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ON HADAMARD DIFFERENTIABILITY AND ITS APPLICATION TO R-ESTIMATION IN LINEAR MODELS

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Abstract. We show that the remainder term of a form of the Taylor expansion, involving the Hadamard derivative, of the functional defined on the space D[0,1]×D[0,1] is uniformly asymptotically negligible over a compact set with respect to weighted empirical processes. We also show that the functional induced by the estimating equation of R-estimators of regression is defined on the space D[0,1]×D[0,1] and that this functional is Hadamard differentiable. Such differentiability property of the estimating equations directly reveals the equivalence relation between the estimating equations of R- and M-estimatiors of regression. Thereby, the equivalence of R- and M-estimators of regression and the asymptotic properties of R-estimators are derived under less stringent conditions in the literature.

1. Introduction.

In nonparametric models, a parameter θ (= T(F)) is regarded as a functional $T(\cdot)$ on a space $\mathfrak{F}$ of distribution functions (d.f.) F. Thus, the same functional of the sample d.f. (i.e., $T(F_n)$) is regarded as a natural estimator of θ. Using a form of the Taylor expansion involving the derivatives of the functional, Von Mises (1947) expressed $T(F_n)$ as

$$T(F_n) = T(F) + T'_F(F_n-F) + \text{Rem}(F_n-F;\ T(\cdot)) \tag{1.1}$$

where T'_F is the derivative of the functional at F and $\text{Rem}(F_n-F;\ T(\cdot))$ is the

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remainder term in this first order expansion. Note that $F_n(x) = n^{-1}\sum_{i=1}^{n} I(X_i \le x)$ is based on n independent and identically distributed random variables (i.i.d.r.v.) $X_1, \cdots, X_n$, each having the d.f. F, and that T'_F is a linear functional. Hence, $T'_F(F_n - F)$ is an average of n i.i.d.r.v.'s. For drawing statistical conclusions (in large samples), T'_F plays the basic role, and in this context, it remains to show that $\text{Rem}(F_n - F;\ T(\cdot))$ is asymptotically negligible to the desired extent. Since a statistical functional induces a functional on the space D[0,1] (of right continuous functions having left hand limits) by: $\tau(G) = T(G \circ F)$, where $G \in D[0,1]$, we can equivalently write (1.1) as

$$\tau(U_n) = \tau(U) + \tau'_U(U_n - U) + \text{Rem}(U_n - U;\ \tau) \tag{1.2}$$

where U_n is the empirical d.f. of the $F(X_i)$, $1 \le i \le n$, and U is the classical uniform d.f. on [0,1] (i.e., $U(t) = t$, $0 \le t \le 1$). Hence, it is equivalent to show that $\text{Rem}(U_n - U; \tau(\cdot))$ is asymptotically negligible. The appropriate differentiability conditions are usually incorporated in this verification. The condition of Fréchet differentiability (viz., Kallianpur and Rao, 1955; Boos and Serfling, 1981, among others), Hadamard (or Compact) differentiability (viz., Reeds, 1976; Fernholz, 1983; Gill, 1989, 1991; Ren and Sen, 1991; among others) and Gâteaux differentiability (viz., Kallianpur, 1963; among others) have been considered by various people. Particularly, using Hadamard differentiability (along with some other regularity conditions), Reeds (1976) showed that

$$\sqrt{n}\ \text{Rem}(U_n - U;\ \tau) \xrightarrow{P} 0, \qquad \text{as } n \to \infty. \tag{1.3}$$

This Von Mises method has been used by Fernholz (1983) and others to study the asymptotic properties of estimators based on i.i.d.r.v.'s.

Here, we consider a different application of Hadamard differentiability. Some statistics, such as (robust) M- and R-estimators in linear models (which are not based on i.i.d.r.v.'s), are defined through some estimating equations. Specifically, consider the simple linear model:

$$X_i = \underset{\sim}{c}_i^T \underset{\sim}{\beta} + e_i, \qquad i \ge 1 \tag{1.4}$$

where the $\underset{\sim}{c}_i = (c_{i1}, \cdots, c_{ip})^T$ are known p-vectors of regression constants, $\underset{\sim}{\beta}$ is the p-vector of unknown (regression) parameters, $p \ge 1$, and e_i are i.i.d.r.v.'s with d.f. F. Then an M-estimator $\hat{\underset{\sim}{\beta}}_M$, suggested by Huber (1973), is defined as a solution (with respect to $\underset{\sim}{\theta}$) of the equations

$$\sum_{i=1}^{n} \underset{\sim}{c}_i\ \varphi(X_i - \underset{\sim}{c}_i^T \underset{\sim}{\theta}) \equiv \underset{\sim}{0}, \tag{1.5}$$

where "$\equiv \underset{\sim}{0}$" means "as near $\underset{\sim}{0}$ as possible" and φ is the score function. An R-estimator $\hat{\underset{\sim}{\beta}}_R$, suggested respectively by Jurečková (1971) and by Jaeckel (1972), is defined as a solution of the equations

$$\sum_{i=1}^{n} a[R(X_i - \underset{\sim}{c}_i^T \underset{\sim}{\theta})]\ (\underset{\sim}{c}_i - \bar{\underset{\sim}{c}}) \equiv \underset{\sim}{0}, \tag{1.6}$$

where $R(X_i - c_i^T\theta)$ is the rank of $X_i - c_i^T\theta$ among $\{X_1 - c_1^T\theta, \cdots, X_n - c_n^T\theta\}$, $a(i) = a_n(i) = \psi(\frac{i}{n+1})$ for a proper score-generating function ψ, $\bar{c} = (\bar{c}_1, \cdots, \bar{c}_p)^T$ is a column of $\bar{D}_n = n^{-1}1_n 1_n^T D_n$ for $D_n = (c_1, \cdots, c_n)^T$. We will see later on that these estimating equations induces a functional defined on a space of distribution functions, and that frequently this statistical functional possesses differentiability properties which provide information about the asymptotic behavior of the estimating equations.

We denote $\|\cdot\|$ for Euclidean norm. Let $Y_i = X_i - c_i^T\beta$ (i.i.d.r.v's with d.f. F), and let $B_n = \sum_{i=1}^n c_i c_i^T = D_n^T D_n$, $C_n = \sum_{i=1}^n (c_i - \bar{c})(c_i - \bar{c})^T = (D_n - \bar{D}_n)^T(D_n - \bar{D}_n)$, $B_n^0 = \mathrm{Diag}(\|d_1\|, \cdots, \|d_p\|)$ for d_i to be the column vector of D_n, $C_n^0 = \mathrm{Diag}(\|\bar{d}_1\|, \cdots, \|\bar{d}_p\|)$ for $\bar{d}_i$ to be the column vector of $(D_n - \bar{D}_n)$, $b_{ni} = (B_n^0)^{-1}c_i = (b_{ni1}, \cdots, b_{nip})^T$, $c_{ni} = (C_n^0)^{-1}(c_i - \bar{c}) = (c_{ni1}, \cdots, c_{nip})^T$, $u = B_n^0(\theta - \beta)$ and $v = C_n^0(\theta - \beta)$, then (1.5) is equivalent to the normalized equations

$$M_n(u) = \sum_{i=1}^n b_{ni}\varphi(Y_i - b_{ni}^T u) \equiv 0, \tag{1.7}$$

and with the assumption $\sum_{i=1}^n a(i) = 0$, R-estimators are defined by the solutions of

$$E_n(v) = \sum_{i=1}^n c_{ni}\ a[R(Y_i - c_{ni}^T v)] \equiv 0. \tag{1.8}$$

Here (1.8) is a normalized version of the estimating equations suggested by Jurečková (1971) or Hettmansperger (1984). We notice that $M_n(u)$ is a linear functional of $\mathfrak{I}_n^*(t,u) = \sum_{i=1}^n b_{ni} I(Y_i \le F^{-1}(t) + b_{ni}^T u)$, $t \in [0,1]$, $u \in R^p$, viz.,

$$M_n(u) = \int_0^1 \varphi(F^{-1}(t))\, d\mathfrak{I}_n^*(t,u) = \Phi(\mathfrak{I}_n^*(\cdot,u)),$$

and $E_n(v)$ is (equivalent to) a functional of $(S_n^*(\cdot,v), F_n^*(\cdot,v))$ (see Theorem 4.3), viz.,

$$E_n(v) \doteq \int_0^1 S_n^*(F_n^{*-1}(t,v), v)\, d\psi(t) = \Psi(S_n^*(\cdot,v), F_n^*(\cdot,v)),$$

where for $t \in [0,1]$, $v \in R^p$,

$$F_n^*(t,v) = \frac{1}{n+1} \sum_{i=1}^n I\{Y_i \le F^{-1}(t) + c_{ni}^T v\}, \tag{1.9}$$

$$S_n^*(t,v) = \sum_{i=1}^n c_{ni}\ I\{Y_i \le F^{-1}(t) + c_{ni}^T v\}, \tag{1.10}$$

and $F_n^{*-1}(t,v) = \inf\{y;\ F_n^*(y,v) \ge t\}$. Since for any fixed u, $\mathfrak{I}_{nk}^*(\cdot,u)$ (the component of $\mathfrak{I}_n^*$) is an element of D[0,1] and $(S_{nk}^*(\cdot,u), F_n^*(\cdot,u))$ $\big($the component of $(S_n^*, F_n^*)\big)$ is an element of D[0,1]×D[0,1], M_n and E_n correspond to a functional defined on the space D[0,1] and D[0,1]×D[0,1], respectively. We show, in Lemma 4.1 that functional Ψ, induced by the estimating equations $E_n(v)$ of R-estimators of regression, is Hadamard differentiable. Such differentiability property of Ψ suggests its asymptotic equivalence relation with a linear functional Ψ', provided a general form of (1.3) holds with respect to $(\underset{\sim}{S}_n^*, F_n^*)$.

