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WEIGHTED EMPIRICAL LIKELIHOOD RATIO CONFIDENCE INTERVALS FOR THE MEAN WITH CENSORED DATA

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Abstract. We propose a procedure to construct the empirical likelihood ratio confidence interval for the mean using a resampling method. This approach leads to the definition of a likelihood function for censored data, called *weighted empirical likelihood function*. With the second order expansion of the log likelihood ratio, a weighted empirical likelihood ratio confidence interval for the mean is proposed and shown by simulation studies to have comparable coverage accuracy to alternative methods, including the nonparametric bootstrap-t. The procedures proposed here apply in a unified way to different types of censored data, such as right censored data, doubly censored data and interval censored data, and computationally more efficient than the bootstrap-t method. An example of a set of doubly censored breast cancer data is presented with the application of our methods.

Key words and phrases: Bootstrap, doubly censored data, interval censored data, leveraged bootstrap, right censored data.

1. Introduction

Since Owen (1998), the empirical likelihood method has been developed to construct tests and confidence sets based on nonparametric likelihood ratios. For more references, see Owen (1990, 1991), DiCiccio *et al.* (1991), Qin and Lawless (1994), Mykland (1995), among others. Studies have shown that empirical likelihood ratio inferences are of comparable accuracy to alternative methods. In this research, we combine the ideas of empirical likelihood and resampling to develop a general method so that the confidence intervals for different types of censored data can be constructed in a unified way.

We begin with a review of work by Owen (1988). Let $X_1, \dots, X_n$ be an independent and identically distributed (i.i.d.) sample from a continuous distribution function (d.f.) F_0. Then, the empirical d.f. F_n based on this sample is the nonparametric maximum likelihood estimator (NPMLE) of F_0, since it maximizes the following likelihood function,

$$L(F) = \prod_{i=1}^{n} (F(X_i) - F(X_i-)) \tag{1.1}$$

over all distribution functions F. The empirical likelihood ratio function (Owen (1988)) is given by

$$R(F) = L(F)/L(F_n), \tag{1.2}$$

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and it is shown that for a constant $c > 0$, $\{\int xdF \mid R(F) \geq c\} = [X_{L,n}, X_{U,n}]$ may be used as confidence region for the mean μ of F_0. Specifically, Owen (1988) showed

$$\lim_{n\to\infty} P\{X_{L,n} \leq \mu \leq X_{U,n}\} = P\{\chi^2_{(1)} \leq -2\log c\}, \tag{1.3}$$

where $\chi^2_{(1)}$ denotes a random variable (r.v.) with chi-squared distribution of degrees of freedom 1. One of the advantages of this method is that there is no need to estimate the variance of the mean estimator. Here we consider how to construct the empirical likelihood ratio confidence interval for the mean with different types of censored data.

In this work, what we have in mind includes the folloing types of censored data.

Right censored sample. One observes (V_i, δ_i), $i = 1, \ldots, n$, with

$$V_i = \begin{cases} X_i & \text{if} \quad X_i \leq Y_i,\ \delta_i = 1 \\ Y_i & \text{if} \quad X_i > Y_i,\ \delta_i = 0 \end{cases} \tag{1.4}$$

where Y_i is the right censoring variable and independent from X_i. This type of censoring has been extensively studied in the literature over the past two decades.

Doubly censored sample. One observes (V_i, δ_i), $i = 1, \ldots, n$, with

$$V_i = \begin{cases} X_i & \text{if} \quad Z_i < X_i \leq Y_i,\ \delta = 1 \\ Y_i & \text{if} \quad X_i > Y_i,\ \delta_i = 2 \\ Z_i & \text{if} \quad X_i \leq Z_i,\ \delta_i = 3 \end{cases} \tag{1.5}$$

where, Y_i and Z_i are right and left censoring variables, respectively, with $P\{Y_i > Z_i\} = 1$, and (Y_i, Z_i) is independent from X_i. This type of censoring has been considered by Turnbull (1974), Chang and Yang (1987), Gu and Zhang (1993), Ren (1995*a*), Mykland and Ren (1996), among others. In practice, doubly censored data have recently occurred in studies of primary breast cancer (Peer *et al.* (1993), Ren and Peer (2000)).

Interval censored sample. One observes (V_i, δ_i), $i = 1, \ldots, n$, with $V_i = (Y_i, Z_i)$ and

$$\delta_i = \begin{cases} 1 & \text{if} \quad Z_i < X_i \leq Y_i, \\ 2 & \text{if} \quad X_i > Y_i \\ 3 & \text{if} \quad X_i \leq Z_i \end{cases} \tag{1.6}$$

where $P\{Y_i > Z_i\} = 1$ and (Y_i, Z_i) is independent from X_i. This type of censoring was considered by Groeneboom and Wellner (1992). In practice, interval censored data have been encountered in AIDS research (Kim *et al.* (1993)).

Clearly, one possible way to construct the empirical likelihood ratio confidence intervals with censored data is to use the likelihood function for a specific censoring model. This requests some careful investigation for each type of censored sample. In particular, the computation of the confidence region and the asymptotic results on the coverage of the confidence region need to be studied for each type censored data. In this paper, we intend to give a unified method which is easily applicable to different types of censored data including all mentioned above.

One may note that the reason that Owen's empirical likelihood method (1988) does not directly apply to censored data is that the complete i.i.d. sample $X_i, \ldots, X_n$ is not available. Since for all types of censored data mentioned above, the NPMLE $\hat{F}_n$ can be numerically computed (see Mykland and Ren (1996) for doubly censored data; Groeneboom and Wellner (1992) for interval censored data) and the strong uniform consistency of $\hat{F}_n$ has been established (see Stute and Wang (1993); Gu and Zhang (1993); Groeneboom and Wellner (1992); among others), one may hope that if for an integer m, an i.i.d. sample $X_1^*, \ldots, X_m^*$ is taken from $\hat{F}_n$, this sample may behave the same asymptotically (for large n) as $X_1, \ldots, X_m$. This resampling method is called the *Leveraged Bootstrap* (LB) (Ren (1995*b*)). For the problem considered in this paper, one may see that using this pseudo complete i.i.d. sample $X_1^*, \ldots, X_m^*$, called the *leveraged bootstrap sample*, the likelihood function and the empirical likelihood ratio function are immediately given by

$$L^*(F) = \prod_{i=1}^{m} (F(X_i^*) - F(X_i^*-)) \tag{1.7}$$

and

$$R^*(F) = L^*(F)/L^*(F_m^*), \tag{1.8}$$

respectively, where F_m^* is the empirical d.f. based on $X_1^*, \ldots, X_m^*$. As Owen (1988), we denote $F \ll F_m^*$ as F with support in $[X_{(1)}^*, X_{(m)}^*]$ and denote

$$\mathbf{F}_{c,m}^* = \{F \mid R^*(F) \geq c, F \ll F_m^*\} \quad \text{for} \quad c > 0. \tag{1.9}$$

We also define

$$X_{L,m}^* = \inf_{F \in \mathbf{F}_{c,m}^*} \int x dF \quad \text{and} \quad X_{U,m}^* = \sup_{F \in \mathbf{F}_{c,m}^*} \int x dF. \tag{1.10}$$

In Section 2, we show that $[X_{L,m}^*, X_{U,m}^*]$ may be used as the confidence interval for μ for censored data (1.4)–(1.6), called the *LB-Empirical Likelihood Ratio Confidence Interval* (LB-ELRCI), which eventually leads to the definition of a likelihood function for censored data, called *weighted empirical likelihood function*. We establish the second order expansion of the log likelihood ratio, based on which without using resampling, a *weighted empirical likelihood ratio confidence interval* (WELRCI) for the mean is proposed. The proofs are deferred to Section 5.

