^{(q/2)}.$$

Let $\Psi(\gamma) = \widehat{b_j^n}(\gamma) - \widehat{s_j^n}(\gamma) - \frac{\Theta}{T_n}\widehat{F_n}(\gamma)$. Note that Ψ is in the span of $S(T_n, K)$. Thus, using (5.8), (5.10), and Lemma 4.3,

$$\int |\gamma - y(\widehat{s_j^n})|^q |\widehat{\Psi}(\gamma)|^2 d\gamma \le CT_n^2 K^q ||\Psi||^2_{L^2(\mathbb{R})} = CT_n^2 K^q \sum_{l=1}^{\lfloor T_n^{2/p}\rfloor \lfloor T_n^{2/q}\rfloor -1} |\sigma_{n,l}|^2$$
$$\le CT_n^2 K^q \left(T_n^2 \frac{\Theta^2}{T_n^4}\right) = CK^q\Theta^2.$$

v.c. Combining the estimates from *v.a* and *v.b* we have

$$\begin{aligned}
\Delta_q(\widehat{b_j^n}) &\le \left(\int |\gamma - y(\widehat{s_j^n})|^q |b_j^n(\gamma)|^2 d\gamma\right)^{\frac{1}{2}} \\
&\le \left(\int |\gamma - y(\widehat{s_j^n})|^q |\widehat{s_j^n}(\gamma)|^2 d\gamma\right)^{\frac{1}{2}} \\
&+ \left(\int |\gamma - y(\widehat{s_j^n})|^q |\widehat{b_j^n}(\gamma) - \widehat{s_j^n}(\gamma)|^2 d\gamma\right)^{\frac{1}{2}} \\
&\le \left(\int |\gamma - y(\widehat{s_j^n})|^q |\widehat{s_j^n}(\gamma)|^2 d\gamma\right)^{\frac{1}{2}} + \left(\int |\gamma - y(\widehat{s_j^n})|^q |\frac{\Theta}{T_n}\widehat{F_n}(\gamma)|^2 \gamma\right)^{\frac{1}{2}} \\
&+ \left(\int |\gamma - y(\widehat{s_j^n})|^q |\widehat{b_j^n}(\gamma) - \widehat{s_j^n}(\gamma) - \frac{\Theta}{T_n}\widehat{F_n}(\gamma)|^2 d\gamma\right)^{\frac{1}{2}} \\
&\le \left(\int |\gamma - y(\widehat{s_j^n})|^q |\widehat{s_j^n}(\gamma)|^2 d\gamma\right)^{\frac{1}{2}} + \Theta CK^{(q/2)} + \Theta CK^{(q/2)} \\
&= \left(\int |\gamma - y(\widehat{s_j^n})|^q |\widehat{s_j^n}(\gamma)|^2 d\gamma\right)^{\frac{1}{2}} + C\Theta K^{(q/2)}.
\end{aligned}$$

Assume Θ was chosen small enough so that $C\Theta K^{q/2} < \frac{\epsilon}{2}$. Thus,

$$\Delta_q(\widehat{b_j^n}) \le C_{q,\varphi} + CK^{(q/2)}\Theta < C_{q,\varphi} + \epsilon. \tag{5.17}$$

vi. Having shown that all the elements of $B = \bigcup_{j=1}^\infty b_j^n$ have the desired localization, it only remains to prove that B is complete. To see this, note that, by (5.2) and the definition of F_n, we have

$$\begin{aligned}
||P_{[B_1,\cdots,B_k]}f_k||^2_{L^2(\mathbb{R})} &= ||P_{[B_1,\cdots,B_{k-1}]}f_k||^2_{L^2(\mathbb{R})} + ||P_{[B_k]}f_k||^2_{L^2(\mathbb{R})} \\
&= ||P_{[B_1,\cdots,B_{k-1}]}f_k||^2_{L^2(\mathbb{R})} + ||P_{[B_k]}(F_k + P_{[B_1,\cdots,B_{k-1}]}f_k)||^2_{L^2(\mathbb{R})} \\
&= 1 - ||F_k||^2_{L^2(\mathbb{R})} + ||P_{[B_k]}F_k||^2_{L^2(\mathbb{R})} \\
&= 1 - ||F_k||^2_{L^2(\mathbb{R})} + \sum_{j=1}^{\lfloor T_k^{2/p}\rfloor\lfloor T_k^{2/q}\rfloor} |\langle F_k, b_j^k\rangle|^2 \\
&= 1 - ||F_k||^2_{L^2(\mathbb{R})} + \lfloor T_k^{2/p}\rfloor\lfloor T_k^{2/q}\rfloor \left(\frac{\Theta}{T_k}||F_k||^2_{L^2(\mathbb{R})}\right)^2 \\
&\geq 1 - ||F_k||^2_{L^2(\mathbb{R})} + (\Theta/2)^2||F_k||^4_{L^2(\mathbb{R})} \\
&\geq (\Theta/2)^2.
\end{aligned}$$

To see the final inequality, let $h(t) = 1 - t^2 + a^2t^4$ be defined on $[0,1]$, where $0 < a < \frac{1}{4}$ is fixed. It is easy to see that $h(t) \geq a^2$. Since $||F_n||_{L^2(\mathbb{R})} \leq 1$ and $\Theta < \frac{1}{4}$, the last step follows.

Now, suppose $y \in L^2(\mathbb{R})$ satisfies $\langle y, b\rangle = 0$ for all $b \in B$. If y is not identically zero, then $\tilde{y} = y/||y||_{L^2(\mathbb{R})}$ is in the unit sphere of $L^2(\mathbb{R})$ and there exists f_{n_k} such that $f_{n_k} \to \tilde{y}$ in $L^2(\mathbb{R})$ as $k \to \infty$. Thus,

$$0 < \frac{\Theta}{2} \leq ||P_{[B_1,\cdots,B_{n_k}]}f_{n_k}||_{L^2(\mathbb{R})} \leq ||P_{[B]}f_{n_k}||_{L^2(\mathbb{R})} \to ||P_{[B]}\tilde{y}||_{L^2(\mathbb{R})} = 0.$$

where the limit is taken as $k \to \infty$. This contradiction shows that the orthonormal set B is complete, and hence it is an orthonormal basis for $L^2(\mathbb{R})$. □

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