The essential difficulty in the study of the asymptotic properties of M- and R-estimators is the nonlinearity of the estimators, and often it is the case that the uniform asymptotic linear approximation of the estimators (Jurečková, 1971, 1977) may provide an easy access to the study. To establish the follwoing Jurečková-uniform asymptotic linearity of M-estimators:

$$(1.11) \qquad \sup_{|\underset{\sim}{u}|\leq K} \left|\underset{\sim}{M}_n(\underset{\sim}{u})-\underset{\sim}{M}_n(\underset{\sim}{0})+\underset{\sim}{Q}_n\underset{\sim}{u}\gamma_\varphi\right| \xrightarrow{P} 0, \qquad \text{as } n\to\infty,$$

where K is any fixed positive real number, $|\cdot|$ stands for uniform norm, $\underset{\sim}{Q}_n = \sum_{i=1}^n \underset{\sim}{b}_{ni}\underset{\sim}{b}_{ni}^T = (\underset{\sim}{B}_n^o)^{-1}\underset{\sim}{B}_n(\underset{\sim}{B}_n^o)^{-1}$ and $\gamma_\varphi = \int F'(x)d\varphi(x) > 0$, Ren and Sen (1991), using Hadamard differentiability, extend the result (1.3) to a more general form with respect to the weighted empirical process $\underset{\sim}{\mathcal{S}}_n^*$ for p = 1, which show that $\text{Rem}(\mathcal{S}_{nk}^*(\cdot,\underset{\sim}{u})-U;\,\tau)$ is uniformly asymptotically negligible for $|\underset{\sim}{u}| \leq K$. A further generalization regarding a functional defined on the space D[0,1]×D[0,1] with respect to the weighted empirical processes $(\underset{\sim}{S}_n^*, F_n^*)$ is established in Theorem 3.1. We would like to point out that our Theorem 3.1 is a general result on Hadamard differentiability. It includes (1.3) as a special case, and it allow us to directly investigate the asymptotic behavior of any statistic which induces a Hadamard differentiable functional of weighted empirical processes. In fact, the L-estimators of regression of Welsh (1987) or Ren (1992) and above estimating equations of R-estimators of regression all induce some Hadamard differentiable functional of $(\underset{\sim}{S}_n^*, F_n^*)$ (see Ren, 1991 and 1992, on L-estimators of regression).

To illustrate the use of our Theorem 3.1, we show, in Theorem 4.3 and Theorem 4.4, that the estimating equations of M- and R-estimators of regression, $\underset{\sim}{M}_n(\underset{\sim}{u})$ and $\underset{\sim}{E}_n(\underset{\sim}{u})$, are asymptotically and uniformly equivalent for proper score functions, because the functional induced by $\underset{\sim}{M}_n(\underset{\sim}{u})$ is linear and because Ψ, the functional induced by $\underset{\sim}{E}_n(\underset{\sim}{u})$, is Hadamard differentiable. By this equivalence of the estimating equations, easily the uniform asymptotic linearity of R-estimators is followed from (1.11) and the asymptotic properties of R-estimators are derived.

Jurečková (1977) studied the relation between M- and R-estimators. In her proof, she required more technical assumptions on the structure of the design matrix and assumed that F has finite Fisher's information. These assumptions are weakened in Theorem 4.4 without requiring any essentially stronger conditions on ψ for the asymptotic equivalence of M- and R-estimators of regression.

The asymptotic normality of R-estimators was studied by Jurečková (1971) for general scores with technical assumptions on the structure of the design matrix $\underset{\sim}{D}_n$ and by Aubuchon (1982) (a proof can be found in Hettmansperger's (1984)) for a specific score. Applying an analogue of the Convexity Lemma (Pollard, 1991), which was established to derive the limit distribution using a technique analogous to the method by Jurečková (1977), Heiler and Willers (1988) relaxed Jurečková's (1971) assumptions on $\underset{\sim}{D}_n$. All these people require that the error distribution function F has finite Fisher's information. In Theorem 4.3, we weaken these conditions for the same study.

2. Notation and Assumptions.

Consider the D[0,1] space endowed with uniform topology and the σ-field of subsets of D[0,1], $\mathfrak{D}$, generated by the open balls. The space C[0,1] of real valued continuous functions, endowed with the uniform topology, is a subspace of D[0,1]. For convenience sake, we list most notations used through this paper as below. For $1 \le j \le p$, $t \in [0,1]$, $\underset{\sim}{u} \in R^p$, we denote

$$E_{nj}(\underset{\sim}{u}) = \sum_{i=1}^{n} c_{nij}\ a[R(Y_i - \underset{\sim}{c}_{ni}^{T}\underset{\sim}{u})],$$

as the components of $\underset{\sim}{E}_n(\underset{\sim}{u})$, and

$$S_{nj}^{*}(t,\underset{\sim}{u}) = \sum_{i=1}^{n} c_{nij} I\{Y_i \le F^{-1}(t) + \underset{\sim}{c}_{ni}^{T}\underset{\sim}{u}\},$$

as the components of $\underset{\sim}{S}_n^{*}(t,\underset{\sim}{u})$. We also let

$$S_{nj}^{*+}(t,\underset{\sim}{u}) = \sum_{i=1}^{n} c_{nij}^{+}\ I\{Y_i \le F^{-1}(t) + \underset{\sim}{c}_{ni}^{T}\underset{\sim}{u})\},$$
$$S_{nj}^{*-}(t,\underset{\sim}{u}) = \sum_{i=1}^{n} c_{nij}^{-}\ I\{Y_i \le F^{-1}(t) + \underset{\sim}{c}_{ni}^{T}\underset{\sim}{u})\},$$

and

$$F_n(t,\underset{\sim}{u}) = E\{F_n^{*}(t,\underset{\sim}{u})\} = \frac{1}{n+1}\ \sum_{i=1}^{n}\ F(F^{-1}(t) + \underset{\sim}{c}_{ni}^{T}\underset{\sim}{u})),$$
$$S_{nj}(t,\underset{\sim}{u}) = E\{S_{nj}^{*}(t,\underset{\sim}{u})\} = \sum_{i=1}^{n} c_{nij}\ F(F^{-1}(t) + \underset{\sim}{c}_{ni}^{T}\underset{\sim}{u})),$$
$$S_{nj}^{+}(t,\underset{\sim}{u}) = E\{S_{nj}^{*+}(t,\underset{\sim}{u})\} = \sum_{i=1}^{n} c_{nij}^{+}\ F(F^{-1}(t) + \underset{\sim}{c}_{ni}^{T}\underset{\sim}{u})),$$
$$S_{nj}^{-}(t,\underset{\sim}{u}) = E\{S_{nj}^{*-}(t,\underset{\sim}{u})\} = \sum_{i=1}^{n} c_{nij}^{-}\ F(F^{-1}(t) + \underset{\sim}{c}_{ni}^{T}\underset{\sim}{u})),$$

where $c_{nij}^{+} = \max\{0, c_{nij}\}$ and $c_{nij}^{-} = -\min\{0, c_{nij}\}$. Then,

$$c_{nij} = c_{nij}^{+} - c_{nij}^{-}, \qquad \underset{\sim}{c}_{ni} = \underset{\sim}{c}_{ni}^{+} - \underset{\sim}{c}_{ni}^{-},$$
$$S_{nj}^{*}(t,\underset{\sim}{u}) = S_{nj}^{*+}(t,\underset{\sim}{u}) - S_{nj}^{*-}(t,\underset{\sim}{u}),$$
$$S_{nj}(t,\underset{\sim}{u}) = S_{nj}^{+}(t,\underset{\sim}{u}) - S_{nj}^{-}(t,\underset{\sim}{u}).$$

We denote, for $n \ge 1$, $\underset{\sim}{Q}_n = \sum_{i=1}^{n} \underset{\sim}{c}_{ni}\underset{\sim}{c}_{ni}^{T} = (\underset{\sim}{C}_n^{o})^{-1}\underset{\sim}{C}_n(\underset{\sim}{C}_n^{o})^{-1}$, then some assumptions, which may be required for our results, are given below:

(A1) $\overline{\lim}_{n\to\infty} \max_{1\le i\le n} \|\underset{\sim}{c}_{ni}\|^2 = 0$;

(A2) There exists a positive definite p×p matrix $\underset{\sim}{Q}$ such that $\lim_{n\to\infty} \underset{\sim}{Q}_n = \underset{\sim}{Q}$.

(B) F has a positive and uniformly continuous derivative F′.

(C1) The *scores* $a_n(i)$, i =1, 2, ⋯, n, are generated by a function $\psi(u)$, $0 < u < 1$ by either of the following two ways:

$$a_n(i) = E\{\psi(U_n^{(i)})\}$$

$$a_n(i) = \psi(i/(n+1)),$$

where $U_n^{(i)}$ denotes the *i*th order statistic in a sample of size n from uniform distribution on (0,1).

(C2) The score-generating function ψ is a finite sum of right continuous and monotone functions on [0,1] with

$$A^2 = \int_0^1 [\psi(t)]^2 dt > 0, \quad \text{and} \quad \int_0^1 \psi(t)dt = 0.$$

(D) φ is bounded, nondecreasing and right or left continuous with $\int \varphi dF = 0$.