In Section 3, we present some simulation results for right censored data, doubly censored data and interval censored data, respectively, and we apply the proposed methods to a set of doubly censored breast cancer data (Peer *et al.* ((1993)). Section 4 includes some concluding remarks.

We note that the nonparametric bootstrrap-t (Efron and Tibshirani (1993), p. 160–163) may be used to construct the confidence intervals for the mean with the censored data mentioned above, and it compares well with empirical likelihood method when there is no censoring (Owen (1988)). However, since the NPMLE $\hat{F}_n$ can only be computed numerically for doubly censored data or interval censored data, say using the EM algorithm, it can be very time-consuming to perform the nonparametric bootstrrap-t for a large sample (see comments on p. 162 of Efron and Tibshirani (1993)). Simulation studies show that our methods proposed in this paper are computationally more efficient and generally has excellent performance in terms of coverage accuracy.

2. Weighted empirical likelihood ratio confidence interval

To treat the ties among $X_1^*, \ldots, X_m^*$, we use the device by Owen (1988) as follows. For any d.f. F, let

$$w_i \geq 0, \qquad \sum_{j:X_j=X_i} w_j = F(X_i^*) - F(X_i^*-), \quad i = 1, \ldots, m \tag{2.1}$$

where the w_i have the from of probabilities attached to observations X_i^*. Then, by Lemma 1 of Owen (1998), we know that for $\boldsymbol{w} = (w_1, \ldots, w_m)$,

$$\mathbf{F}_{c,m}^* = \left\{ F \middle| \prod_{i=1}^m m\ w_i \geq c, \text{for } \boldsymbol{w} \text{ satisfying(2.1)} \right\}.$$

Therefone, in (1.10) we have

$$X_{L,m}^* = \inf_{\boldsymbol{w} \in \Lambda_{c,m}} \sum_{i=1}^m w_i X_i^* \quad \text{and} \quad X_{U,m}^* = \sup_{\boldsymbol{w} \in \Lambda_{c,m}} \sum_{i=1}^m w_i X_i^* \tag{2.2}$$

where

$$\Lambda_{c,m} = \left\{ \boldsymbol{w} \middle| \prod_{i=1}^m m\ w_i \geq c, w_i \geq 0, \sum_{i=1}^m w_i = 1 \right\}. \tag{2.3}$$

The computtation of $X_{L,m}^*$ and $X_{U,m}^*$ is described in Section 2 of Owen (1988).

Note that the NPMLE $\hat{F}_n$ for censored data (1.4)–(1.6) is not always a proper distribution function. In this study, we will always adjust $\hat{F}_n$ to a proper d.f. by setting $\hat{F}_n = 1$ at largest observation in the datta set, so that any observation $X_i^* = \hat{F}_n^{-1}(U_i)$ in a leveraged bootstrap sample is always well defined for any uniform r.v. U_i on $[0, 1]$. This kind of odjustment of the NPMLE $\hat{F}_n$ is a generally adopted convention for censored data (Efron (1967) and Miller (1976)).

The following theorem investigates the asymptotic property of the interval $[X_{L,m}^*, X_{U,m}^*]$, called the *LB-Empirical Likelihood Ratio Confidence Interval* (LB-ELRCI), with the proof deferred to Section 5. Let $\|\cdot\|$ denote the supremum norm and let

$$0 < \mu = \int x dF_0(x) < \infty, \qquad 0 < \sigma^2 = \int (x-\mu)^2 dF_0(x) < \infty \tag{2.4}$$

$$\mu_n = \int x d\hat{F}_n(x), \qquad \sigma_n^2 = \int (x - \mu_n)^2 d\hat{F}_n(x).$$

THEOREM 1. *Assume $m \to \infty$ and $m/n \to \gamma \in [0, \infty)$, as $n \to \infty$, and assume*
(A1) $\|\hat{F}_n - F_0\| \to^P$, *as* $n \to \infty$;
(A2) *for some* $0 < \tau^2 < \infty$, $\sqrt{n}(\mu_n - \mu) \to^D \mathrm{N}(0, \tau^2)$, *as* $n \to \infty$;
(A3) $\sigma_n^2 \to^P \sigma^2$, *as* $n \to \infty$;
(A4) $E\{\int x^4 d\hat{F}_n(x)\} \leq M_0 < \infty$ *for all* $n \geq 1$.
Then,

$$\lim_{n\to\infty} P\{X_{L,m}^* \leq \mu \leq X_{U,m}^*\} = P\left\{ \chi_{(1)}^2 \leq \frac{-2\log c}{1 + (\gamma\tau^2/\sigma^2)} \right\}. \tag{2.5}$$

From Theorem 1, one may see that if $m = o(n)$, we have $\gamma = 0$ in (2.5), thus (2.5) is the same as (1.3) which was obtained by Owen (1988) for the complete data case. We may say that the leveraged bootstrap is consistent with $m = o(n)$. However with some modifications in the proofs of our Theorem 1 and Owen's Theorem 1(1988), one can easily show that $[X^*_{U,m} - X^*_{L,m}] = O_p(1/\sqrt{m})$. Hence, with $m = o(n)$, the width of the $(1-\alpha)100\%$ confidence interval $[X^*_{L,m}, X^*_{U,m}]$ is wider than that by the nonparametric bootstrap method based on assumption (A2):

$$\mu_n \pm z_{\alpha/2}\hat{s}_n \tag{2.6}$$

where for $0 < \alpha < 1$, $z_{\alpha/2}$ is the $(1 - \frac{\alpha}{2})$ 100-th percentile of the standard normal distribution, and $\hat{s}_n = O_p(1/\sqrt{n})$ is the standard error of the mean estimator μ_n and may be computed by the nonparametric bootstrap method (Efron and Tibshirani (1993) p. 47). Keeping this in mind, in practice we may simply use $m = n$ as the sample size of the pseudo i.i.d. sample $X^*_1, \ldots, X^*_m$, which gives $\gamma = 1$ in (2.5) and

$$\lim_{n\to\infty} P\{X^*_{L,n} \le \mu \le X^*_{U,n}\} = P\left\{\chi^2_{(1)} \le \frac{-2\log c}{1 + (\tau^2/\sigma^2)}\right\}. \tag{2.7}$$

Note that when there is no censoring, we have $\tau^2 = \sigma^2$ in (2.7) and (2.7) becomes

$$\lim_{n\to\infty} P\{X^*_{L,n} \le \mu \le X^*_{U,n}\} = P\{\chi^2_{(1)} \le -\log c\}.$$

Compared with (1.3), this implies that length of the LB-ELRCI is wider than the ELRCI $[X_{L,n}, X_{U,n}]$ given by Owen (1988)

Note that Theorem 1 gives the asymptotic property of the LB-ELRCI for one leveraged bootstrap sample. If we take N leveraged bootstrap samples from $\hat{F}_n$, then clearly for each sample $(X^*_{k1}, \ldots, X^*_{kn})$, $k = 1, \ldots, N$, an LB-ELRCI $[X^*_{L_k}, X^*_{U_k}]$ can be computed by (2.2), and one may expect to improve the efficiency of (2.7) using the idea of the 'best' sample (Ren (1995*b*)) in this case. Specifically, among N leveraged bootstrap samples, find a sample $X^*_{1,N}, \ldots, X^*_{n,N}$ and its empirical d.f. $F^*_{n,N}$ such that

$$\|F^*_{n,N} - \hat{F}_n\| = \min_{1\le k\le N} \|F^*_{kn} - \hat{F}_n\|, \tag{2.8}$$

where F^*_{kn} is the empirical d.f. of $X^*_{k1}, \ldots, X^*_{kn}$, and use the interval $[X^*_{L,n,N}, X^*_{U,n,N}]$ computed by (2.2) based on this 'best' i.i.d. sample $X^*_{1,N}, \ldots, X^*_{n,N}$ as the confidence interval for μ. The intuition behind such use of the 'best' i.i.d. sample is that we consider that $F^*_{n,N}$ is the one closest to F_0 among N leveraged bootstrap samples, because by (2.8), $F^*_{n,N}$ is the one closest to $\hat{F}_n$. In the next theorem, we investigate the asymptotic properties of $[X^*_{L,n,N}, X^*_{U,n,N}]$ with proofs deferred to Section 5.

