AN OPTIMAL EXAMPLE FOR THE BALIAN-LOW UNCERTAINTY PRINCIPLE

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Abstract. We analyze the time-frequency concentration of the Gabor orthonormal basis $\mathcal{G}(f,1,1)$ constructed by Høholdt, Jensen, and Justesen. We prove that their window function f has near optimal time and frequency localization with respect to a non-symmetric version of the Balian-Low Theorem. In particular, we show that if $(p,q) = (3/2, 3)$, then $\int |t|^{p-\epsilon}|f(t)|^2 dt < \infty$ and $\int |\gamma|^{q-\epsilon}|\widehat{f}(\gamma)|^2 d\gamma < \infty$, for $0 < \epsilon \leq 3/2$, but that both integrals are infinite if $\epsilon = 0$.

1. Introduction

Given a square integrable function $g \in L^2(\mathbb{R})$ and constants $a, b > 0$, the associated *Gabor system*, $\mathcal{G}(g,a,b) = \{g_{m,n}\}_{m,n\in\mathbb{Z}}$, is defined by

$$g_{m,n}(t) = e^{2\pi i a m t} g(t - bn).$$

Gabor systems are of considerable interest for their ability to give frame decompositions for many function spaces, [18], [13], [14], [4]. A collection $\{e_n\}_{n\in\mathbb{Z}} \subseteq L^2(\mathbb{R})$ is a *frame* for $L^2(\mathbb{R})$ if there exist constants $0 < A \leq B < \infty$ such that

$$\forall f \in L^2(\mathbb{R}), \quad A||f||^2_{L^2(\mathbb{R})} \leq \sum_{n\in\mathbb{Z}} |\langle f, e_n\rangle|^2 \leq B||f||^2_{L^2(\mathbb{R})}.$$

If $\mathcal{G}(g,a,b)$ is a frame for $L^2(\mathbb{R})$, we shall refer to it as a *Gabor frame* for $L^2(\mathbb{R})$; if it is an orthonormal basis for $L^2(\mathbb{R})$ we refer to it as a *Gabor orthonormal basis* for $L^2(\mathbb{R})$.

A key property of Gabor systems is the fact that one can construct Gabor frames, $\mathcal{G}(g,a,b)$, for $L^2(\mathbb{R})$ such that the *window function* g has excellent time and frequency localization. For example, if $0 < ab < 1$ and $g(t) = e^{-t^2}$, then $\mathcal{G}(g,a,b)$ is an oversampled Gabor frame for $L^2(\mathbb{R})$, e.g., [18], Chapter 7. Overcompleteness is a very important part of such well localized constructions and can provide robustness and numerical stability in applied settings. On the other hand, if $g \in L^2(\mathbb{R})$, and $\mathcal{G}(g,a,b)$ is an orthonormal basis for $L^2(\mathbb{R})$, then one must have $ab = 1$, e.g., [18], Corollary 7.5.2. If one wishes to construct Gabor orthonormal bases, i.e., non-redundant frames, then there are severe restrictions on the window function's time and frequency localization. The Balian-Low Theorem makes this precise. We use

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the Fourier transform defined by $\widehat{g}(\gamma) = \int g(t)e^{-2\pi i\gamma t}dt$, where our convention is that the integral without specific limits denotes the integral over $\mathbb{R}$.

Theorem 1.1 (Balian-Low)**.** *Let $g \in L^2(\mathbb{R})$. If*

$$\int |t|^2|g(t)|^2dt < \infty \quad \textit{and} \quad \int |\gamma|^2|\widehat{g}(\gamma)|^2d\gamma < \infty,$$

then $\mathcal{G}(g,1,1)$ is not an orthonormal basis for $L^2(\mathbb{R})$.

The Balian-Low Theorem has undergone numerous extensions and generalizations since the early references [2], [23], [3], [8]. For example, it holds in higher dimensions for rather general time-frequency lattices, and also holds if one replaces "orthonormal basis" by "Riesz basis". For recent work related to the Balian-Low Theorem see [1], [5], [6], [7], [9], [10], [16], [19], [11]. The issue of sharpness or optimality in the Balian-Low Theorem was investigated in [6]. There, it was shown that the following result holds true.

Theorem 1.2. *If $\frac{1}{p} + \frac{1}{q} = 1$, where $1 < p, q < \infty$, and $d > 2$, then there exists a function $g \in L^2(\mathbb{R})$ such that $\mathcal{G}(g,1,1)$ is an orthonormal basis for $L^2(\mathbb{R})$ and*

$$\int \frac{1+|t|^p}{\log^d(2+|t|)}|f(t)|^2dt < \infty \quad \textit{and} \quad \int \frac{1+|\gamma|^q}{\log^d(2+|\gamma|)}|\widehat{f}(\gamma)|^2d\gamma < \infty.$$

Letting $(p,q) = (2,2)$ in Theorem 1.2 shows how to construct Gabor orthonormal bases which are essentially optimally localized with respect to the Balian-Low Theorem. In particular, the bases constructed come within a logarithmic factor of satisfying the forbidden localization hypotheses of the Balian-Low Theorem.

Since Theorem 1.2 also constructs Gabor orthonormal bases for values of (p,q) other than $(2,2)$, it is natural to ask whether there are versions of the Balian-Low Theorem for the weights (t^p, γ^q). The best that is known is the following.