Remark 1. It can be shown that (A1) and (A2) are implied by the conditions on the design matrix required by Jurečková (1971, 1977) and Aubuchon (1982). (A1) is a special case of (V′) of Heiler and Willers (1988). Hence, (A1) is weaker and easier to check than (V) of Heiler and Willers (1988). However, Heiler and Willers (1988) do not require any limiting condition on $(\underset{\sim}{D}_n - \overline{\underset{\sim}{D}}_n)$, i.e., (A2) in our case. This is because they consider the asymptotic normality of (equivalently) $\underset{\sim}{Q}_n^{-1/2} \underset{\sim}{C}_n^o(\hat{\underset{\sim}{\beta}}_R - \underset{\sim}{\beta})$, instead of our $\underset{\sim}{C}_n^o(\hat{\underset{\sim}{\beta}}_R - \underset{\sim}{\beta})$ (see Theorem 4.3). Note that $\underset{\sim}{C}_n^o$ is a diagonal matrix with diagonal elements to be the norm of the column vector of $(\underset{\sim}{D}_n - \overline{\underset{\sim}{D}}_n)$. Hence, our $\underset{\sim}{C}_n^o(\hat{\underset{\sim}{\beta}}_R - \underset{\sim}{\beta})$ is more general than $\sqrt{n}(\hat{\underset{\sim}{\beta}}_R - \underset{\sim}{\beta})$ considered by Jurečková (1971) and Aubuchon (1982), and the advantage of considering $\underset{\sim}{C}_n^o(\hat{\underset{\sim}{\beta}}_R - \underset{\sim}{\beta})$ is that its asymptotic normality clearly implies the convergence rate in probability for each component of $\hat{\underset{\sim}{\beta}}_R$ is just the norm of the column vector of $(\underset{\sim}{D}_n - \overline{\underset{\sim}{D}}_n)$, respectively.

Remark 2. Jurečková (1971, 1977), Aubuchon (1982) and Heiler and Willers (1988) all require F has finite Fisher's information for the study of the equivalence of M- and R-estimators of regression or the asymptotic normality of R-estimators of regression. Note that, by Theorem 4.2 of Huber (1981, page 77), finite Fisher's information implies that the density function F′ = f has to be absolutely continuous. In comparison, our condition (B) on F is weaker. The condition of finite Fisher's information usually comes along with the study of the efficiency of the estimator. However, the efficiency of R-estimator (even for location model) cannot be achieved when F is not symmetric (Huber, 1981, page 70). Hence, just for the study of asymptotic normality of R-estimators and its equivalence relation with M-estimators, the condition of finite Fisher's information does not seem necessary. In fact, Bickel (1973) and Welsh (1987) do not require such condition on F for L-estimators of regression. In practice, weaker conditions on F increases the usability of the estimators, since F is usually unknown and not necessarily symmetric. The examples of F with infinite Fisher's information can be easily found. For instance, if the error distribution is the mixture of double gamma distribution, say, $F(x) = \lambda F_1(x) + (1-\lambda)F_2(x)$, where $0<\lambda<1$ and F_i is the double gamma d.f. with p.d.f.: $f_i(x) = \beta_i^{\alpha_i}[2\Gamma(\alpha_i)]^{-1}|x|^{\alpha_i-1}e^{-\beta_i|x|}$, $\alpha_i > 0$, $\beta_i > 0$, i = 1, 2, then $I(f) = \infty$ for certain α_i and β_i (for example: when $\beta_1 = \beta_2 = 1$, $\alpha_2 = 1$, $1 < \alpha_1 < 3/2$). Note that our condition (B) holds though for such F.

Remark 3. (C2) implies that ψ is square integrable and is essentially the same as those conditions required by Jurečková (1971, 1977) and by Heiler and Willers (1988). The standardization condition of ψ, $\int_0^1 \psi(t)dt = 0$, implies $\sum_{i=1}^{n} a_n(i) = 0$.

To prove our results in this paper, the definition of Hadamard differentiability is given as below.

DEFINITION. Let V and W be the topological vector spaces and L(V,W) be the set of continuous linear transformation from V to W. Let $\mathcal{A}$ be an open set of V, a functional τ: $\mathcal{A} \to W$ is *Hadamard Differentiable* (or *Compact Differentiable*) at $S \in \mathcal{A}$ if there exists $\tau'_S \in L(V,W)$ such that for any compact set Γ of V,

$$\lim_{t\to 0} \frac{\tau(S+tH)-\tau(S)-\tau'_S(tH)}{t} = 0$$

uniformly for any $H \in \Gamma$. The linear function τ'_S is called the *Hadamard Derivative* of τ at S.

For our current study, we consider the functional τ defined on the space $D[0,1]\times D[0,1]$, and denote

$$\mathrm{Rem}(tH; \tau) = \tau(S+tH)-\tau(S)-\tau'_S(tH),$$

where S = (U, U) and $H \in D[0,1]\times D[0,1]$.

3. On Hadamard Differentiability

THEOREM 3.1. Suppose τ: $D[0,1]\times D[0,1] \to R$ is a functional and is Hadamard differentiable at (U,U). Assume (A1) and (B). Then, for any K>0, $1 \le k \le p$, as $n \to \infty$

$$\sup_{|\underset{\sim}{u}| \le K} \Big| \sum_{i=1}^{n} c_{nik}^{+} \, \mathrm{Rem}\Big(F_n^*(\cdot,\underset{\sim}{u}) - U(\cdot), \frac{S_{nk}^{*+}(\cdot,\underset{\sim}{u})}{\sum_{i=1}^{n} c_{nik}^{+}} - U(\cdot); \tau\Big) \Big| \xrightarrow{P} 0, \tag{3.1}$$

and

$$\sup_{|\underset{\sim}{u}| \le K} \Big| \sum_{i=1}^{n} c_{nik}^{-} \, \mathrm{Rem}\Big(F_n^*(\cdot,\underset{\sim}{u}) - U(\cdot), \frac{S_{nk}^{*-}(\cdot,\underset{\sim}{u})}{\sum_{i=1}^{n} c_{nik}^{-}} - U(\cdot); \tau\Big) \Big| \xrightarrow{P} 0. \tag{3.2}$$

Therefore, we have as $n \to \infty$

$$\sup_{|\underset{\sim}{u}| \le K} \Big| \sum_{i=1}^{n} c_{nik}^{+} \Big\{\tau\Big(F_n^*(\cdot,\underset{\sim}{u}), \frac{S_{nk}^{*+}(\cdot,\underset{\sim}{u})}{\sum_{i=1}^{n} c_{nik}^{+}}\Big) - \tau\big(U(\cdot), U(\cdot)\big)\Big\} -$$

$$- \sum_{i=1}^{n} c_{nik}^{+} \, \tau'_{(U,U)}\Big(F_n^*(\cdot,\underset{\sim}{u}) - U(\cdot), \frac{S_{nk}^{*+}(\cdot,\underset{\sim}{u})}{\sum_{i=1}^{n} c_{nik}^{+}} - U(\cdot)\Big)\Big| \xrightarrow{P} 0, \tag{3.3}$$

and

$$\sup_{|\underset{\sim}{u}| \le K} \Big| \sum_{i=1}^{n} c_{nik}^{-} \Big\{\tau\Big(F_n^*(\cdot,\underset{\sim}{u}), \frac{S_{nk}^{*-}(\cdot,\underset{\sim}{u})}{\sum_{i=1}^{n} c_{nik}^{-}}\Big) - \tau\big(U(\cdot), U(\cdot)\big)\Big\} -$$

$$- \sum_{i=1}^{n} c_{nik}^{-} \, \tau'_{(U,U)}\Big(F_n^*(\cdot,\underset{\sim}{u}) - U(\cdot), \frac{S_{nk}^{*-}(\cdot,\underset{\sim}{u})}{\sum_{i=1}^{n} c_{nik}^{-}} - U(\cdot)\Big)\Big| \xrightarrow{P} 0. \tag{3.4}$$

The proof of Theorem 3.1 will be given later. We first notice that the following corollary follows immediately from Theorem 3.1 because $\sum_{i=1}^{n} c_{nik} = \sum_{i=1}^{n} c_{nik}^{+} - \sum_{i=1}^{n} c_{nik}^{-} = 0$ for $1 \le k \le p$.

COROLLARY 3.2. Suppose τ: $D[0,1] \times D[0,1] \to R$ is a functional and is Hadamard differentiable at (U,U). Assume (A1) and (B). Then, for any $1 \le k \le p$ and $K>0$, as $n \to \infty$

$$\sup_{|\underset{\sim}{u}| \le K} \Big| \sum_{i=1}^{n} c_{nik}^{+} \tau\Big(F_n^*(\cdot,\underset{\sim}{u}), \frac{S_{nk}^{*+}(\cdot,\underset{\sim}{u})}{\sum_{i=1}^{n} c_{nik}^{+}}\Big) - \sum_{i=1}^{n} c_{nik}^{-} \tau\Big(F_n^*(\cdot,\underset{\sim}{u}), \frac{S_{nk}^{*-}(\cdot,\underset{\sim}{u})}{\sum_{i=1}^{n} c_{nik}^{-}}\Big) -$$

$$- \tau'_{(U,U)}\big(0, S_{nk}^*(\cdot,\underset{\sim}{u})\big)\Big| \xrightarrow{P} 0. \tag{3.5}$$

Before we start proving Theorem 3.1, we need to notice that

$$\sum_{i=1}^{n} c_{nik}^{2} = 1, \qquad \sum_{i=1}^{n} c_{nik} = \sum_{i=1}^{n} c_{nik}^{+} - \sum_{i=1}^{n} c_{nik}^{-} = 0, \qquad \text{for } 1 \le k \le p \tag{3.6}$$

and a few facts implied by assumption (A1) and (B). Since

$$1 = \sum_{i=1}^{n} c_{nik}^2 \le \{\max_{1\le i\le n}|c_{nik}|\} \sum_{i=1}^{n}|c_{nik}| = 2\{\max_{1\le i\le n}|c_{nik}|\} \sum_{i=1}^{n} c_{nik}^{+},$$

(A1) implies

$$\sum_{i=1}^{n} c_{nik}^{+} = \sum_{i=1}^{n} c_{nik}^{-} \to \infty, \qquad \text{as } n \to \infty. \tag{3.7}$$

(B) implies that F' is bounded and uniformly continuous. We can also easily see that it suffices to establish (3.1) for the proof of Theorem 3.1.