Denote $Y_{(i)}$'s and $Z_{(i)}$'s as the order statistics for $Y_1, \ldots, Y_n$ and $Z_1, \ldots, Z_n$ respectively. If $V_i = (Y_i, Z_i)$, such as for interval censored data (1.6), then as a convention, we denote

$$\begin{aligned} &V_{(n)} \equiv \max\{Y_{(n)}, Z_{(n)}\} \quad \text{and} \quad V_{(1)} \equiv \min\{Y_{(1)}, Z_{(1)}\} \\ &E|V|^q < \infty \quad \text{if and only if} \quad E|Y|^q < \infty \text{ and } E|Z|^q < \infty, \end{aligned} \tag{2.9}$$

where $q > 0$.

THEOREM 2. *Assume* (A1)–(A4) *in Theorem* 1 *and assume*
(A5) $E|V|^3 < \infty$

(A6) *there exists* $0 < \beta < \frac{1}{6}$ *such that* $n^{-\beta}(V_{(n)} - V_{(1)}) \max\{|V_{(1)}|, |V_{(n)}|\} = o_p(1)$. *Then,*

(i) *for* $\mu^*_{n,N} = \int x dF^*_{n,N}$ *and* $\delta^*_{n,N} = \sqrt{n}(\mu^*_{n,N} - \mu_n)/\sigma_n$, *we have*

$$\text{(2.10)} \qquad \lim_{\substack{n\to\infty \\ N\to\infty}} P\{X^*_{L,n,N} \le \mu \le X^*_{U,n,N}\} = \lim_{\substack{n\to\infty \\ N\to\infty}} P\left\{\left((1+o_p(1))\delta^*_{n,N} + \frac{\sqrt{n}(\mu_n - \mu)}{\sigma_n}\right)^2 \le -2\log c\right\}$$

where $o_p(1)$ *converges to* 0 *in probability as* $n \to \infty$, $N \to \infty$;

(ii) $\delta^*_{n,N} \to^{P_n} o_{n,p}(1)$, *as* $N \to \infty$, *and*

$$\text{(2.11)} \qquad \lim_{n\to\infty} \lim_{N\to\infty} P\{X^*_{L,n,N} \le \mu \le X^*_{U,n,N}\} = P\left\{\chi^2_{(1)} \le -\frac{2\log c}{\tau^2/\sigma^2}\right\},$$

where P_n *denotes the conditional probability given* $\hat{F}_n$ *and* $o_{n,p}(1)$ *converges to* 0 *in probability as* $n \to \infty$.

Now, when there is no censoring, we have $\hat{F}_n = F_n$ and $\tau^2 = \sigma^2$ in (2.11), and we can easily show

$$\|F^*_{n,N} - F_n\| \overset{P_n}{\to} 0 \quad \text{and} \quad [X^*_{L,n,N}, X^*_{U,n,N}] \overset{P_n}{\to} [X_{L,n}, X_{U,n}], \quad \text{as } N \to \infty$$

thus in turn, (2.11) becomes

$$\lim_{n\to\infty} \lim_{N\to\infty} P\{X^*_{L,n,N} \le \mu \le X^*_{u,n,N}\} = \lim_{n\to\infty} E\{\lim_{N\to\infty} P_n\{X^*_{L,n,N} \le \mu \le X^*_{U,n,N}\}\}$$
$$= \lim_{n\to\infty} P\{X_{L,n} \le \mu \le X_{U,n}\} = P\{\chi^2_{(1)} \le -2\log c\},$$

which coincides with Owen's (1.3). This indicates that applying the idea of the 'best' sample (Ren (1995*b*)), we indeed improve (2.7) with more leveraged bootstrap samples used in the proposed procedure. Following Remark 1 below, we outline the steps to construct the LB-ELRCI for the mean with censored data.

Remark 1. $\delta^*_{n,N}$ in (2.10) may converge to 0 rather slowly depending on the rate of $(V_{(n)} - V_{(1)})$. Thus from (A2), we know that (2.10) gives

$$\text{(2.12)} \qquad P\{X^*_{L,n,N} \le \mu \le X^*_{U,n,N}\} \ge P\left\{\chi^2_{(1)} \le \frac{(\sqrt{-2\log c} - |\delta^*_{n,N}|^2}{(\tau^2/\sigma^2_n)}\right\},$$

for large n and N, which may be used to set the constant c for a given confidence level in practice. One may note that since $\delta^*_{n,N}$ converges to 0, the limit of the left hand side of the inequality in (2.12) is the same as that of the right hand side. Thus, the use of $\delta^*_{n,N}$ in (2.12) gives a slightly conservative coverage of the confidence interval, which should show for a moderate sample size n, but makes no difference for a very large n.

Constructing LB-ELRCI for the mean.

(S1) Compute the NPMLE $\hat{F}_n$ using censored data;

(S2) Compute μ_n and σ_n^2 in (2.4);

(S3) Use nonparametric bootstrap method to compute the standard error $\hat{s}_n$ in (2.6) for the mean estimator μ_n;

(S4) Take N i.i.d. samples from $\hat{F}_n$: $(X_{k1}^*, \ldots, X_{kn}^*)$, $k = 1, \ldots, N$, and find the 'best' sample $X_{1,N}^*, \ldots, X_{n,N}^*$ with an empirical d.f. $F_{n,N}^*$ satisfying (2.8);

(S5) For a given confidence level, compute $c > 0$ in (2.12) with $\tau \approx \sqrt{n}\hat{s}_n$;

(S6) Using c computed in (S5) and the 'best' sample $(X_{1,N}^*, \ldots, X_{n,N}^*)$ obtained in (S4), compute the LB-ELRCI $[X_{L,n,N}^*, X_{U,n,N}^*]$ given by (2.2).

Remark 2. The idea of 'best' sample and (2.11) also suggest that we may choose a 'best' sample without using resampling. Simulation studies show that the following alternative step may be used to replace above (S4) in the proposed procedure above and it performs well.

(S4') Find a 'sample' $X_1^*, \ldots, X_n^*$ such that it is one of the possible i.i.d. samples of size n from $\hat{F}_n$ and has an empirical d.f. F^* satisfying $0 \leq \hat{F}_n - F^* \leq 1/n$.

Since $\hat{F}_n$ is a step function for censored data (1.4)–(1.6), (S4') can be done by a simple algorithm and is more efficient computationally than (S4). Our studies also show that the use of $0 \leq F^* - \hat{F}_n \leq 1/n$ in (S4') makes almost no difference.