Theorem 1.3. *Suppose $\frac{1}{p} + \frac{1}{q} = 1$ with $1 < p < \infty$ and let $\epsilon > 0$. If*

$$\int |t|^{(p+\epsilon)}|g(t)|^2dt < \infty \quad \textit{and} \quad \int |\gamma|^{(q+\epsilon)}|\widehat{g}(\gamma)|^2d\gamma < \infty,$$

then $\mathcal{G}(g,1,1)$ is not an orthonormal basis for $L^2(\mathbb{R})$.

The above theorem follows by combining Theorem 4.4 of [12] and Theorem 1 in [17]. By the Balian-Low Theorem, one may set $\epsilon = 0$ if $(p,q) = (2,2)$. A version of the Balian-Low Theorem for the case $(p,q) = (1,\infty)$ is given in [7].

2. Overview

Theorem 1.2 constructively produces Gabor orthonormal bases which are almost optimally localized with respect to the Balian-Low Theorem and Theorem 1.3. However, these bases do not have simple expressions. The main aim of this paper is to study the elegant Gabor orthonormal basis constructed by Høholdt, Jensen, and Justesen in [22], and to show that it is almost optimally localized with respect to Theorem 1.3 for a certain choice of (p,q). Their basis has a simpler, more explicit, form than those in [6], and gives insight into other ingredients needed for constructing well localized Gabor bases. The key ingredients in the constructions in [22] and [6] are functions which possess unimodular Zak transforms with small singular supports. For perspective, we remark that [21] provides several examples of functions with Zak transforms with few zeros which are used to construct tight Gabor frames. These examples could provide further insight into the study of optimality in the Balian-Low Theorem, and merit future investigation.

The remainder of the paper is organized as follows. In Section 3, we recall the basis of Høholdt, Jensen, and Justesen, and state our main result Theorem 3.3. In Section 4 we prove the time localization estimates for the basis, and in Section 5 we prove the frequency localization estimates. We end with some relevant remarks in Section 6.

3. The Gabor basis of Høholdt, Jensen, and Justesen

The Zak transform is an important tool in the analysis and construction of Gabor systems, e.g., [18], Chapter 8. Given $g \in L^2(\mathbb{R})$, the *Zak transform* is formally defined by

$$\forall (t,\gamma) \in \mathbb{R} \times \mathbb{R}, \quad Zg(t,\gamma) = \sum_{n \in \mathbb{Z}} g(t-n) e^{2\pi i n \gamma}.$$

With the above definition, the Zak transform satisfies the *quasiperiodicity relations*:

$$\forall k \in \mathbb{Z}, \quad Zf(x, \gamma + k) = Zf(x,\gamma),$$

and

$$\forall k \in \mathbb{Z}, \quad Zf(x+k, \gamma) = Zf(x,\gamma)\ e^{2\pi i k \gamma},$$

e.g., see [18], Section 8.2. Thus, the Zak transform Zf of a function $f \in L^2(\mathbb{R})$ is a locally square integrable function defined on all of $\mathbb{R}^2$ and is uniquely determined by its values on $Q \equiv [0,1)^2$. Therefore, Z defines a unitary operator from $L^2(\mathbb{R})$ to $L^2(Q)$, and its inverse $Z^{-1} : L^2(Q) \to L^2(\mathbb{R})$ is formally given by

$$\forall t \in \mathbb{R}, \quad (Z^{-1}F)(t) = \int_0^1 F(t,\gamma) d\gamma.$$

The utility of the Zak transform for constructing Gabor bases stems from the following result, e.g., [18], Corollary 8.3.2, which forms the foundation for the constructions in both [6] and [22], cf., [21].

Theorem 3.1. *Let $g \in L^2(\mathbb{R})$. Then, $\mathcal{G}(g,1,1)$ is an orthonormal basis for $L^2(\mathbb{R})$ if and only if $|Zg(t,\gamma)| = 1$ for a.e. $(t,\gamma) \in Q$.*

This shows that constructing Gabor orthonormal bases is equivalent to constructing unimodular functions on $L^2(Q)$. Høholdt, Jensen, and Justesen consider the function $F \in L^2(Q)$ defined by

$$\forall (t,\gamma) \in Q, \quad F(t,\gamma) = \frac{1 + \alpha(t) e^{2\pi i \gamma}}{1 + \alpha(t) e^{-2\pi i \gamma}}, \tag{3.1}$$

where $\alpha : [0,1] \to [0,1]$ is a measurable function. In [22], the function α was chosen as $\alpha(t) = \sin(\frac{\pi}{2} t)$, since this was shown to minimize $\int |\gamma|^2 |(\widehat{Z^{-1}F})(\gamma)|^2 d\gamma$.