Since $[0,1]\times[-K,K]^p$ is divided into finite pieces by smooth curves l: $t = F(Y_i - \underset{\sim}{c}_{ni}^{\mathsf{T}}\underset{\sim}{u})$, $1 \le i \le n$, and the values of $S_{nk}^{*+}(t,\underset{\sim}{u})$ and $F_n^*(t,\underset{\sim}{u})$ are simply constants in each piece, we can smooth $S_{nk}^{*+}(t,\underset{\sim}{u})$ and $F_n^*(t,\underset{\sim}{u})$ through those pieces, respectively. Let $\bar{S}_{nk}^{*+}(t,\underset{\sim}{u})$ and $\bar{F}_n^*(t,\underset{\sim}{u})$ be the continuous version in $(t,\underset{\sim}{u})$ of $S_{nk}^{*+}(t,\underset{\sim}{u})$ and $F_n^*(t,\underset{\sim}{u})$ for $1 \le k \le p$, respectively, and let E_q denotes the space $[0,1]^q$ for a positive integer q, we have the following lemma.

LEMMA 3.3. For any $k = 1, 2, \cdots, p$, we have, as $n \to \infty$

$$\sup_{(t,\underset{\sim}{u})\in E_1\times[-K,K]^p} |\bar{S}_{nk}^{*+}(t,\underset{\sim}{u}) - S_{nk}^{*+}(t,\underset{\sim}{u})| \le (p+1)\max_{1\le i\le n} c_{nik}^{+}, \qquad \text{a.s.} \tag{3.8}$$

and

$$\sup_{(t,\underset{\sim}{u})\in E_1\times[-K,K]^p} |\bar{F}_n^*(t,\underset{\sim}{u}) - F_n^*(t,\underset{\sim}{u})| \le \frac{(p+1)}{(n+1)}, \qquad \text{a.s.} \tag{3.9}$$

Proof. The proof is just the general form of the one of Lemma 4.1 by Ren and Sen (1991). More specifically, with probability one, there are no more than (p+1) smooth curves l: $t = F(Y_i - \underset{\sim}{c}_{ni}^{\mathsf{T}}\underset{\sim}{u})$, $1 \le i \le n$, intersect at one point for each $n \ge 1$. Therefore, with probability one, the largest jump of S_{nk}^{*+} and F_n^* are no larger than $(p+1)\max_{1\le i\le n} c_{nik}^{+}$ and $(p+1)/(n+1)$, respectively. □

For $\underset{\sim}{u} = (u_1, \cdots, u_p)^{\mathsf{T}}$, $\underset{\sim}{v} = (v_1, \cdots, v_p)^{\mathsf{T}} \in R^p$, we denote

$$\underset{\sim}{u} \preceq \underset{\sim}{v} \quad \text{iff} \quad u_i \le v_i, \quad \text{for } i = 1, 2, \cdots, p.$$

We also denote $R_+ = [0,\infty)$, and $\omega_h(\delta) = \sup_{|\underset{\sim}{x}-\underset{\sim}{y}|\le\delta} |h(\underset{\sim}{x}) - h(\underset{\sim}{y})|$ for a function h defined on R^p.

LEMMA 3.4. Assume (B) and assume for any $n \ge 1$ and $1 \le i \le n$, $\alpha_{ni} \in R_+$, $\underset{\sim}{\beta}_{ni} \in$

R_+^p and $\gamma_{ni} \in R_+^p$ satisfy

$$\max_{1\le i\le n}\{\alpha_{ni}^2\} \to 0, \quad \max_{1\le i\le n}\{\|\beta_{ni}\|^2\} \to 0, \quad \max_{1\le i\le n}\{\|\gamma_{ni}\|^2\} \to 0, \qquad \text{as } n\to\infty,$$

and

$$\sum_{i=1}^n \alpha_{ni}^2 \le 1, \quad \sum_{i=1}^n \|\beta_{ni}\|^2 \le 1, \quad \sum_{i=1}^n \|\gamma_{ni}\|^2 \le 1.$$

Then,

$$\lim_{\delta\to 0} \overline{\lim_{n\to\infty}} P\Big(\omega_{V_n}(\delta)\ge\epsilon \Big) = 0, \tag{3.10}$$

and

$$\sup\Big\{|V_n(t,u,v)-V_n(t,0,0)|;\ t\in[0,1],\ |u|\le K,\ |v|\le K\Big\} \xrightarrow{P} 0, \quad \text{as } n\to\infty, \tag{3.11}$$

where for $K > 0$, $t \in [0,1]$, u and $v \in R^p$ with $|u|\le K$, $|v|\le K$, $V_n(t,u,v) = J_n^*(t,u,v) - J_n(t,u,v)$ and $J_n^*(t,u,v)=\sum_{i=1}^n \alpha_{ni} I\{Y_i\le\beta_{ni}^T u-\gamma_{ni}^T v+F^{-1}(t)\}$, $J_n(t,u,v) = E\{J_n^*(t,u,v)\}$.

Therefore, for any $1\le k\le p$ and any $K>0$, assume (A1) and (B), then, for any $\epsilon>0$

$$\lim_{\delta\to 0} \overline{\lim_{n\to\infty}} P\Big(\omega_{W_{nk}}(\delta)\ge\epsilon \Big) = 0, \tag{3.12}$$

where $W_{nk}(t,u) = \Big(\sum_{i=1}^n c_{nik}^+[F_n^*(t,u)-F_n(t,u)],\ [S_{nk}^{*+}(t,u)-S_{nk}^+(t,u)]\Big)$, $|u|\le K$.

Proof. Without loss of the generality, we may assume that $K = 1$. For any $\delta>0$, we choose positive integers m_δ and m_n such that $m_\delta=[\delta^{-1/(3p)}]$ and $m_n=[\lambda_n^{-1/2}]$, where $\lambda_n=\max\Big\{\max_{1\le i\le n}\|\beta_{ni}\|,\ \max_{1\le i\le n}\|\gamma_{ni}\|,\ \max_{1\le i\le n}\alpha_{ni}\Big\}$. For any u, v x, $y \in R^p$ there exist u', u'', v', v'', x', x'', y', $y'' \in L_p(m_\delta)=\{m_\delta^{-1}(j_1, \cdots, j_p)^T|\ j_i = 0, \pm1, \pm2, \cdots, \pm m_\delta;\ i =1, 2, \cdots, p\}$ such that

$$u' \preceq u \preceq u'', \quad v' \preceq v \preceq v'', \quad x' \preceq x \preceq x'', \quad y' \preceq y \preceq y''$$

and

$$|u-u'|\le m_\delta^{-1}, \quad |u''-u|\le m_\delta^{-1}, \quad |v-v'|\le m_\delta^{-1}, \quad |v''-v|\le m_\delta^{-1},$$

$$|x-x'|\le m_\delta^{-1}, \quad |x''-x|\le m_\delta^{-1}, \quad |y-y'|\le m_\delta^{-1}, \quad |y''-y|\le m_\delta^{-1}.$$

Also, for any $s\in[0,1]$, there exists $k \in \{0, 1, 2, \cdots, (m_n-1)\}$ such that $\frac{k}{m_n} \le s \le \frac{(k+1)}{m_n}$. Since $J_n^*(t,u,v)$ is nondecreasing and nonincreasing in u and v, respectively, we have

$$V_n(t,u,v) - V_n(s,x,y) = \{J_n^*(t,u,v)-J_n(t,u,v)\}-\{J_n^*(s,x,y)-J_n(s,x,y)\}$$

$$\le \{J_n^*(t,u'',v')-J_n(t,u,v)\}-\{J_n^*(s,x',y'')-J_n(s,x,y)\}$$

$$= \{V_n(t,\underline{u}'',\underline{v}')-V_n(s,\underline{x}',\underline{y}'')\} + \{J_n(t,\underline{u}'',\underline{v}')-J_n(t,\underline{u},\underline{v})\}-\{J_n(s,\underline{x}',\underline{y}'')-J_n(s,\underline{x},\underline{y})\}$$

$$= \{V_n(t,\underline{u}'',\underline{v}')-V_n(s,\underline{u}'',\underline{v}')\} + \{V_n(s,\underline{u}'',\underline{v}')-V_n(\tfrac{k}{m_n},\underline{u}'',\underline{v}')\} +$$

$$+ \{V_n(\tfrac{k}{m_n},\underline{u}'',\underline{v}')-V_n(\tfrac{k}{m_n},\underline{x}',\underline{y}'')\} + \{V_n(\tfrac{k}{m_n},\underline{x}',\underline{y}'')-V_n(s,\underline{x}',\underline{y}'')\} +$$

$$+ \{J_n(t,\underline{u}'',\underline{v}')-J_n(t,\underline{u},\underline{v})\}-\{J_n(s,\underline{x}',\underline{y}'')-J_n(s,\underline{x},\underline{y})\}$$

and similarly

$$V_n(t,\underline{u},\underline{v}) - V_n(s,\underline{x},\underline{y}) \geq \{V_n(t,\underline{u}',\underline{v}'')-V_n(s,\underline{u}',\underline{v}'')\} + \{V_n(s,\underline{u}',\underline{v}'')-V_n(\tfrac{k}{m_n},\underline{u}',\underline{v}'')\} +$$

$$+ \{V_n(\tfrac{k}{m_n},\underline{u}',\underline{v}'')-V_n(\tfrac{k}{m_n},\underline{x}'',\underline{y}')\} + \{V_n(\tfrac{k}{m_n},\underline{x}'',\underline{y}')-V_n(s,\underline{x}'',\underline{y}')\} +$$

$$+ \{J_n(t,\underline{u}',\underline{v}'')-J_n(t,\underline{u},\underline{v})\}-\{J_n(s,\underline{x}'',\underline{y}')-J_n(s,\underline{x},\underline{y})\}.$$