In Section 3, our simulation studies show that the proposed LB-ELRCI with (S4') or (S4) generally performs well, which leads us to take a closer look at Theorem 2. Suppose that the NPMLE $\hat{F}_n$ for censored data is given by

$$\hat{F}_n(x) = \sum_{i=1}^{n_0} \hat{p}_i I\{W_i \leq x\} \tag{2.13}$$

where $W_1 < W_2 < \cdots < W_{n_0}$ are distinct observations among V_i's. Then, for a large N, the 'best' i.i.d. sample $X_{1,N}^*, \ldots, X_{n,N}^*$ from $\hat{F}_n$ should have an empirical d.f. $F_{n,N}^* = \sum_{i=1}^{n_0} \frac{n_i}{n} I\{W_i \leq x\}$, where n_i is the number of $X_{i,N}^*$'s equal to W_i and $\frac{n_i}{n} \approx \hat{p}_i$. Thus, for this 'best' sample, the likelihood function (1.7) satisfies

$$\begin{aligned} L^*(F) &= \prod_{i=1}^{n} (F(X_{i,N}^*) - F(X_{i,N}^*-)) \\ &= \prod_{i=1}^{n_0} (F(W_i) - F(W_i-))^{n_i} \approx \prod_{i=1}^{n_0} (F(W_i) - F(W_i-))^{n\hat{p}_i}. \end{aligned}$$

This means that the idea of the 'best' sample in LB-ELRCI procedure is approximating the following likelihood function:

$$\hat{L}(F) = \prod_{i=1}^{n_0} (F(W_i) - F(W_i-))^{n\hat{p}_i}, \tag{2.14}$$

called *weighted empirical likelihood function* for censored data. One may note that for the complete i.i.d. sample case, we have $V_i = X_i$, $1 \leq i \leq n$ and the weighted empirical likelihood function (2.14) coincides with the empirical likelihood function (1.1) by Owen (1988), because the NPMLE $F_n(x) = n^{-1}\sum_{i=1}^n I\{X_i \le x\} = \sum_{i=1}^{n_0} \frac{n_i}{n} I\{W_i \le x\}$ and

$$\hat{L}(F) = \prod_{i=1}^{n_0}(F(W_i) - F(W_i-))^{n(n_i/n)} = \prod_{i=1}^{n_0}(F(W_i) - F(W_i-))^{n_i} = L(F).$$

It is easy to show that $\hat{L}(F)$ is maximized at $\hat{F}_n$. Thus, the *weighted empirical likelihood ratio*

$$\hat{R}(F) = \hat{L}(F)/\hat{L}(\hat{F}_n), \tag{2.15}$$

may be used to construct the confidence interval for the mean, called the *weighted empirical likelihood ratio confidence interval* (WELRCI). The next theorem gives the asymptotic property of this confidence interval: $[\hat{X}_{L,n}, \hat{X}_{U,n}] = \{\int x dF \mid \hat{R}(F) \ge c\}$, where $c > 0$ is a constant.

THEOREM 3. *Let $p_i = F(W_i) - F(W_i-)$, $1 \le i \le n_0$, and let*

$$r(\mu) = \sup\left\{\prod_{i=1}^{n_0}(p_i/\hat{p}_i)^{n\hat{p}_i} \middle| \sum_{i=1}^{n_0} p_i W_i = \mu, p_i \ge 0, \sum_{i=1}^{n_0} p_i = 1\right\}. \tag{2.16}$$

Assume (A1)–(A3) *and* (A5) *in Theorem* 1 *and Theorem* 2, *and assume*

(A7) $\int x^4 d\hat{F}_n(x) = O_p(1)$;

(A8) $\hat{\mu}_{3n} = \int (x-\mu_n)^3 d\hat{F}_n(x) \to^P E(X-\mu)^3, \quad$ *as* $\quad n \to \infty$.

Then,

$$-2\log r(\mu) = \frac{n(\mu_n-\mu)^2}{\sigma_n^2}\left(1 + \frac{2(\mu_n-\mu)\hat{\mu}_{3n}}{3\sigma_n^4}\right) + O_p(n^{-1}), \tag{2.17}$$

and

$$\lim_{n\to\infty} P\{\hat{X}_{L,n} \le \mu \le \hat{X}_{U,n}\} = \lim_{n\to\infty} P\{-2\log r(\mu) \le -2\log c\} = P\left\{\chi^2_{(1)} \le -\frac{2\log c}{\tau^2/\sigma^2}\right\}. \tag{2.18}$$

Remark 3. Based on the second order expansion of the log likelihood ratio given by (2.17), we suggest the WELRCI for the mean be constructed as follows in practice. Let $\mathbf{C}_n = n(\mu_n - \mu)^2/\sigma_n^2$ and $Z_n = \sqrt{n}(\mu_n - \mu)/\tau$, then without loss of generality, assuming that $\hat{\mu}_{3n} \ge 0$ in probability, we have

$$P\left\{\mathbf{C}_n\left(1 + \frac{2|\hat{\mu}_{3n}|\tau}{3\sigma_n^4\sqrt{n}} z_\rho\right) \le -2\log c\right\}$$
$$\le P\left\{\mathbf{C}_n\left(1 + \frac{2\hat{\mu}_{3n}\tau}{3\sigma_n^4\sqrt{n}} Z_n\right) \le -2\log c\right\} + o(1) + P\{Z_n > z_\rho\}$$

where z_ρ is the $(1-\rho)$ 100-th percentile of the standard normal distribution. Therefore,

$$\lim_{n\to\infty} P\{\hat{X}_{L,n} \le \mu \le \hat{X}_{U,n}\} \tag{2.19}$$
$$= \lim_{n\to\infty} P\left\{\frac{n(\mu_n-\mu)^2}{\sigma_n^2}\left(1 + \frac{2(\mu_n-\mu)\hat{\mu}_{3n}}{3\sigma_n^4}\right) \le -2\log c\right\}$$

$$= \lim_{n\to\infty} P\left\{\mathbf{C}_n \left(1 + \frac{2\hat{\mu}_{3n}\tau}{3\sigma_n^4\sqrt{n}} Z_n\right) \le -2\log c\right\}$$

$$\ge \lim_{n\to\infty} P\left\{\mathbf{C}_n \left(1 + \frac{2|\hat{\mu}_{3n}|\tau}{3\sigma_n^4\sqrt{n}} z_p\right) \le -2\log c\right\} - \rho$$

$$\overset{(2.20)}{=} P\left\{\frac{\mathbf{C}_n}{\tau^2/\sigma_n^2} \le z_{\gamma/2}^2\right\} - \rho = \lim_{n\to\infty} P\{\chi_{(1)}^2 \le z_{\gamma/2}^2\} - \rho = 1 - \gamma - \rho,$$

where the constant c is set to have

$$-2\log c = z_{\gamma/2}^2 \left\{\left(1 + \frac{2|\hat{\mu}_{3n}|\tau}{3\sigma_n^4\sqrt{n}} z_\rho\right) \frac{\tau^2}{\sigma_n^2}\right\} \tag{2.20}$$

and τ is estimated by $\sqrt{n}\hat{s}_n$ as in (2.6). The confidence interval based on (2.20) is slightly conservative, but performs well in simulation studies as shown in Section 3. One may note that it is not clear if the Bartlett-correction (DiCiccio *et al.* (1991)) holds generally for censored data considered here, while (2.19)–(2.20), with a 'correction' term $O(n^{-1/2})$ instead of $O(n^{-1})$, has a similar form and holds generally for various types of censored data.

Remark 4. The censoring mechanism of the data is reflected by the term τ^2 in (2.11) and (2.18), and by the weights $\hat{p}_i$ of the NPMLE in the weighted empirical likelihood function (2.14). Thus in our proposed methods, τ^2 and σ^2 need to be estimated, which is the price we pay for the generality of our approach. Nonetheless, simulation studies in Section 3 show that this does not appear to affect the coverage and the length of the LB-ELRCI and WELRCI.