Definition 3.2. *Let $f \in L^2(\mathbb{R})$ be the function defined by (3.1), where*

$$f = Z^{-1} F \quad \text{and} \quad \alpha(t) = \sin(\frac{\pi}{2} t). \tag{3.2}$$

It was proven in [22] *that* $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, *and that* f *is explicitly defined by*
(3.3)
$$f(t) = \begin{cases} 0, & \text{if } t \in (-\infty, -1], \\ \sin(\frac{\pi}{2}(t+1)), & \text{if } t \in (-1, 0], \\ (-1)^n \cos^2(\frac{\pi}{2}(t-n)) \sin^n(\frac{\pi}{2}(t-n)), & \text{if } t \in (n, n+1],\ n = 0, 1, 2, \ldots. \end{cases}$$

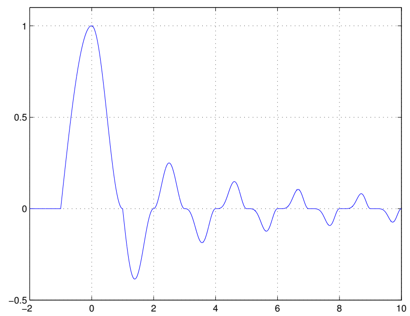

Figure 1. Graph of the function f.

It is easy to verify that $|Zf(t,\gamma)| = |F(t,\gamma)| = 1$ for a.e. $(t,\gamma) \in Q$, and hence $\mathcal{G}(f,1,1)$ is an orthonormal basis for $L^2(\mathbb{R})$. We may now state our main result as follows.

Theorem 3.3. *Let* $f \in L^2(\mathbb{R})$ *be the window function defined by Definition 3.2. For every* $0 < \epsilon \leq 3/2$, f *satisfies*

$$\int |t|^{3/2-\epsilon} |f(t)|^2 dt < \infty \quad \text{and} \quad \int |\gamma|^{3-\epsilon} |\widehat{f}(\gamma)|^2 d\gamma < \infty. \tag{3.4}$$

Moreover,

$$\int |t|^{3/2} |f(t)|^2 dt = \infty \quad \text{and} \quad \int |\gamma|^{3} |\widehat{f}(\gamma)|^2 d\gamma = \infty. \tag{3.5}$$

In particular, the Gabor orthonormal basis $\mathcal{G}(f,1,1)$ is almost optimally localized with respect to Theorem 1.3 with $(p,q) = (3/2, 3)$.

4. Time localization estimates

In this section we derive the time localization estimates in Theorem 3.3.

Theorem 4.1. *Let* $f \in L^2(\mathbb{R})$ *be the function defined in Definition 3.2 and let* $a > 0$. *Then*

$$\int |t|^a |f(t)|^2 dt < \infty \quad \text{if and only if} \quad a < 3/2.$$

Proof. A direct calculation shows that for $n = 0, 1, 2, \cdots$

$$\int_n^{n+1} |f(t)|^2 dt = \int_0^1 \cos^4(\frac{\pi}{2}t) \sin^{2n}(\frac{\pi}{2}t) dt = \frac{2}{\pi} \left(\frac{3}{4n^2 + 12n + 8} \right) \int_0^{\pi/2} \sin^{2n} u \, du. \tag{4.1}$$

One can also calculate that

$$\frac{2}{\pi}\int_0^{\pi/2} \sin^{2n} u \, du = \frac{(1)(3)(5)(7)\cdots(2n-1)}{(2)(4)(6)(8)\cdots(2n)} \equiv P_n. \tag{4.2}$$

By taking the natural log of P_n and using Taylor approximations for $\ln(1-x)$ near $x = 0$ to estimate the resulting sum, it is straightforward to show that

$$P_n \sim \frac{1}{\sqrt{n}}. \tag{4.3}$$

Equivalently, we could use Stirling's formula for Gamma function to show (4.3). Here and subsequently $A \sim B$ means that $A \lesssim B \lesssim A$, where $A \lesssim B$, in turn, means that there exists an absolute constant C such that $A \leq CB$. When necessary, we shall point out any dependence of the implicit constants on other parameters. Therefore,

$$\begin{aligned}\int_1^\infty |t|^a |f(t)|^2 dt &\geq \sum_{n=1}^\infty n^a \int_n^{n+1} |f(t)|^2 dt \\ &= \sum_{n=1}^\infty n^a \left(\frac{3}{4n^2+12n+8}\right) P_n \gtrsim \sum_{n=1}^\infty n^{a-5/2}.\end{aligned}$$

In particular,

$$a \geq 3/2 \implies \int |t|^a |f(t)|^2 dt = \infty.$$

Also, using (4.1), (4.2), and (4.3), we obtain the estimate,

$$\int_1^\infty |t|^a |f(t)|^2 dt \lesssim \sum_{n=1}^\infty \frac{(n+1)^a}{n^{5/2}}.$$

Since f is bounded on $[-1, 1]$, and $f = 0$ on $(-\infty, -1)$, it follows that

$$0 < a < 3/2 \implies \int |t|^a |f(t)|^2 dt < \infty.$$

□

5. Frequency localization estimates

In this section we derive the frequency localization estimates in Theorem 3.3.