Therefore,

$$\omega_{V_n}(\delta) \leq 3 \max_{\underline{u},\ \underline{v}\in L_p(m_\delta)}\left\{ \sup_{|s-t|\leq\delta} \left|V_n(t,\underline{u},\underline{v})-V_n(s,\underline{u},\underline{v})\right|\right\} +$$

$$+ \max_{0\leq k\leq(m_n-1)}\left\{\sup\left\{\left|V_n(\tfrac{k}{m_n},\underline{u},\underline{v})-V_n(\tfrac{k}{m_n},\underline{x},\underline{y})\right|;\ |(\underline{u},\underline{v})\text{-}(\underline{x},\underline{y})|\leq(\delta+2m_\delta^{-1}),\right.\right.$$

$$\left.\left.\underline{u},\underline{v},\underline{x},\underline{y}\in L_p(m_\delta)\right\}\right\} +$$

$$+ 2 \sup\left\{|J_n(t,\underline{u},\underline{v})-J_n(t,\underline{x},\underline{y})|;\ t\in[0,1],\ |(\underline{u},\underline{v})-(\underline{x},\underline{y})|\leq m_\delta^{-1}\right\}.$$

For any $\epsilon > 0$,

$$P\left\{\omega_{V_n}(\delta) \geq \epsilon\right\} \leq \sum_{\underline{u},\ \underline{v}\in L_p(m_\delta)} P\left\{ \sup_{|s-t|\leq\delta} \left|V_n(t,\underline{u},\underline{v})-V_n(s,\underline{u},\underline{v})\right| \geq \tfrac{\epsilon}{9}\right\} +$$

$$+ \sum_{k=0}^{(m_\delta-1)} P\left\{ \sup_{\underline{u},\underline{v},\underline{x},\underline{y}\in L_p(m_\delta)} \left\{\left|V_n(\tfrac{k}{m_n},\underline{u},\underline{v})-V_n(\tfrac{k}{m_n},\underline{x},\underline{y})\right|;\ |(\underline{u},\underline{v})-(\underline{x},\underline{y})|\leq(\delta+2m_\delta^{-1})\right\}\geq\tfrac{\epsilon}{3}\right\}+$$

$$+ P\left\{\sup\left\{|J_n(t,\underline{u},\underline{v})-J_n(t,\underline{x},\underline{y})|;\ t\in[0,1],\ |(\underline{u},\underline{v})-(\underline{x},\underline{y})|\leq m_\delta^{-1}\right\} \geq \tfrac{\epsilon}{6}\right\}. \tag{3.13}$$

By the proof of Corollary 2 of Shorack and Wellner (page 112, 1986), we have

$$\sum_{\underline{u},\ \underline{v}\in L_p(m_\delta)} P\left\{ \sup_{|s-t|\leq\delta} \left|V_n(t,\underline{u},\underline{v})-V_n(s,\underline{u},\underline{v})\right| \geq \tfrac{\epsilon}{9}\right\}$$

$$\leq \sum_{\underline{u},\ \underline{v}\in L_p(m_\delta)} P\left\{ \sup_{|s-t|\leq m_\delta^{-3p}} \left|V_n(t,\underline{u},\underline{v})-V_n(s,\underline{u},\underline{v})\right| \geq \tfrac{\epsilon}{9}\right\}$$

$$\leq \frac{(2m_\delta+1)^{2p}}{(\epsilon/9)^4} \left(\max_{1\leq i\leq n} \alpha_{ni}^2 +\right.$$

$$+M \max_{1\le i\le n}\left\{ \sup_{|(\underset{\sim}{u},\underset{\sim}{v})|\le 1,\ |s-t|\le m_\delta^{-3p}} \left|F\left(\underset{\sim}{\beta}_{ni}^T\underset{\sim}{u}-\underset{\sim}{\gamma}_{ni}^T\underset{\sim}{v}+F^{-1}(t)\right)-F\left(\underset{\sim}{\beta}_{ni}^T\underset{\sim}{u}-\underset{\sim}{\gamma}_{ni}^T\underset{\sim}{v}+F^{-1}(s)\right)\right|\right\}\Bigg)$$

$$\le \frac{(2m_\delta+1)^{2p}}{(\epsilon/9)^4}\left(\max_{1\le i\le n}\alpha_{ni}^2 + M\left\{m_\delta^{-3p}+2M_1p\left(\max_{1\le i\le n}\|\underset{\sim}{\beta}_{ni}\|+\max_{1\le i\le n}\|\underset{\sim}{\gamma}_{ni}\|\right)\right\}\right)$$

$$\to \frac{(2m_\delta+1)^{2p}}{(\epsilon/9)^4} M m_\delta^{-3p}, \qquad \text{as } n\to\infty, \tag{3.14}$$

where M and M_1 are constants, and

$$\sum_{k=0}^{(m_n-1)} P\left\{ \sup_{\underset{\sim}{u},\underset{\sim}{v},\underset{\sim}{x},\underset{\sim}{y}\in L_p(m_\delta)} \left\{\left|V_n(\tfrac{k}{m_n},\underset{\sim}{u},\underset{\sim}{v})-V_n(\tfrac{k}{m_n},\underset{\sim}{x},\underset{\sim}{y})\right|;\ |(\underset{\sim}{u},\underset{\sim}{v})-(\underset{\sim}{x},\underset{\sim}{y})|\le(\delta+2m_\delta^{-1})\right\}\ge\tfrac{\epsilon}{3}\right\}$$

$$\le \sum_{k=0}^{(m_n-1)} \sum_{\underset{\sim}{u},\underset{\sim}{v}\in L_p(m_\delta)} \sum_{\mu=1}^{2p} P\left\{ \sup_{0\le\rho_\mu\le N_\delta^{-1}} \left|V_n\left(\tfrac{k}{m_n},(\underset{\sim}{u},\underset{\sim}{v})\right)-V_n\left(\tfrac{k}{m_n},(\underset{\sim}{u},\underset{\sim}{v})+\underset{\sim}{e}_\mu\rho_\mu\right)\right| \ge \tfrac{\epsilon}{6p}\right\}$$

$$\le \frac{2pM_2m_n(2m_\delta+1)^{2p}}{(\epsilon/6p)^4}\left\{\frac{M_1}{N_\delta}\lambda_n + \max_{1\le i\le n}\alpha_{ni}^2\right\}$$

$$\le \frac{2pM_2(2m_\delta+1)^{2p}}{(\epsilon/6p)^4}\left\{\frac{M_1}{N_\delta}\lambda_n^{1/2} + \lambda_n^{3/2}\right\} \to 0, \qquad \text{as } n\to\infty, \tag{3.15}$$

where $N_\delta=[m_\delta/3]$, $\underset{\sim}{e}_\mu$ denotes the μ-the unit vector and M_2 is a constant. Therefore, (3.10) follows from (3.13) through (3.15), $\lim_{\delta\to 0} m_\delta=\infty$, and the fact:

$$\sup\left\{|J_n(t,\underset{\sim}{u},\underset{\sim}{v})-J_n(t,\underset{\sim}{x},\underset{\sim}{y})|;\ t\in[0,1],\ |(\underset{\sim}{u},\underset{\sim}{v})-(\underset{\sim}{x},\underset{\sim}{y})|\le m_\delta^{-1}\right\} \to 0, \quad \text{as } n\to\infty.$$

Similarly, (3.11) can be easily shown.