Remark 5. The computation of the WELRCI $[\hat{X}_{L,n}, \hat{X}_{U,n}]$ can be obtained by solving the following optimization problems:

$$\hat{X}_{L,n} = \min \sum_{i=1}^{n_0} p_i W_i \quad \text{and} \quad \hat{X}_{U,n} = \max \sum_{i=1}^{n_0} p_i W_i$$

both subject to: $p_i \ge 0$, $\sum_{i=1}^{n_0} p_i = 1$, $\prod_{i=1}^{n_0} (p_i/\hat{p}_i)^{n\hat{p}_i} \ge c$. Let

$$h(\lambda) = -n \sum_{i=1}^{n_0} \hat{p}_i \log \left\{(W_i - \lambda)\left(\sum_{i=1}^{n_0} \frac{\hat{p}_i}{W_i - \lambda}\right)\right\} - \log c,$$

and W_{i1} be the smallest W_j's with $\hat{p}_j > 0$ W_{i2} the largest W_j's with $\hat{p}_j > 0$. It can be shown that $h(\lambda)$ is monotone in λ and that the solutions are given by

$$\hat{X}_{L,n} = \sum_{i=1}^{n_0} \left(\frac{\hat{p}_i W_i}{W_i - \lambda}\right)\left(\sum_{i=1}^{n_0} \frac{\hat{p}_i}{W_i - \lambda}\right)^{-1}, \quad \text{for} \quad \lambda < W_{i1} \text{ with } h(\lambda) = 0,$$

and

$$\hat{X}_{U,n} = \sum_{i=1}^{n_0} \left(\frac{\hat{p}_i W_i}{\lambda - W_i}\right)\left(\sum_{i=1}^{n_0} \frac{\hat{p}_i}{\lambda - W_i}\right)^{-1}, \quad \text{for} \quad \lambda < W_{i2} \text{ with } h(\lambda) = 0.$$

Table 1. 90% C.I. for the mean with right censored exponential data.

sample Size $n = 100$	Coverage	Mean Length of C.I.	s.d. Length of C.I.	Mean $\lvert\delta^*_{n,N}\rvert$	s.d. $\lvert\delta^*_{n,N}\rvert$
$\mu_n \pm 1.645\hat{s}_n$	.869	.368	.095	-	-
Bootstrap-t	.900	.428	.167	-	-
LB-ELRCI ($N = 100$)	.900	.423	.138	.269	.214
LB-ELRCI ($N = 10000$)	.901	.407	.122	.182	.146
LB-ELRCI ($S4'$)	.907	.419	.112	.201	.054
WELRCI	.912	.435	.139	-	-
Sample Size $n = 500$					
$\mu_n \pm 1.645\hat{s}_n$	.896	.174	.027	-	-
LB-ELRCI (S4')	.923	.186	.029	.115	.026
WELRCI	.928	.193	.003	-	-
Sample Size $n = 1000$					
$\mu_n \pm 1.645\hat{s}_n$	.888	.123	.016	-	-
LB-ELRCI(S4')	.900	.130	.016	.089	.019
WELRCI	.913	.135	.019	-	-

$X \sim$ Exp(1)(74.8%), $Y \sim$ Exp(3)(25.2%).

Table 2. 90% C.I. for the mean with right censored normal data.

Sample Size $n = 100$	Coverage	Mean Length of C.I.	s.d. Length of C.I.	Mean $\lvert\delta^*_{n,N}\rvert$	s.d.$\lvert\delta^*_{n,N}\rvert$
$\mu_n \pm 1.645\hat{s}_n$	.888	.368	.047	-	-
Bootstrap-t	.901	.391	.050	-	-
LB-ELRCI ($N = 100$)	.910	.410	.062	.210	.165
LB-ELRCI ($N = 10000$)	.896	.396	.054	.135	.100
LB-ELRCI (S4')	.910	.411	.053	.201	.039
WELRCI	.903	.390	.052	-	-
Sample Size $n = 500$					
$\mu_n \pm 1.645\hat{s}_n$	.906	.166	.014	-	-
LB-ELRCI (S4')	.922	.176	.015	.109	.012
WELRCI	.917	.172	.015	-	-
Sample Size $n = 1000$					
$\mu_n \pm 1.645\hat{s}_n$	.885	.118	.009	-	-
LB-ELRCI (S4')	.898	.123	.009	.084	.008
WELRCI	.895	.122	.010	-	-

$X \sim$ N(0, 1)(67.2%), $Y \sim$ N(1, 4)(32.8%).

3. Simulation results and examples

This section considers the application of the WELRCI for the mean, which is compared with other methods, including the LB-ELRCI described in Section 2. We denote Exp(μ) as the exponential distribution with mean μ, $N(\mu, \sigma^2)$ as the normal distribution

Table 3. 90% C.I. for the mean with right censored lognormal data.

$n = 100$	Coverage	Mean Length of C.I.	s.d. Length of C.I.
$\mu_n \pm 1.645\hat{s}_n$	.808	.795	.398
Bootstrap-t	.869	1.158	1.243
LB-ELRCI(S4')	.869	.930	.490
WELRCI	.887	1.049	.650

$X \sim$ LN$(0, 1)(67.2\%)$, $Y \sim$ LN$(1, 4)(32.8\%)$.

Table 4. 90% C.I. for the mean with doubly censored exponential data.

$n = 100$	Coverage	Mean Length of C.I.	s.d. Length of C.I.
$\mu_n \pm 1.645\hat{s}_n$	.873	.422	.130
LB-ELRCI(S4')	.909	.478	.150
WELRCI	.923	.517	.210

$X \sim$ Exp$(1)(55.7\%)$, $Y \sim$ Exp$(3)(25.2\%)$, $Z = \frac{2}{3}Y - 2.5(19.1\%)$.

Table 5. 90% C.I. for the mean with doubly censored normal data.

$n = 100$	Coverage	Mean Length of C.I.	s.d. Length of C.I.
$\mu_n \pm 1.645\hat{s}_n$	.892	.378	.047
LB-ELRCI(S4')	.924	.423	.052
WELRCI	.910	.402	.053

$X \sim$ N$(0, 1)(53.4\%)$, $Y \sim$ N$(1, 4)(32.8\%)$, $Z = \frac{2}{3}Y - 2.5(13.8\%)$.

with mean μ and variance σ^2, and $LN(\mu, \sigma^2)$ as the lognormal distribution.

In Table 1, 1000 right censored samples of size 100 are taken from X ~Exp(1), Y ~Exp(3) (note that the percentages of right censored and uncensored observations are given at the bottom of Table1, respectively), and for each sample, a 90% LB-ELRCI for the mean is computed using (S1)–(S6) given in Section 2 with $N = 100$ and $N = 10,000$ in (S4), respectively, where the standard error $\hat{s}_n$ is estimated based on 100 nonparametric bootstrap samples. For these 1000 right censored exponential samples, 90% LB-ELRCI using (S4') instead of (S4), 90% confidence intervals (2.6) and 90% WELRCI with $\gamma = .09$ and $\rho = .01$ in (2.20) are also computed. In each case, the coverage for the mean of X by 1000 confidence intervals is displayed in Table 1, and the simulation mean and standard deviation (s.d.) of the length of these confidence intervals are displayed as well.

As mentioned in Section 1, the nonparametric bootstrap-t method (Efron and Tibshirani (1993), p. 160–163) can also be used to construct the confidence interval for the mean. In our studies here, this method is applied to above 1000 right censored exponential samples, where 1000 nonparametric bootstrap samples are used for the computation of percentiles and 30 nested bootstrap samples are used for the estimation of the standard error. Table 1 also includes some results for sample size $n = 500$ and $n = 1000$, respectively.

Table 6. 90% C.I. for the mean with doubly censored lognormal data.

$n = 100$	Coverage	Mean Length of C.I.	s.d. Length of C.I.
$\mu_n \pm 1.645\hat{s}_n$	.840	1.292	1.176
LB-ELRCI(S4')	.874	1.424	1.224
WELRCI	.910	1.972	2.349

$X \sim$ LN(0, 1)(31.8%), $Y \sim$ LN(1, 4)(32.6%), $Z = \frac{2}{3}Y - 2.5$(35.5%).

Table 7. 90% C.I. for the mean with interval censored exponential data.