Theorem 5.1. *Let $f \in L^2(\mathbb{R})$ be the function defined in Definition 3.2 and let $0 < a$. Then*

$$\int |\gamma|^a |\widehat{f}(\gamma)|^2 d\gamma < \infty \quad \textit{if and only if} \quad a < 3.$$

It will be convenient to view Theorem 5.1 in terms of Sobolev spaces. Given $s > 0$, the *homogeneous Sobolev space* of order s, denoted by $\dot{H}^s(\mathbb{R})$, consists of all $g \in L^2(\mathbb{R})$ such that $||g||^2_{\dot{H}^s(\mathbb{R})} \equiv \int |\gamma|^{2s} |\widehat{g}(\gamma)|^2 d\gamma < \infty$. For later convenience, we also define $\langle f, g\rangle_{\dot{H}^s(\mathbb{R})} = \int |\gamma|^{2s} \widehat{f}(\gamma)\overline{\widehat{g}(\gamma)} d\gamma$. Theorem 5.1 now says that $0 < s < 3/2$ implies $f \in \dot{H}^s(\mathbb{R})$, and that $s \geq 3/2$ implies $f \notin \dot{H}^s(\mathbb{R})$. The following result, e.g., [24], Chapter 8, gives a useful alternate characterization of $\dot{H}^s(\mathbb{R})$. It is used in the proof of Lemma 5.6.

Lemma 5.2. *If $0 < s < 2$ and $f \in \dot{H}^s(\mathbb{R})$, then there exists $C_s > 0$ such that*

$$||f||^2_{\dot{H}^s(\mathbb{R})} = C_s \int\int \frac{|f(x+t)+f(x-t)-2f(x)|^2}{|t|^{1+2s}} dxdt.$$

Lemma 5.2 can be proven by applying Parseval's Theorem to the inner integral. A similar calculation gives the following result used in the proof of Lemma 5.8.

Lemma 5.3. *If $0 < s < 2$ and $f, g \in \dot{H}^s(\mathbb{R})$ then there exists $C_s > 0$ such that*

$$\langle f, g\rangle_{\dot{H}^s(\mathbb{R})} = C_s \int\int \frac{(f(x+t)+f(x-t)-2f(x))\overline{(g(x+t)+g(x-t)-2g(x))}}{|t|^{2s+1}} dxdt.$$

We shall use the following lemma directly in Theorem 5.1.

Lemma 5.4. *If $3 \leq a$ then $f \notin \dot{H}^{a/2}(\mathbb{R})$.*

Proof. If $3 \leq a < 4$ then for $0 < \eta$ small

$$||f||^2_{\dot{H}^{a/2}(\mathbb{R})} \sim \int\int \frac{|f(x+t)+f(x-t)-2f(x)|^2}{|t|^{1+a}} dxdt \geq \int_0^\eta \int_{-t}^0 \frac{\sin^2(\frac{\pi}{2}(x+t))}{|t|^{1+a}} dxdt$$
$$\gtrsim \int_0^\eta \int_{-t}^0 \frac{(x+t)^2}{|t|^{1+a}} dxdt \gtrsim \int_0^\eta \frac{t^3}{|t|^{1+a}} dt = \infty.$$

Since $f \in L^1(\mathbb{R})$, it now also follows that $f \notin \dot{H}^{a/2}(\mathbb{R})$ for all $3 \leq a$. □

We now prove that $0 < a < 3$ implies $f \in \dot{H}^{a/2}$. Since this is more involved than our prior estimates, we split it up into several lemmas (Lemmas 5.9 and 5.10). Lemma 5.5 is used in Lemma 5.6, which, in turn, is used in the proof of Lemma 5.7. Lemmas 5.7 and 5.8 allow us to prove Lemma 5.9.

Lemma 5.5. *For $n \geq 3$, let $f_n(t) = \mathbf{1}_{(n,n+1]}(t) f(t)$, where $\mathbf{1}_S(t)$ denotes the characteristic function of a set $S \subseteq \mathbb{R}$. The functions f_n have the following properties:*

(1) *f_n is continuous and differentiable on $\mathbb{R}$.*
(2) *$f_n''(t)$ exists for all $t \in \mathbb{R}\backslash\{n+1\}$.*
(3) *$||f_n||^2_{L^2(\mathbb{R})} \lesssim 1/n^{5/2}$ and $||f_n||^{1/2}_{L^{1/2}(\mathbb{R})} \lesssim 1/n$.*
(4) *If $0 < \delta < 3/2$, then $||f_n||^{1/2+\delta}_{L^{1/2+\delta}(\mathbb{R})} \lesssim 1/n^{1+\delta}$.*
(5) *$\forall t \in \mathbb{R}\backslash\{n+1\}$, $|f_n''(t)| \lesssim 1$.*

The implicit constants in (3) and (4) are independent of n, and the implicit constant in (5) is independent of t and n.

Proof. The first two items can be verified by direct calculations. The estimate for $||f_n||^2_{L^2(\mathbb{R})}$ in (3) has already been done in the proof of Theorem 4.1. In fact,

$$||f_n||^2_{L^2(\mathbb{R})} = \left(\frac{3}{4n^2+12n+8}\right) P_n \lesssim \frac{1}{n^{2.5}}.$$

The estimate for $||f_n||^{1/2}_{L^{1/2}(\mathbb{R})}$ in (3) holds since

$$||f_n||^{1/2}_{L^{1/2}(\mathbb{R})} = \int_n^{n+1} \cos(\frac{\pi}{2}(t-n)) \sin^{\frac{n}{2}}(\frac{\pi}{2}(t-n)) dt = \frac{4}{\pi(n+2)}.$$