Note that the upper-bound 1 for $\sum_{i=1}^n\alpha_{ni}^2$, $\sum_{i=1}^n\|\underset{\sim}{\beta}_{ni}\|^2$ and $\sum_{i=1}^n\|\underset{\sim}{\gamma}_{ni}\|^2$ may be replaced by any finite positive constant. Therefore, (3.12) follows from (3.10) because for $(\alpha_{ni}, \underset{\sim}{\beta}_{ni}, \underset{\sim}{\gamma}_{ni}) = (\sum_{i=1}^n c_{nik}^+/(n+1),\ \underset{\sim}{c}_{ni}^+,\ \underset{\sim}{c}_{ni}^-)$, $\sum_{i=1}^n c_{nik}^+[F_n^*(t,\underset{\sim}{u})-F_n(t,\underset{\sim}{u})] = V_{nk}(t,\underset{\sim}{u},\underset{\sim}{u})$ and for $(\alpha_{ni}, \underset{\sim}{\beta}_{ni}, \underset{\sim}{\gamma}_{ni})=(c_{nik}^+, \underset{\sim}{c}_{ni}^+, \underset{\sim}{c}_{ni}^-)$, $[S_{nk}^{*+}(t,\underset{\sim}{u})-S_{nk}^+(t,\underset{\sim}{u})]=V_{nk}(t,\underset{\sim}{u},\underset{\sim}{u})$. □

LEMMA 3.5. For any $1\le k\le p$, assume (A1) and (B). Let

$$T_{nk}(t,\underset{\sim}{u}) = \left(\sum_{i=1}^n c_{nik}^+(\overline{F}_n^*(t,(2\underset{\sim}{u}-\underset{\sim}{I})K)-t),\ [\overline{S}_{nk}^{*+}(t,(2\underset{\sim}{u}-\underset{\sim}{I})K)-t\sum_{i=1}^n c_{nik}^+]\right),$$

for $(t,\underset{\sim}{u})\in E_{p+1}$ and let $\{P_{nk};\ n\ge 1\}$ be the sequence of probability measures corresponding to T_{nk}, $n\ge 1$. Then, $\{P_{nk}\}$ is relatively compact.

Proof. Using (3.12) of Lemma 3.4, the proof is analogous to the one of Proposition 4.3 by Ren and Sen (1991). More specifically, for $1 \le k \le p$, let

$$H_{nk}(t,\underset{\sim}{u}) = \sum_{j=1}^{n} c_{njk}^{+}\Big\{\sum_{i=1}^{n} \frac{1}{n+1}\, \underset{\sim}{c}_{ni}^{T}\, (2\underset{\sim}{u}-\underset{\sim}{I})F'(F^{-1}(t))K\Big\},$$

$$G_{nk}(t,\underset{\sim}{u}) = \sum_{i=1}^{n} c_{nik}^{+}\underset{\sim}{c}_{ni}^{T}(2\underset{\sim}{u}-\underset{\sim}{I})F'(F^{-1}(t))K,$$

it suffices to establish (4.8) of Proposition 4.3 by Ren and Sen (1991) for H_{nk} and G_{nk}, respectively, i.e.,

(3.16) $$\sup_{(t,\underset{\sim}{u})\in E_{p+1}} \Big|[S_{nk}^{+}(t,\, (2\underset{\sim}{u}-\underset{\sim}{I})K) - t\sum_{i=1}^{n} c_{nik}^{+}] - G_{nk}(t,\underset{\sim}{u})\Big| \to 0, \quad \text{as } n\to\infty,$$

and

(3.17) $$\sup_{(t,\underset{\sim}{u})\in E_{p+1}} \Big|\sum_{i=1}^{n} c_{nik}^{+}\, [F_n(t,\, (2\underset{\sim}{u}-\underset{\sim}{I})K) - t] - H_{nk}(t,\underset{\sim}{u})\Big| \to 0, \quad \text{as } n\to\infty.$$

Since

$$\sup_{(t,\underset{\sim}{u})\in E_{p+1}} \Big|\sum_{i=1}^{n} c_{nik}^{+}\, [F_n(t,\, (2\underset{\sim}{u}-\underset{\sim}{I})K) - t] - H_{nk}(t,\underset{\sim}{u})\Big|$$

$$= \sup_{(t,\underset{\sim}{u})\in E_{p+1}} \Big| \frac{1}{n+1}\sum_{i=1}^{n} c_{nik}^{+}\Big\{\sum_{j=1}^{n} [F(F^{-1}(t)+\underset{\sim}{c}_{nj}^{T}(2\underset{\sim}{u}-\underset{\sim}{I})K) - t - \tfrac{t}{n}]\Big\} - H_{nk}(t,\underset{\sim}{u})\Big|$$

$$\le \frac{\sum_{i=1}^{n} c_{nik}^{+}}{n+1} + \sup_{(t,\underset{\sim}{u})\in E_{p+1}} \Big|\frac{1}{n+1} \sum_{i=1}^{n} c_{nik}^{+}\Big\{\sum_{j=1}^{n} [F'(\xi) - F'(F^{-1}(t))]\underset{\sim}{c}_{nj}^{T}(2\underset{\sim}{u}-\underset{\sim}{I})K)\Big\}\Big|$$

$$\le \frac{\sum_{i=1}^{n} c_{nik}^{+}}{n+1} + \frac{pnK}{n+1} \sup_{(t,\underset{\sim}{u})\in E_{p+1}} |F'(\xi) - F'(F^{-1}(t))|,$$

and similarly

$$\sup_{(t,\underset{\sim}{u})\in E_{p+1}} \Big|[S_{nk}^{+}(t,\, (2\underset{\sim}{u}-\underset{\sim}{I})K) - t\sum_{i=1}^{n} c_{njk}^{+}] - G_{nk}(t,\underset{\sim}{u})\Big|$$

$$\le pK \sup_{(t,\underset{\sim}{u})\in E_{p+1}} |F'(\xi) - F'(F^{-1}(t))|,$$

where ξ is between $F^{-1}(t)$ and $F^{-1}(t)+\underset{\sim}{c}_{nj}^{T}(2\underset{\sim}{u}-\underset{\sim}{I})K$, and since

$$\frac{\sum_{i=1}^{n} c_{nik}^{+}}{n+1} \le \frac{\sqrt{n}}{n+1} \to 0, \qquad \text{as } n\to\infty,$$

(3.16) and (3.17) follow from (A1) and the uniform continuity of F'. □

Let Γ be a set in $D[0,1]\times D[0,1]$ and $H\in D[0,1]\times D[0,1]$, define

$$\text{dist}(H, \Gamma) = \inf_{G\in\Gamma} \| H - G \|.$$

LEMMA 3.6. Let Q: $D[0,1]\times D[0,1]\times R\to R$ and suppose that for any compact set Γ in $D[0,1]\times D[0,1]$,

$$\lim_{t\to 0} Q(H, t) = 0 \tag{3.18}$$

uniformly for $H\in\Gamma$. Let $\epsilon>0$ and let α_n, β_n be sequences of real numbers such that $\alpha_n\to 0$, $\beta_n\to 0$, as $n\to\infty$. Then, for any compact set Γ in $D[0,1]\times D[0,1]$, there exists a positive integer N such that, if $\text{dist}(H, \Gamma)\le\alpha_n$, then

$$|Q(H, \beta_n)| < \epsilon, \qquad n\ge N.$$

Proof. The proof is similar to the one of Lemma 4.4 of Ren and Sen (1991). □

<u>Proof of Theorem 3.1</u>. By Lemma 3.5, the sequence of probability measures $\{P_{nk}\}$ is relatively compact in $C[0,1]^{p+1}\times C[0,1]^{p+1}$, where $P_{nk}(A)=P(T_{nk}\in A)$. Since $C[0,1]^{p+1}\times C[0,1]^{p+1}$ is complete and separable, by Prohorov's theorem (Billingsley 1968, Theorem 6.2), $\{P_{nk}\}$ is tight, i.e., for any $\epsilon>0$, there exists a compact set Γ in $C[0,1]^{p+1}\times C[0,1]^{p+1}$ such that $P\{T_{nk}\in\Gamma\}>1-\frac{\epsilon}{2}$, $n\ge 1$. By Lemma 3.3, we have

$$P\Big\{T_{nk}\in\Gamma,\ \Big\|\Big(\sum_{i=1}^{n} c^{+}_{nik}\bar{F}^{*}_{n}(\cdot,\cdot),\ \bar{S}^{*+}_{nk}(\cdot,\cdot)\Big)-\Big(\sum_{i=1}^{n} c^{+}_{nik}F^{*}_{n}(\cdot,\cdot),\ S^{*+}_{nk}(\cdot,\cdot)\Big)\Big\|\le$$

$$\le(p+1)\max\Big\{\frac{\sum_{i=1}^{n}c^{+}_{ink}}{n+1},\ \max_{1\le i\le n} c^{+}_{nik}\Big\}\Big\} \ge 1-\epsilon, \qquad \text{for large n.} \tag{3.19}$$