$n = 100$	Coverage	Mean Length of C.I.	s.d. Length of C.I.
$\mu_n \pm 1.645\hat{s}_n$	.858	.430	.084
LB-ELRCI(S4')	.904	.474	.102
WELRCI	.933	.524	.142

$X \sim$ Exp(1), $Y \sim$ Exp(3), $Z = \frac{2}{3}Y - 2.5$; $\delta = 1 : 55.9\%$, $\delta = 2 : 25.0\%$, $\delta = 3 : 19.1\%$.

Table 8. 90% C.I. for the mean with interval censored normal data.

$n = 100$	Coverage	Mean Length of C.I.	s.d. Length of C.I.
$\mu_n \pm 1.645\hat{s}_n$	.874	.461	.057
LB-ELRCI(S4')	.893	.497	.062
WELRCI	.900	.501	.062

$X \sim$ N(0, 1), $Y \sim$ N(1, 4), $Z = \frac{2}{3}Y - 2.5$; $\delta = 1 : 53.6\%$, $\delta = 2 : 32.7\%$, $\delta = 3 : 13.6\%$.

The same studies in Table 1 are repeated in Table 2 for right censored normal samples. For right censored lognormal samples, Table 3 compares the performance of the confidence interval based on (2.6), nonparametric bootstrap-t, LB-ELRCI with (S4') and WELRCI.

From Table 1–3, we can see : (1) the coverage and the length of the LB-ELRCI's with large N or with (S4') used are basically the same as those by the nonparametric bootstrap-t; (2) as n increases, the quantity $|\delta^*_{n,N}|$ decreases, and the length of the LB-ELRCI's gets closer and closer to that of the confidence intervals constructed by the usual asymptotic method (2.6); (3) WELRCI compares well with bootstrap-t and LB-ELRCI and performs better than the usual asymptotic method (2.6).

Except bootstrap-t (because it is very time-consuming), the studies in Table 3 are conducted in Tables 4–9 for doubly censored samples and interval censored samples with exponential, normal and lognormal distributions, respectively. Note that it is known that constructing confidence intervals for lognormal distributions is a hard problem, and here we consider various types of censored lognormal distributions in our studies. Clearly, in all cases WELRCI gives the best coverage among all methods considered here for the lognormal data, and the confidence interval based on (2.6) performs very poorly with interval censored lognormal data even for larger sample size $n = 200$ (see Table 9).

Next, we apply the proposed WELRCI and LB-ELRCI methods to a doubly censored

Table 9. 90% C.I. for the mean with interval censored lognormal data.

$n = 200$	Coverage	Mean Length of C.I.	s.d. Length of C.I.
$\mu_n \pm 1.645\hat{s}_n$	.778	.703	.305
LB-ELRCI(S4')	.845	.776	.391
WELRCI	.881	2.662	34.378

$X \sim \text{LN}(0,1)$, $Y \sim \text{LN}(1,4)$, $Z = \frac{2}{3}Y - 2.5$; $\delta = 1 : 31.9\%$, $\delta = 2 : 32.9\%$, $\delta = 3 : 35.2\%$.

Table 10. 90% C.I. for the mean with breast cancer data.

WELRCI	LB-ELRCI ($N = 1000$)	LB-ELRCL(S4')	$\mu \pm 1.645\hat{s}_n$
[61.96, 64.92]	[62.04, 64.91]	[62.05, 65.04]	[62.07, 64.85]

data set encountered in a practical situation.

Example 1. In a recent study of primary breast cancer (Peer *et al.* (1993)), a doubly censored sample is encountered. The age (in years), X, at which a tumor volume is developed, is observed among 236 woman with age ranging from 41-84 years. From 1981 to 1990, serial screening mammograms with a mean screening interval of 2 years were obtained. Among the tumor volumes detected by the screening mammograms, 45 women had tumor volumes observed at the first screening mammograms—yielding left censored observations, 79 did not have tumor volumes observed at the last screening mammograms—yielding right censored observations, and 112 were observed to grow tumor during the period of the serial screening mammograms—yielding uncensored observations. The statistical inference on X should indicate the effect of the frequency of the screening mammograms in detection of early atage of canser (Ren and Peer (2000)). For this doubly censored data set, the confidence interval (2.6), WELRCI and LB-ELRCI for the mean of X are constructed, respectively, and the results are displayed in Table 10. One may note that the confidence intervals in Table 10 do not differ very much, though the lengths of the WELRCI and LB-ELRCI are a little bit wider as expected based on the simulation studies above.

4. Conclusions

Using the idea of leveraged bootstrap, a new method of constructing confidence intervals for the mean with various types of censored data, called *LB-Empirical Likelihood Confidence Intervals*, is proposed in this paper. The investigation of this method leads to the discovery of the use of (S4') in the proposed and the discovery of the *weighted empirical likelihood ratio confidence interval*, which do not need to take any leveraged bootstrap samples in their computations. Simulation studies show that the proposed WELRCI, though from (2.19) theoretically slightly conservative based on the second order expansion of the log likelihood ratio, compares very well (even for censored lognormal samples) with the nonparametric bootstrap-t method in terms of coverage accuracy and the length of the confidence interval, and is computationally for more efficient for doubly censored data or interval censored data.

Another advantage of the proposed methods in this paper is that they are easily applicable to different types of censored data and they do not require (no more than the bootstrap-t) the case by case study on the computation and asymptotic properties of the empirical likelihood confidence bands for different types of censored data. In fact, the proposed WELRCI method (under some conditions) directly applies to any incomplete data for which the mean estimator based on the NPMLE is asymptotically normal.

5. Proofs

PROOF OF THEOREM 1. First, it can be shown that (A1) implies that as $n \to \infty$,

$$X^*_{(1)} < \mu < X^*_{(m)}, \qquad \text{in probability.} \tag{5.1}$$

Thus, in probability

$$r^*_m(x) = \sup\left\{ \prod_{i=1}^m m w_i \,\Big|\, \sum_{i=1}^m w_i X^*_i = x, \text{ for } \boldsymbol{w} \text{ satisfying (2.1)} \right\} \tag{5.2}$$

always exists. Nothing that $\Lambda_{c,m}$ is compact and convex, from (2.2), (2.3), the definition of $\gamma^*_m(x)$ and the Intermediate Value Theorem, we can show that in probability,

$$X^*_{L,m} \le \mu \le X^*_{U,m} \qquad \text{if and only if} \quad r^*_m(\mu) \ge c. \tag{5.3}$$

From the proof of Theorem 1 by Owen (1998), we have

$$\log R_0 \equiv \log r^*_m(\mu) = -\sum_{i=1}^m \log(1 + \lambda_0 Y^*_i), \tag{5.4}$$

where $Y^*_i = X^*_i - \mu$, $i = 1, \ldots, m$, and $\lambda_0 \in (-1/Y^*_{(m)}, -1/Y^*_{(1)})$ is a unique solution of

$$g(\lambda) \equiv \frac{1}{m} \sum_{i=1}^m \frac{Y^*_i}{(1 + \lambda Y^*_i)} = 0. \tag{5.5}$$

Therefore, by (5.3) it suffices to show that $-2\log R_0 \to (1 + \frac{\gamma\tau^2}{\sigma^2})\chi^2_{(1)}$ in distribution.