The fourth item follows from (3) and the following standard interpolation formula, e.g, [15], Proposition 6.10,

$$||f||_{L^q(\mathbb{R})} \leq ||f||^{\lambda}_{L^p(\mathbb{R})} ||f||^{1-\lambda}_{L^r(\mathbb{R})},$$

where

$$0 < p < q < r \leq \infty \text{ and } \lambda = \frac{1/q - 1/r}{1/p - 1/r}.$$

To prove (5) first note that, for $n < t < n+1$,

$$\begin{aligned} f_n''(t-n) =&(-1)^n \left(-(5n+2)\frac{\pi^2}{4} \sin^n(\frac{\pi}{2}t) \cos^2(\frac{\pi}{2}t) \right. \\ &\left. + 2\frac{\pi^2}{4} \sin^{n+2}(\frac{\pi}{2}t) + \frac{\pi^2}{4} n(n-1) \cos^4(\frac{\pi}{2}t) \sin^{n-2}(\frac{\pi}{2}t) \right). \end{aligned}$$

Therefore, $f_n''(t-n) = (-1)^n h_n(u)$, where $u = \sin^2(\frac{\pi}{2}(t-n))$, and

$$h_n(u) = \frac{\pi^2}{4} u^{n/2-1} \left[2u^2 - (5n+2)u(1-u) + n(n-1)(1-u)^2 \right].$$

Straightforward, but tedious, calculations show that $|h_n(t)| \leq C$ on $(n, n+1)$, for some constant C independent of t and n. Since $f_n''(t) = 0$ on $\mathbb{R}\backslash(n, n+1]$, we conclude that $|f_n''(t)| \lesssim 1$ on $\mathbb{R}\backslash\{n+1\}$. □

Lemma 5.6. *Assume* $0 < a < 3$*, and let* $\epsilon = 3 - a$*. Then*

$$\forall n \geq 3, \quad ||f_n||^2_{\dot{H}^{a/2}(\mathbb{R})} \lesssim \frac{1}{n^{1+\epsilon/4}}.$$

The implicit constant is independent of n*.*

Proof. We shall estimate $||f_n||^2_{\dot{H}^{a/2}(\mathbb{R})}$ by using the double integral in Lemma 5.2. Let $B = \{t \in \mathbb{R} : |t| < 1\}$, and note that

$$\int_{\mathbb{R}\backslash B} \int_{\mathbb{R}} \frac{|f_n(x+t) + f_n(x-t) - 2f_n(x)|^2}{|t|^{a+1}} dxdt \lesssim ||f_n||^2_{L^2(\mathbb{R})} \int_{\mathbb{R}\backslash B} \frac{1}{|t|^{a+1}} dt \lesssim \frac{1}{n^{5/2}}.$$

It remains for us to estimate

$$\int_B \int_{\mathbb{R}} \frac{|f_n(x+t) + f_n(x-t) - 2f_n(x)|^2}{|t|^{a+1}} dxdt.$$

We write this as the sum of two integrals, over $[0,1]\times\mathbb{R}$ and $[-1,0]\times\mathbb{R}$, respectively. Since the estimates for both integrals are similar, it suffices to consider the first, which, in turn, is estimated by breaking it up into the following four integrals.

$$I_1 = \int_0^1 \int_{-\infty}^{n-t}, \; I_2 = \int_0^1 \int_{n-t}^{n+1-t}, \quad I_3 = \int_0^1 \int_{n+1-t}^{n+1+t}, \; I_4 = \int_0^1 \int_{n+1+t}^{\infty}.$$

First, note that the support properties of f_n imply that $I_1 = 0$ and $I_4 = 0$.

Next note that if $x+t, x-t, x$ are all less than $n+1$, then Lemma 5.5 and the mean value theorem imply

$$|f_n(x+t) + f_n(x-t) - 2f_n(x)| \lesssim |t|^2, \tag{5.1}$$

where the implicit constant is independent of x, t and n. To estimate I_2 note that, by (5.1) and Lemma 5.5,

$$\begin{aligned} W_2(t) &\equiv \int_{n-t}^{n+1-t} |f_n(x+t)+f_n(x-t)-2f_n(x)|^2 dx \\ &\lesssim |t|^{3-\epsilon/2} \int_{n-t}^{n+1-t} |f_n(x+t)+f_n(x-t)-2f_n(x)|^{1/2+\epsilon/4} dx \\ &\lesssim |t|^{3-\epsilon/2} \; ||f_n||^{1/2+\epsilon/4}_{L^{1/2+\epsilon/4}(\mathbb{R})} \lesssim \frac{|t|^{3-\epsilon/2}}{n^{1+\epsilon/4}}. \end{aligned}$$

It now follows that

$$I_2 = \int_0^1 \frac{W_2(t)}{|t|^{a+1}} dt = \int_0^1 \frac{W_2(t)}{|t|^{4-\epsilon}} dt \lesssim 1/n^{1+\epsilon/4} \int_0^1 \frac{1}{|t|^{1-\epsilon/2}} dt \lesssim 1/n^{1+\epsilon/4}.$$

To estimate I_3, define

$$W_3(t) = \int_{n+1-t}^{n+1+t} |f_n(x+t)+f_n(x-t)-2f_n(x)|^2 dx.$$

Note that by the definition of f_n and its support properties,

$$\begin{aligned} W_3(t) &= \int_{n+1-t}^{n+1+t} |f_n(x-t)-2f_n(x)|^2 dx \lesssim \int_{n+1-2t}^{n+1} |f_n(x)|^2 dx \\ &\lesssim \int_{n+1-2t}^{n+1} \cos^4\left(\frac{\pi}{2}(x-n)\right) dx \lesssim |t|^5. \end{aligned}$$