Let $\Gamma_1=\{T_{nk}(\cdot,\underset{\sim}{u});\ T_{nk}\in\Gamma,\ \underset{\sim}{u}\in E_p\}$, then Γ_1 is a compact set in $C[0,1]\times C[0,1]$, and is also a compact set in $D[0,1]\times D[0,1]$ because $C[0,1]$ is a subspace of $D[0,1]$. Since $T_{nk}\in\Gamma$ implies $T_{nk}(\cdot,\underset{\sim}{u})\in\Gamma_1$ for any $\underset{\sim}{u}\in E_p$, i.e., for any $\underset{\sim}{u}\in E_p$,

$$\Big(\sum_{i=1}^{n}c^{+}_{nik}(\bar{F}^{*}_{n}(\cdot,(2\underset{\sim}{u}-\underset{\sim}{I})K)-U(\cdot)),\ [\bar{S}^{*+}_{nk}(\cdot,(2\underset{\sim}{u}-\underset{\sim}{I})K-U(\cdot)\sum_{i=1}^{n}c^{+}_{nik}]\Big)\in\Gamma_1, \tag{3.20}$$

and since

$$\Big\|\Big(\sum_{i=1}^{n} c^{+}_{nik}\bar{F}^{*}_{n}(\cdot,\cdot),\bar{S}^{*+}_{nk}(\cdot,\cdot)\Big)-\Big(\sum_{i=1}^{n} c^{+}_{nik}F^{*}_{n}(\cdot,\cdot),S^{*+}_{nk}(\cdot,\cdot)\Big)\Big\|\le(p+1)\max\Big\{\frac{\sum_{i=1}^{n}c^{+}_{nik}}{n+1},\ \max_{1\le i\le n} c^{+}_{nik}\Big\}$$

implies

$$\Big\|\Big(\sum_{i=1}^{n} c^{+}_{nik}\bar{F}^{*}_{n}(\cdot,(2\underset{\sim}{u}-\underset{\sim}{I})K),\ \bar{S}^{*+}_{nk}(\cdot,(2\underset{\sim}{u}-\underset{\sim}{I})K)\Big)-\Big(\sum_{i=1}^{n} c^{+}_{nik}F^{*}_{n}(\cdot,(2\underset{\sim}{u}-\underset{\sim}{I})K),\ S^{*+}_{nk}(\cdot,(2\underset{\sim}{u}-\underset{\sim}{I})K)\Big)\Big\|$$

$$\le (p+1)\max\Big\{\frac{\sum_{i=1}^{n}c^{+}_{nik}}{n+1},\ \max_{1\le i\le n} c^{+}_{nik}\Big\}, \qquad \text{for any } \underset{\sim}{u}\in E_p, \tag{3.21}$$

then, by (3.19) and the fact that (3.20) and (3.21) imply

$$\operatorname{dist}\Big\{\Big(\sum_{i=1}^{n} c_{nik}^{+}[F_n^*(\cdot,(2\underset{\sim}{u}-\underset{\sim}{I})K)-U(\cdot)],\ [S_{nk}^{*+}(\cdot,(2\underset{\sim}{u}-\underset{\sim}{I})K)-U(\cdot)\sum_{i=1}^{n} c_{nik}^{+}]\Big),\ \Gamma_1\Big\} \le$$

$$\le (p+1)\max\left\{\frac{\sum_{i=1}^{n} c_{nik}^{+}}{n+1},\ \max_{1\le i\le n} c_{nik}^{+}\right\},\qquad \text{for any } \underset{\sim}{u}\in E_p,$$

we have, for $n\ge 1$,

$$P\Bigg(\operatorname{dist}\Big\{\Big(\sum_{i=1}^{n} c_{nik}^{+}[F_n^*(\cdot,(2\underset{\sim}{u}-\underset{\sim}{I})K)-U(\cdot)],\ [S_{nk}^{*+}(\cdot,(2\underset{\sim}{u}-\underset{\sim}{I})K)-U(\cdot)\sum_{i=1}^{n} c_{nik}^{+}]\Big),\ \Gamma_1\Big\}\le$$

$$\text{(3.22)}\qquad \le (p+1)\max\left\{\frac{\sum_{i=1}^{n} c_{nik}^{+}}{n+1},\ \max_{1\le i\le n} c_{nik}^{+}\right\},\ \text{for any } \underset{\sim}{u}\in E_p\Bigg) > 1-\epsilon.$$

Since τ: $D[0,1]\times D[0,1]\to R$ is Hadamard differentiable at (U,U), by the definition of Hadamard differentiability, (3.18) holds for $Q(H,t)=\mathrm{Rem}(tH)/t$. By Lemma 3.6, (3.7) and (A1), for the above compact set Γ_1, there exists a positive integer N such that, for $n\ge N$ if $\operatorname{dist}(H,\Gamma_1)\le(p+1)\max\left\{\frac{\sum_{i=1}^{n} c_{nik}^{+}}{n+1},\ \max_{1\le i\le n} c_{nik}^{+}\right\}$, then

$$\Big|\sum_{i=1}^{n} c_{nik}^{+}\ \mathrm{Rem}\Big(\frac{H}{\sum_{i=1}^{n} c_{nik}^{+}};\ \tau\Big)\Big| < \epsilon.$$

Hence, taking $H=\Big(\sum_{i=1}^{n} c_{nik}^{+}[F_n^*(\cdot,(2\underset{\sim}{u}-\underset{\sim}{I})K)-U(\cdot)],[S_{nk}^{*+}(\cdot,(2\underset{\sim}{u}-\underset{\sim}{I})K)-U(\cdot)\sum_{i=1}^{n} c_{nik}^{+}]\Big)$ for $n\ge N$ and $\underset{\sim}{u}\in E_p$, we have that

$$\operatorname{dist}\Big\{\Big(\sum_{i=1}^{n} c_{nik}^{+}[F_n^*(\cdot,(2\underset{\sim}{u}-\underset{\sim}{I})K)-U(\cdot)],\ [S_{nk}^{*+}(\cdot,(2\underset{\sim}{u}-\underset{\sim}{I})K)-U(\cdot)\sum_{i=1}^{n} c_{nik}^{+}]\Big),\ \Gamma_1\Big\}\le$$

$$\le (p+1)\max\left\{\frac{\sum_{i=1}^{n} c_{nik}^{+}}{n+1},\ \max_{1\le i\le n} c_{nik}^{+}\right\},\ \text{for any } \underset{\sim}{u}\in E_p$$

implies for $\underset{\sim}{u}\in E_p$

$$\text{(3.23)}\quad \Big|\sum_{i=1}^{n} c_{nik}^{+}\ \mathrm{Rem}\Big([F_n^*(\cdot,(2\underset{\sim}{u}-\underset{\sim}{I})K)-U(\cdot)],\ [\frac{S_{nk}^{*+}(\cdot,\ (2\underset{\sim}{u}-\underset{\sim}{I})K)}{\sum_{i=1}^{n} c_{nik}^{+}}-U(\cdot)];\ \tau\Big)\Big| < \epsilon.$$

Since (3.23) implies

$$\sup_{|\underset{\sim}{u}|\le K}\Big|\sum_{i=1}^{n} c_{nik}^{+}\ \mathrm{Rem}\Big([F_n^*(\cdot,\ \underset{\sim}{u})-U(\cdot)],\ [\frac{S_{nk}^{*+}(\cdot,\ \underset{\sim}{u})}{\sum_{i=1}^{n} c_{nik}^{+}}-U(\cdot)];\ \tau\Big)\Big| \le \epsilon,$$

by (3.22) we have, for $n\ge N$,

$$1-\epsilon < P\Big\{ \sup_{|\underset{\sim}{u}|\le K} \Big| \sum_{i=1}^{n} c_{nik}^{+} \operatorname{Rem}\Big([F_n^*(\cdot,\underset{\sim}{u})-U(\cdot)],\ [\frac{S_{nk}^{*+}(\cdot,\underset{\sim}{u})}{\sum_{i=1}^{n} c_{nik}^{+}}-U(\cdot)];\ \tau\Big)\Big| \le \epsilon \Big\}.$$

□

4. On R-estimators in Linear Models.

We will see in Theorem 4.3 that the estimating equations of R-estimators $\underset{\sim}{E}_n(\underset{\sim}{y})$ induces the following functional Ψ: $D[0,1]\times D[0,1]\to R$, defined by

$$\Psi(G,\ H) = \int_0^1 G(H^{-1}(t))\ d\psi(t), \quad G,\ H\in D[0,1], \tag{4.1}$$

where $H^{-1}(y)=\inf\{1,\ x;\ H(x)\ge y\}$. We show, in the following lemma, that Ψ is Hadamard differentiable.

LEMMA 4.1. Assume (C2), then the functional Ψ defined by (4.1) is Hadamard differentiable at (U,U) with derivative

$$\Psi'_{(U,U)}(G,\ H) = \int_0^1 [G(t)-H(t)]\ d\psi(t).$$

Proof. The functional Ψ can be expressed as a composition of the following Hadamard differentiable transformations:

γ_1: $D[0,1]\times D[0,1]\to D[0,1]\times L^1[0,1]$ defined by $\gamma_1(G,H) = (G,H^{-1})$, is, by Proposition 6.1.1 of Fernholz (1983), Hadamard differentiable at (S, U) with derivative

$$\gamma'_{1(S,U)}(G,\ H) = (G,\ -H),$$

for any $S\in D[0,1]$.

γ_2: $D[0,1]\times L^1[0,1]\to L^1[0,1]$ defined by $\gamma_2(G,H)=G(H)$, is, by Proposition 6.1.6 of Fernholz (1983), Hadamard differentiable at (U,S) with Hadamard derivative

$$\gamma'_{2(U,S)}(G,\ H) = G\circ S + H$$

for a differentiable S with range [0,1] and derivative bounded away from zero and infinity.

γ_3: R_{γ_2} (the range of γ_2) $\to$ R, defined by $\gamma_3(H)=\int_0^1 H(t)d\psi(t)$, is Fréchet differentiable at any S with derivative

$$\gamma'_{3_S}(H) = \int_0^1 H(t)\ d\psi(t),$$

because it is a linear and continuous functional.

We have

$$\Psi(G,\ H) = \gamma_3(\gamma_2(\gamma_1(G,\ H))).$$

Hence, by chain rule (Fernholz, 1983), Ψ is Hadamard differentiable at (U, U) with derivative

$$\Psi'_{(U,U)}(G,\ H) = \gamma'_{3_U}\circ\gamma'_{2(U,U)}\circ\gamma'_{1(U,U)}(G,\ H)$$

$$= \gamma'_{3_U} \circ \gamma'_{2_{(U,U)}}(G, -H) = \gamma'_{3_U}(G-H) = \int_0^1 [G(t)-H(t)]\, d\psi(t). \qquad \square$$

For convenience sake, we give the following theorem for the general uniform asymptotic linearity of M-estimators.