From the Markov inequality and (A4), we have

$$\max_{1\le i\le m} |Y^*_i| < m^{1/3}, \qquad \text{in probability.} \tag{5.6}$$

Let

$$\bar{X}^*_m = m^{-1} \sum_{i=1}^m X^*_i \qquad \text{and} \qquad S^2 = m^{-1} \sum_{i=1}^m (X^*_i - \mu)^2.$$

Then (A2) and the assumption $m = O(n)$ imply

$$\sqrt{m}(\bar{X}^*_m - \mu) = \sqrt{m}(\bar{X}^*_m - \mu_n) + \sqrt{m/n}\sqrt{n}(\mu_n - \mu) = O_p(1), \tag{5.7}$$

and (A4), (A2) and (A3) give

$$S^2 \xrightarrow{P} \sigma^2, \qquad \text{as} \quad n \to \infty. \tag{5.8}$$

Using the same argument in the proof of Theorem 1 in Owen (1988), by (5.6)–(5.8) we can show that $\lambda_0 = \{(\bar{X}_m^* - \mu)/S^2\}r_0 = O_p(m^{1/2})$, where $r_0 = 1 + o_p(1)$. Since (A4) implies $m^{-1}\sum_{i=1}^m |X_i^* - \mu|^3 = O_p(1)$, thus applying the argument of Owen ((1988), p. 242) almost line by line for his $-2\log R_0$ to our $-2\log R_0$ given by (5.4), we have

$$-2\log R_0 = \{\sqrt{m}(\bar{X}_m^* - \mu)/S\}^2 + o_p(1), \quad \text{as} \quad m \to \infty.$$

Since in (5.7) the conditional distribution of $\sqrt{m}(\bar{X}_m^* - \mu_n)$ converges to $N(0, \sigma^2)$ as $n \to \infty$, the proof follows from the fact that (A2) gives

$$\sqrt{m}(\bar{X}_m^* - \mu)/S \xrightarrow{D} Z + \frac{\sqrt{\gamma\tau}}{\sigma} Z_0, \quad \text{as} \quad n \to \infty,$$

where Z and Z_0 are two independent standard normal r.v.'s. □

PROOF OF THEOREM 2. (i) Let F_1^* be the empirical d.f. based on one leveraged bootstrap sample $X_1^*, \ldots, X_n^*$ and $F_{n,N}^*$ the empirical d.f. satisfying (2.8) based on the 'best' sample $X_{1,N}^*, \ldots, X_{n,N}^*$. From Shorack and Wellner ((1986), p. 12), we know

$$P_n\{\|F_{n,N}^* - \hat{F}_n\| \geq K/\sqrt{n}\} = (P_n\{\sqrt{n}\|F_1^* - \hat{F}_n\| \geq K\})^N \leq (58e^{-2K^2})^N,$$

where $n \geq 1$ and K is a constant such that $58/e^{2K^2} < 1$. Thus, we have that

$$\|F_{n,N}^* - \hat{F}_n\| \leq K/\sqrt{n}, \quad \text{as} \quad N \to \infty \tag{5.9}$$

in probability. From (A1) and (5.9), it can be shown that in probability

$$X_{(1),N}^* < \mu < X_{(n),N}^*, \quad \text{as} \quad n \to \infty, \; N \to \infty. \tag{5.10}$$

Let

$$r_{n,N}^*(x) = \sup\left\{\prod_{i=1}^n nw_i \,\middle|\, \sum_{i=1}^n w_i X_{i,N}^* = x, \text{ for } \boldsymbol{w} \text{ Satisfying (2.1)}\right\}.$$

Since (5.10) implies that $r_{n,N}^*(\mu)$ exists, from the proof of Theorem 1, we know that in probability,

$$X_{L,n,N}^* \leq \mu \leq X_{U,n,N}^* \quad \text{if and only if} \quad r_{n,N}^*(\mu) \geq c, \tag{5.11}$$

as $n \to \infty$, $N \to \infty$ Also from the proof of Theorem 1, we have

$$\log R_{0,N} \equiv \log r_{n,N}^*(\mu) = -\sum_{i=1}^n \log(1 + \lambda_{0,N}(X_{i,N}^* - \mu)), \tag{5.12}$$

where $\lambda_{0,N} \in (-(X_{(n),N}^* - \mu)^{-1}, -(X_{(1),N}^* - \mu)^{-1})$ is a unique solution of

$$g_N(\lambda) \equiv \frac{1}{n}\sum_{i=1}^n \frac{(X_{i,N}^* - \mu)}{(1 + \lambda(X_{i,N}^* - \mu))} = 0.$$

Nothing that for right censored data (1.4) and doubly censored data (1.5), $V_i \in \mathbb{R}$ for $1 \leq i \leq n$, and $X_{i,N}^* \in \{V_1, \ldots, V_n\}$, $1 \leq i \leq n$, (Efron (1967), Mykland and Ren

(1996)), thus from (A5), Theorem 3.2.1. of Chung (1974) and Borel-Cantelli Lemma, we can show that in probability,

$$\max_{1\le i\le n} |X^*_{i,N} - \mu| \le 2n^{1/3}, \quad \text{as} \quad n \to \infty. \tag{5.13}$$

For interval censored data (1.6), we have $V_i = (Y_i, Z_i) \in \mathbb{R}^2$ and (5.13) also holds, because of $X^*_{i,N} \in \{Y_1, Z_1, \ldots, Y_n, Z_n\}$, $1 \le i \le n$, (Groeneboom and Wellner (1992)) and because of (A5), convention (2.9) and

$$\max_{1\le i\le n} \{|X^*_{i,N} - \mu|\} \le \max_{1\le i\le n} \{|Y_i - \mu|\} + \max_{1\le i\le n} \{|Z_i - \mu|\}.$$

Let $[a_n, b_n]$ be the support of $\hat{F}_n$, then according to convention (2.9), the support of $F^*_{n,N}$ must be included by $[a_n, b_n] \subset [V_{(1)}, V_{(n)}]$ for censored data (1.4)–(1.6). Also let

$$\bar{X}^*_{n,N} = n^{-1}\sum_{i=1}^{n} X^*_{i,N} = \mu^*_{n,N} \quad \text{and} \quad S^2_N = n^{-1}\sum_{i=1}^{n}(X^*_{i,N} - \mu)^2,$$

then some straightforward calculation, (A2), (A6) and (5.9) give

$$n^{(1/2-\beta)}(\mu^*_{n,N} - \mu) \xrightarrow{P} 0 \quad \text{and} \quad S^2_N \xrightarrow{P} \sigma^2, \tag{5.14}$$

as $n \to \infty$, $N \to \infty$. Hence, applying the argument for g in the proof of Theorem 1 to g_N here, we have $\lambda_{0,N} = O_p(n^{-q})$, as $n \to \infty$, $N \to \infty$, for $q = \frac{1}{2} - \beta$.

Note that from (5.9), (A4) and (A6), we have that as $n \to \infty$, $N \to \infty$,

$$\begin{aligned} n^{-1}\sum_{i=1}^{n} |X^*_{i,N} - \mu|^3 &\le 8\left\{n^{-1}\sum_{i=1}^{n} |X^*_{i,N}|^3 + |\mu|^3\right\} \\ &= 8\left\{\int_{a_n}^{b_n} |x|^3 dF^*_{n,N}(x) + |\mu|^3\right\} \le 8\left\{\int_{a_n}^{b_n} x^4 dF^*_{n,N}(x) + 1 + |\mu|^3\right\} \\ &= 8\left\{4\int_{a_n}^{b_n} x^3(\hat{F}_n(x) - F^*_{n,N}(x))dx + \int_{a_n}^{b_n} x^4 d\hat{F}_n(x) + 1 + |\mu|^3\right\} \\ &\le 32(b_n - a_n)^3 \max\{|b_n|^3, |a_n|^3\}\|\hat{F}_n - F^*_{n,N}\| + O_p(1) = o_p(1) + O_p(1). \end{aligned} \tag{5.15}$$

Applying the argument of Owen ((1988), p. 242) almost line by line for his $-2\log R_0$ to our $-2\log R_{0,N}$ given by (5.12), from $\lambda_{0,N} = O_p(n^{-q})$ and (5.15) we have that

$$-2\log R_{0,N} = (2r_{0,N} - r^2_{0,N})\{\sqrt{n}(\bar{X}^*_{n,N} - \mu)/S_N\}^2 + o_p(1), \tag{5.16}$$

as $n \to \infty$, $N \to \infty$ where $r_{0,N} = 1 + o_p(1)$. Therefore, (2.10) follows from (5.11), (5.12), (5.14) and (5.16).