Moreover, we also have

$$W_3(t) \lesssim ||f_n||^2_{L^2(\mathbb{R})} \lesssim \frac{1}{n^{5/2}}.$$

Thus, in order to estimate W_3 we may use the fact that for $x, y > 0$ and $\alpha \in [0,1]$, $\min\{x,y\} \le x^\alpha y^{1-\alpha}$. When $\alpha = (6-\epsilon)/10$, we obtain:

$$W_3(t) \lesssim \frac{|t|^{3-\epsilon/2}}{n^{1+\epsilon/4}}.$$

Thus

$$I_3 = \int_0^1 \frac{W_3(t)}{|t|^{a+1}} dt = \int_0^1 \frac{W_3(t)}{|t|^{4-\epsilon}} dt \lesssim 1/n^{1+\epsilon/4}.$$

□

Lemma 5.7. *Assume* $0 < a < 3$ *and let* $\epsilon = 3 - a$. *If* $3 \le m, n$, *and* $|m-n| = 1$, *then*

$$|\langle f_n, f_m \rangle_{\dot{H}^{a/2}(\mathbb{R})}| \lesssim \frac{1}{n^{1+\epsilon/4}}.$$

The implicit constant is independent of n *and* m.

Proof. Without loss of generality assume $m = n+1$. It follows from Lemma 5.6 that

$$\begin{aligned} |\langle f_n, f_m \rangle_{\dot{H}^{a/2}(\mathbb{R})}| &\le ||f_n||_{\dot{H}^{a/2}(\mathbb{R})} ||f_m||_{\dot{H}^{a/2}(\mathbb{R})} \\ &\lesssim \left(\frac{1}{n^{1+\epsilon/4}}\right)^{\frac{1}{2}} \left(\frac{1}{m^{1+\epsilon/4}}\right)^{\frac{1}{2}} \le \frac{1}{n^{1+\epsilon/4}}. \end{aligned}$$

□

Lemma 5.8. *Let* $0 < a$*. If* $3 \le m, n$ *and* $1 < |m-n|$*, then*

$$|\langle f_n, f_m \rangle_{\dot{H}^{a/2}(\mathbb{R})}| \lesssim \frac{1}{|m-n|^a |n|^{5/4} |m|^{5/4}}.$$

The implicit constant is independent of m *and* n*.*

Proof. Without loss of generality assume $0 < n < m-1$. Let $S_{m,n} = \{t \in \mathbb{R} : |t| > (m-n-1)/2\}$ and let

$$F_m(x,t) = f_m(x+t) + f_m(x-t) - 2f_m(x).$$

Note that

$$t \notin S_{m,n} \implies F_m(x,t)F_n(x,t) = 0.$$

Also,

$$\int |F_m(x,t)F_n(x,t)| dx \lesssim ||f_n||_{L^2(\mathbb{R})} ||f_m||_{L^2(\mathbb{R})}.$$

Therefore, by Lemma 5.3,

$$\begin{aligned} |\langle f_n, f_m \rangle_{\dot{H}^{a/2}(\mathbb{R})}| &\le \int_{S_{m,n}} \int_{\mathbb{R}} \frac{|F_m(x,t)F_n(x,t)|}{|t|^{a+1}} dx dt \\ &\lesssim ||f_n||_{L^2(\mathbb{R})} ||f_m||_{L^2(\mathbb{R})} \int_{S_{m,n}} \frac{1}{|t|^{a+1}} dt \\ &\lesssim \frac{||f_n||_{L^2(\mathbb{R})} ||f_m||_{L^2(\mathbb{R})}}{|m-n|^a} \lesssim \frac{1}{n^{5/4} m^{5/4} |m-n|^a}. \end{aligned}$$

□

To estimate the norm $||f||_{\dot{H}^{a/2}(\mathbb{R})}$ we first break f up into the two parts $F_1(t) = f(t)\mathbf{1}_{(-1,3]}(t)$ and $F_2(t) = f(t)\mathbf{1}_{(3,\infty)}(t)$. Since $f = 0$ on $(-\infty, -1]$ we have $f = F_1 + F_2$. We have the following estimate for F_2.

Lemma 5.9. *If* $0 < a < 3$ *then* $||F_2||^2_{\dot{H}^{a/2}(\mathbb{R})} < \infty$*.*

Proof. Let $\epsilon = 3 - a$, and note that $F_2 = \sum_{n=3}^{\infty} f_n$. Define

$$S_1 = \{(m,n) \in \mathbb{Z}^2 : m, n \ge 3 \text{ and } |m-n| = 1\}$$

and

$$S_2 = \{(m,n) \in \mathbb{Z}^2 : m, n \ge 3 \text{ and } |m-n| > 1\}.$$

By Lemmas 5.6, 5.7, and 5.8 we have

$$\begin{aligned} ||F_2||^2_{\dot{H}^{a/2}(\mathbb{R})} &= \left|\left| \sum_{n=3}^{\infty} f_n \right|\right|^2_{\dot{H}^{a/2}(\mathbb{R})} \le \sum_{m=3}^{\infty} \sum_{n=3}^{\infty} |\langle f_m, f_n \rangle_{\dot{H}^{a/2}(\mathbb{R})}| \\ &= \sum_{n=3}^{\infty} ||f_n||^2_{\dot{H}^{a/2}(\mathbb{R})} + \sum_{(m,n)\in S_1} |\langle f_n, f_m \rangle_{\dot{H}^{a/2}(\mathbb{R})}| + \sum_{(m,n)\in S_2} |\langle f_n, f_m \rangle_{\dot{H}^{a/2}(\mathbb{R})}| \\ &\lesssim \sum_{n=1}^{\infty} \frac{1}{n^{1+\epsilon/4}} + \sum_{n=1}^{\infty} \frac{1}{n^{1+\epsilon/4}} + \sum_{(m,n)\in S_2} \frac{1}{n^{5/4} m^{5/4} |m-n|^{2a}} < \infty. \end{aligned}$$

□

In view of Lemma 5.9, and since

$$||f||_{\dot{H}^{a/2}(\mathbb{R})} \leq ||F_1||_{\dot{H}^{a/2}(\mathbb{R})} + ||F_2||_{\dot{H}^{a/2}(\mathbb{R})},$$

it only remains to estimate $||F_1||_{\dot{H}^{a/2}(\mathbb{R})}$. Note that by the definition of f, F_1 is compactly supported, continuous on $\mathbb{R}$, and infinitely differentiable away from $x_1 = -1, x_2 = 0, x_3 = 1, x_4 = 2$, and $x_5 = 3$. Moreover the first derivative of F_1 also exists at x_2, x_3, x_5. However, the second derivative of F_1 does not exist at any of the points x_1, x_2, x_3, x_4, x_5.

It therefore suffices to estimate $||\varphi_j F_1||_{\dot{H}^{a/2}(\mathbb{R})}$ for $j = 1, 2, 3, 4, 5$, where φ_j is an infinitely differentiable function satisfying

$$\varphi_j(x) = 1 \text{ for } |x - x_j| < 2\nu, \quad \text{and} \quad \varphi_j(x) = 0 \text{ for } |x - x_j| > 4\nu,$$

with $0 < \nu$ sufficiently small.

We present a proof that is analogous to our previous estimates and uses Lemma 5.2. Alternately, one can proceed more directly and use an argument involving integration by parts.

Lemma 5.10. *Let $0 < a < 3$ and let $\varphi_j F_1$ be as above for $j = 1, 2, 3, 4, 5$. Then*

$$||\varphi_j F_1||_{\dot{H}^{a/2}(\mathbb{R})} < \infty, \quad j = 1, 2, 3, 4, 5.$$

Consequently, $||F_1||_{\dot{H}^{a/2}(\mathbb{R})} < \infty$.

Proof. We shall only show the estimate for $||\varphi_3 F_1||_{\dot{H}^{a/2}(\mathbb{R})}$, since the other four estimates proceed along similar lines.

Let $h(t) = (\varphi_3 F_1)(t - 1)$. We need to estimate the double integral

$$||\varphi_3 F_1||^2_{\dot{H}^{a/2}(\mathbb{R})} = ||h||^2_{\dot{H}^{a/2}(\mathbb{R})} \sim \int\int \frac{|h(x+t) + h(x-t) - 2h(x)|^2}{|t|^{1+a}} dx dt.$$

Let ν be as in the definition of φ_3 above, and note that if $B_\nu = \{t \in \mathbb{R} : |t| < \nu\}$ then

$$\int_{\mathbb{R}\backslash B_\nu} \int_{\mathbb{R}} \frac{|h(x+t) + h(x-t) - 2h(x)|^2}{|t|^{1+a}} dx dt \lesssim ||h||^2_{L^2(\mathbb{R})} \int_{\mathbb{R}\backslash B_\nu} \frac{1}{|t|^{1+a}} dt < \infty.$$

Next note that $h(t)$ is infinitely differentiable away from $t = 0$, and has bounded first and second derivatives on $\mathbb{R}\backslash\{0\}$. Therefore, if $x+t, x-t$ and x are all positive, or all negative, then it follows from the mean value theorem that

$$|h(x+t) + h(x-t) - 2h(x)| \lesssim |t|^2. \tag{5.2}$$

Likewise, if $x + t$ and x are both positive or both negative, then

$$|h(x+t) - h(x)| \lesssim |t|. \tag{5.3}$$

The implicit constants in (5.2) and (5.3) are independent of x and t.

To estimate the remaining integral

$$\int_{-\nu}^{\nu} \int_{\mathbb{R}} \frac{|h(x+t) + h(x-t) - 2h(x)|^2}{|t|^{1+a}} dx dt,$$

we break this integral up over the domains $[\nu, 0] \times \mathbb{R}$ and $[-\nu, 0] \times \mathbb{R}$. Since both integrals are similar we only show estimates for the first, which, in turn, we estimate by considering the integrals

$$J_1 = \int_0^\nu \int_t^\infty, \quad J_2 = \int_0^\nu \int_0^t, \quad J_3 = \int_0^\nu \int_{-t}^0, \quad J_4 = \int_0^\nu \int_{-\infty}^{-t}.$$

It follows from (5.2), and the compact support of h, that

$$J_1 = \int_0^\nu \int_t^\infty \frac{|h(x+t)+h(x-t)-2h(x)|^2}{|t|^{1+a}} dxdt$$
$$= \int_0^\nu \int_t^{5\nu} \frac{|h(x+t)+h(x-t)-2h(x)|^2}{|t|^{1+a}} dxdt \lesssim \int_0^\nu \frac{|t|^4}{|t|^{1+a}} dt < \infty.$$

The estimate for $J_4 < \infty$ is similar.