(ii) First, note that among all possible empirical d.f.'s based on complete i.i.d. samples from $\hat{F}_n$, there exists one F^*_n such that

$$\|F^*_n - \hat{F}_n\| \le 1/n \quad \text{and} \quad \|F^*_{n,N} - F^*_n\| \xrightarrow{P_n} 0, \quad \text{as} \quad N \to \infty. \tag{5.17}$$

From the discussions in the proof of (i) above, we know that the support of $F^*_{n,N}$ must be included by the support of $\hat{F}_n$; that is the support of $F^*_{n,N}$ is included by $[a_n, b_n] \subset [V_{(1)}, V_{(n)}]$ for censored data (1.4)–(1.6). Thus,

$$\sqrt{n}|\mu^*_{n,N} - \mu^*_n| = \sqrt{n}\left|\int_{a_n}^{b_n} (F^*_{n,N} - F^*_n)dx\right| \overset{P_n}{\to} 0, \quad \text{as} \quad N \to \infty, \tag{5.18}$$

where $\mu^*_n = \int x dF^*_n(x)$. Moreover, from (A6) and (5.17) we know that as $n \to \infty$,

$$\sqrt{n}|\mu^*_n - \mu_n| = \sqrt{n}\left|\int_{a_n}^{b_n} (F^*_n - \hat{F}_n)dx\right| \le \sqrt{n}(b_n - a_n)\|F^*_n - \hat{F}_n\| = o_{n,p}(1). \tag{5.19}$$

Hence, (5.18), (5.19) and (5.14) imply

$$\delta^*_{n,N} \overset{P_n}{\to} o_{n,p}(1), \quad \text{as} \quad N \to \infty. \tag{5.20}$$

Therefore, the proof follows from that fact that (2.10), (5.20), (A2), (5.14) and the Dominated Convergence Theorem (DCT) imply

$$\begin{aligned}
&\lim_{n\to\infty} \lim_{N\to\infty} P\{X^*_{L,n,N} \le \mu \le X^*_{U,n,N}\} \\
&\overset{DCT}{=} \lim_{n\to\infty} E\left\{\lim_{N\to\infty} P_n\left\{\left((1 + o_p(1))\delta^*_{n,N} + \frac{\sqrt{n}(\mu_n - \mu)}{\sigma_n}\right)^2 \le -2\log c\right\}\right\} \\
&= \lim_{n\to\infty} P\left\{\left((1 + o_p(1))o_{n,p}(1) + \frac{\sqrt{n}(\mu_n - \mu)}{\sigma_n}\right)^2 \le -2\log c\right\}. \qquad \square
\end{aligned}$$

Proof of Theorem 3. Since $W_1 \le X^*_{(1)} < X^*_{(m)} \le W_{n_0}$ in (5.1), then (A1) implies

$$W_1 < \mu < W_{n_0}, \quad \text{in probability} \tag{5.21}$$

as $n \to \infty$. From Remark 4 in Section 2, it can be shown that in probability, $\hat{X}_{L,n} \le \mu \le \hat{X}_{U,n}$ if and only if

$$r(\mu) = \sup\left\{\prod_{i=1}^{n_0}(p_i/\hat{p}_i)^{n\hat{p}_i} \,\middle|\, \sum_{i=1}^{n_0} p_i W_i = \mu, p_i \ge 0, \sum_{i=1}^{n_0} p_i = 1\right\} \ge c. \tag{5.22}$$

Following the proof of Theorem 1 by Owen (1988), from (A2), (A3) and (A5) we have

$$\max_{1\le i\le n_0} |W_i - \mu| = O_p(n^{1/3}) \tag{5.23}$$

and

$$\log r(\mu) = -n\sum_{i=1}^{n_0} \hat{p}_i \log(1 + \lambda_0(W_i - \mu)) \tag{5.24}$$

where

$$\lambda_0 = O_p(n^{-1/2}) \tag{5.25}$$

is the solution of

$$g(\lambda) \equiv \sum_{i=1}^{n_0} \frac{\hat{p}_i(W_i - \mu)}{1 + \lambda(W_i - \mu)} = 0. \tag{5.26}$$

Let

$$(5.27)\quad S^2 = \sum_{i=1}^{n_0} \hat{p}_i (W_i - \mu)^2 = \sigma_n^2 + (\mu_n - \mu)^2 = \sigma_n^2 + O_p(n^{-1}),$$

and

$$(5.28)\quad \hat{\mu}_3 = \sum_{i=1}^{n_0} \hat{p}_i (W_i - \mu)^3 = \hat{\mu}_{3n} + 3(\mu_n - \mu)\sigma_n^2 + (\mu_n - \mu)^3 = \hat{\mu}_{3n} + O_p(n^{-1/2}).$$

From (5.23), (5.25) and (5.26), the Taylor expansion gives

$$(5.29)\qquad \begin{aligned} 0 = g(\lambda_0) &= \sum_{i=1}^{n_0} \frac{\hat{p}_i (W_i - \mu)}{1 + \lambda_0 (W_i - \mu)} \\ &= \sum_{i=1}^{n_0} \hat{p}_i (W_i - \mu)[1 - \lambda_0 (W_i - \mu) + \lambda_0^2 (W_i - \mu)^2 + \xi_i^3] \\ &= (\mu_n - \mu) - \lambda_0 S^2 + \lambda_0^2 \hat{\mu}_3 + O_p(n^{-3/2}), \end{aligned}$$

where $|\xi_i| \le |\lambda_0 (W_i - \mu)|$ and by (A7)

$$\left| \sum_{i=1}^{n_0} \hat{p}_i (W_i - \mu) \xi_i^3 \right| \le |\lambda_0|^3 \sum_{i=1}^{n_0} \hat{p}_i (W_i - \mu)^4 = O_p(n^{-3/2}).$$

From (A2), (5.25), (5.27)–(5.28) and (A8), we can show that (5.29) implies

$$(5.30)\qquad \lambda_0 = \frac{(\mu_n - \mu)}{S^2} + \frac{(\mu_n - \mu)^2 \hat{\mu}_3}{S^6} + O_p(n^{-3/2}).$$

Thus, the Taylor expansion, (5.24) and (5.30) give

$$\begin{aligned} -2 \log r(\mu) &= 2n \sum_{i=1}^{n_0} \hat{p}_i \log(1 + \lambda_0 (W_i - \mu)) \\ &= 2n \sum_{i=1}^{n_0} \hat{p}_i \left(\lambda_0 (W_i - \mu) - \frac{\lambda_0^2 (W_i - \mu)^2}{2} + \frac{\lambda_0^3 (W_i - \mu)^3}{3} + \frac{\eta_i}{4} \right) \\ &= \frac{n(\mu_n - \mu)^2}{S^2} \left\{ 1 + \frac{2(\mu_n - \mu)\hat{\mu}_3}{3S^4} \right\} + O_p(n^{-1}), \end{aligned}$$

where $|\eta_i| \le |\lambda_0 (W_i - \mu)|^4$, and by (A7)

$$\left| \sum_{i=1}^{n_0} \hat{p}_i \eta_i \right| \le |\lambda_0|^4 \sum_{i=1}^{n_0} \hat{p}_i |W_i - \mu|^4 = O_p(n^{-2}).$$

Therefore, (2.17) follows from (5.22), (5.27) and (5.28). □

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