To estimate J_2, note that $J_2 \lesssim J_{2,1} + J_{2,2}$ where

$$J_{2,1} = \int_0^\nu \int_0^t \frac{|h(x+t)-h(x)|^2}{|t|^{a+1}} dxdt \quad \text{and} \quad J_{2,2} = \int_0^\nu \int_0^t \frac{|h(x-t)-h(x)|^2}{|t|^{a+1}} dxdt.$$

It follows from (5.3) that

$$J_{2,1}(t) \lesssim \int_0^\nu \int_0^t \frac{|t|^2}{|t|^{1+a}} dxdt = \int_0^\nu \frac{|t|^3}{|t|^{a+1}} dt < \infty.$$

Next, define

$$H(x,t) = \cos^2(\frac{\pi}{2}(x-t-1)) + \cos^2(\frac{\pi}{2}x)\sin(\frac{\pi}{2}x)$$
$$= \sin^2(\frac{\pi}{2}(x-t)) + \cos^2(\frac{\pi}{2}x)\sin(\frac{\pi}{2}x),$$

and note that

$$\int_0^t |H(x,t)|^2 dx \lesssim \int_0^t |(x-t)^2|^2 dx + \int_0^t |x|^2 dx \lesssim |t|^3.$$

It now follows that

$$J_{2,2} = \int_0^\nu \int_0^t \frac{|H(x,t)|^2}{|t|^{a+1}} dxdt \lesssim \int_0^\nu \frac{|t|^3}{|t|^{a+1}} dt < \infty.$$

Therefore,

$$J_2 \lesssim J_{2,1} + J_{2,2} < \infty.$$

By using calculations, similar to those used to deal with J_2, one can also show that $J_3 < \infty$. We can now conclude that $||\varphi_3 F_1||_{\dot{H}^{a/2}(\mathbb{R})} = ||h||_{\dot{H}^{a/2}(\mathbb{R})} < \infty$.

The estimates for $||\varphi_j F_2||_{\dot{H}^{a/2}(\mathbb{R})}$, $j = 1, 2, 4, 5$, proceed along similar lines as above. □

Proof of Theorem 5.1. Combining Lemmas 5.9 and 5.10 shows that if $0 < a < 3$ then

$$\left(\int |\gamma|^a |\widehat{f}(\gamma)|^2 d\gamma\right)^{1/2} = ||f||_{\dot{H}^{a/2}(\mathbb{R})} \leq ||F_1||_{\dot{H}^{a/2}(\mathbb{R})} + ||F_2||_{\dot{H}^{a/2}(\mathbb{R})} < \infty.$$

Together with Lemma 5.4 this completes the proof of Theorem 5.1. □

Our main result Theorem 3.3 now follows by combining Theorem 4.1 and Theorem 5.1.

6. Concluding remarks

1. Throughout this remark we shall assume that Zak transforms have been quasiperiodically extended to $L^2_{\text{loc}}(\mathbb{R}^2)$. A key idea in the construction of [6] was to choose the Gabor window function g so that $|Zg| = 1$ *a.e.* and such that Zg has minimal singular support. In fact, the function Zg used in [6] was locally C^∞ on $\mathbb{R}^2$ except at one point in each square $S_{j,k} = (j, j+1] \times (k, k+1], j, k \in \mathbb{Z}$.

By comparison, one can show that the quasiperiodic extension of Høholdt, Jensen, and Justesen's function $F = Zf$ defined in (3.1) is continuous on $\mathbb{R}^2$ except at the set $\{(j, k + 1/2) : j, k \in \mathbb{Z}\}$. However, F is non-differentiable on the set $\{(t, j) : t \in \mathbb{R}, j \in \mathbb{Z}\}$. In this regard, the construction in [6] provides a Gabor orthonormal basis $\mathcal{G}(g, 1, 1)$ such that Zg has more smoothness than $F = Zf$ in (3.1).

2. We have shown that the basis of Høholdt, Jensen, and Justesen is almost optimally localized with respect to the (p, q) Balian-Low result in Theorem 1.3 when $(p, q) = (3/2, 3)$. It would be interesting to see whether Høholdt, Jensen, and Justesen's method of construction can be extended to provide optimality for other values of (p, q). With respect to further potentially optimal examples, Janssen in [20] and [21] provides several other families of functions which have the Zak transforms with minimal singular support. These include Gaussians, hyperbolic secants, and two-sided exponentials. The operation $Z^{-1}(Zg/|Zg|)$ applied to these functions yields examples of Gabor orthonormal bases for $L^2(\mathbb{R})$. The examples of Janssen are analogous in nature to the examples in [22] and [6], but they possess more symmetry in their decay properties. At the present, [6] provides the only construction which has been proven to be optimal for general values of the time and frequency localization parameters (p,q).

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