THEOREM 4.2. Assume (B) and (D). Let

$$(4.2) \qquad \mathcal{M}_n(\underset{\sim}{u}) = \sum_{i=1}^n \underset{\sim}{\alpha}_{ni}\, \varphi(Y_i - \underset{\sim}{\beta}_{ni}^T \underset{\sim}{u}),$$

where $\underset{\sim}{u}$, $\underset{\sim}{\alpha}_{ni}$, $\underset{\sim}{\beta}_{ni} \in R^p$ with

$$\max_{1\le i\le n} \|\underset{\sim}{\alpha}_{ni}\|^2 \to 0, \qquad \max_{1\le i\le n} \|\underset{\sim}{\beta}_{ni}\|^2 \to 0, \qquad \text{as } n\to\infty,$$

and for a constant $M>0$,

$$\sum_{i=1}^n \|\underset{\sim}{\alpha}_{ni}\|^2 \le M, \quad \sum_{i=1}^n \|\underset{\sim}{\beta}_{ni}\|^2 \le M.$$

Then, for any $K>0$,

$$(4.3) \qquad \sup_{|\underset{\sim}{u}|\le K} \left| [\mathcal{M}_n(\underset{\sim}{u}) - \mathcal{M}_n(\underset{\sim}{0})] + \sum_{i=1}^n \underset{\sim}{\alpha}_{ni} \underset{\sim}{\beta}_{ni}^T \underset{\sim}{u}\, \gamma_\varphi \right| \xrightarrow{P} 0, \qquad \text{as } n\to\infty$$

where $\gamma_\varphi = \int F' d\varphi > 0$. Therefore, assume (A1) and (A2) for $\underset{\sim}{b}_{ni}$ and $\underset{\sim}{Q}_n = \sum_{i=1}^n \underset{\sim}{b}_{ni} \underset{\sim}{b}_{ni}^T$,

$$(4.4) \qquad \sup_{|\underset{\sim}{u}|\le K} \left| \underset{\sim}{M}_n(\underset{\sim}{u}) - \underset{\sim}{M}_n(\underset{\sim}{0}) + \underset{\sim}{Q}_n \underset{\sim}{u}\, \gamma_\varphi \right| \xrightarrow{P} 0, \qquad \text{as } n\to\infty,$$

and furthermore,

$$(4.5) \qquad \underset{\sim}{B}_n^o(\hat{\underset{\sim}{\beta}}_M - \underset{\sim}{\beta}) = \gamma_\varphi^{-1} \underset{\sim}{Q}_n^{-1} \underset{\sim}{M}_n(\underset{\sim}{0}) + o_P(1), \qquad \text{as } n\to\infty.$$

Proof. Let for $1\le j\le p$,

$$V_{nj}^*(t,\underset{\sim}{u}) = \sum_{i=1}^n \alpha_{nij} I\{Y_i \le F^{-1}(t) + \underset{\sim}{\beta}_{ni}^T \underset{\sim}{u}\} \quad \text{and} \quad V_{nj}(t,\underset{\sim}{u}) = E\{V_{nj}^*(t,\underset{\sim}{u})\},$$

$$V_{nj-}^*(t,\underset{\sim}{u}) = \sum_{i=1}^n \alpha_{nij} I\{Y_i < F^{-1}(t) + \underset{\sim}{\beta}_{ni}^T \underset{\sim}{u}\}, \quad V_{nj=}^*(t,\underset{\sim}{u}) = \sum_{i=1}^n \alpha_{nij} I\{Y_i = F^{-1}(t) + \underset{\sim}{\beta}_{ni}^T \underset{\sim}{u}\},$$

then, if φ is left continuous,

$$\int_0^1 V_{nj}^*(t,\underset{\sim}{u})\, d\varphi(F^{-1}(t)) = \sum_{i=1}^n \alpha_{nij} \int_{F(Y_i - \underset{\sim}{\beta}_{ni}^T \underset{\sim}{u})}^1 d\varphi(F^{-1}(t)) = \varphi(+\infty) \sum_{i=1}^n \alpha_{nij} - \mathcal{M}_{nj}(\underset{\sim}{u}).$$

and if φ is right continuous,

$$\int_0^1 V_{nj-}^*(t,\underset{\sim}{u})\, d\varphi(F^{-1}(t)) = \sum_{i=1}^n \alpha_{nij} \int_{F(Y_i - \underset{\sim}{\beta}_{ni}^T \underset{\sim}{u})}^1 d\varphi(F^{-1}(t)) = \varphi(+\infty) \sum_{i=1}^n \alpha_{nij} - \mathcal{M}_{nj}(\underset{\sim}{u}).$$

Since, from Lemma 3.4, we easily have

$$\sup_{t\in[0,1],\ |\underset{\sim}{u}|\le K} |V_{nj=}^*(t,\underset{\sim}{u})| \xrightarrow{P} 0, \qquad \text{as } n\to\infty,$$

hence, as $n\to\infty$,

$$\sup_{t\in[0,1],\ |\underset{\sim}{u}|\le K} \left| \mathcal{M}_{nj}(\underset{\sim}{u}) - \mathcal{M}_{nj}(\underset{\sim}{0}) + \int_0^1 \{V^*_{nj}(t,\underset{\sim}{u}) - V^*_{nj}(t,\underset{\sim}{0})\}\, d\varphi(F^{-1}(t)) \right| \xrightarrow{P} 0.$$

Therefore, in order to show (4.3), it suffices to show for $1\le j\le p$, as $n\to\infty$

(4.6) $\sup\left\{ \left| [V^*_{nj}(t,\underset{\sim}{u}) - V^*_{nj}(t,\underset{\sim}{0})] - \sum_{i=1}^n \alpha_{nij} \underset{\sim}{\beta}^T_{ni} \underset{\sim}{u} F'(F^{-1}(t)) \right|;\ t\in[0,1],\ |\underset{\sim}{u}|\le K \right\} \xrightarrow{P} 0.$

By (3.11), we have

$$\sup\left\{ \left| [V^*_{nj}(t,\underset{\sim}{u}) - V_{nj}(t,\underset{\sim}{u})] - [V^*_{nj}(t,\underset{\sim}{0})] - V_{nj}(t,\underset{\sim}{0})] \right|;\ t\in[0,1],\ |\underset{\sim}{u}|\le K \right\} \xrightarrow{P} 0, \quad \text{as } n\to\infty.$$

Hence, by the uniform continuity of F', (4.6) follows easily from

$$\sup\left\{ \left| [V_{nj}(t,\underset{\sim}{u}) - V_{nj}(t,\underset{\sim}{0})] - \sum_{i=1}^n \alpha_{nij} \underset{\sim}{\beta}^T_{ni} \underset{\sim}{u} F'(F^{-1}(t)) \right|;\ t\in[0,1],\ |\underset{\sim}{u}|\le K \right\} \to 0, \quad \text{as } n\to\infty.$$

Assume (A1) and (A2) for $\underset{\sim}{b}_{ni}$ and $\underset{\sim}{Q}_n$, (4.4) is obvious by (4.3) for $\underset{\sim}{\alpha}_{ni} = \underset{\sim}{\beta}_{ni} = \underset{\sim}{b}_{ni}$. To show (4.5), we consider a function ρ such that $\rho' = \varphi$, then ρ is a convex function. From (4.4) and a similar proof of Lemma 1 by Jaeckel (1972), we have for any $K>0$,

(4.7) $$\sup_{|\underset{\sim}{u}|\le K} \left| D(\underset{\sim}{Y} - \underset{\sim}{D}_n \underset{\sim}{u}) - Q(\underset{\sim}{Y} - \underset{\sim}{D}_n \underset{\sim}{u}) \right| \xrightarrow{P} 0, \quad \text{as } n\to\infty,$$

where $D(\underset{\sim}{Y} - \underset{\sim}{D}_n \underset{\sim}{u}) = \sum_{i=1}^n \rho(Y_i - \underset{\sim}{c}^T_{ni} \underset{\sim}{u})$ and $Q(\underset{\sim}{Y} - \underset{\sim}{D}_n \underset{\sim}{u}) = D(\underset{\sim}{Y}) - \underset{\sim}{u}^T \underset{\sim}{M}_n(\underset{\sim}{0}) + 2^{-1} \gamma_\varphi \underset{\sim}{u}^T \underset{\sim}{M}_n(\underset{\sim}{0}) \underset{\sim}{u}$. Note that, by the theorem of Hájek and Šidák (1967, page 153), $\tilde{\underset{\sim}{u}}_n = \gamma_\varphi^{-1} \underset{\sim}{Q}_n^{-1} \underset{\sim}{M}_n(\underset{\sim}{0})$ is asymptotically normal: $N_p(\underset{\sim}{0},\ \sigma^2 \gamma_\varphi^{-2} \underset{\sim}{Q}^{-1})$ for $\sigma^2 = \int \varphi^2(x) dF(x) > 0$, and $\tilde{\underset{\sim}{u}}_n$ uniquely minimizes $Q(\underset{\sim}{Y} - \underset{\sim}{D}_n \underset{\sim}{u})$, and that $D(\underset{\sim}{Y} - \underset{\sim}{D}_n \underset{\sim}{u})$ is convex in $\underset{\sim}{u}$. Hence, using (4.7), (4.6) follows from the proof of Theorem 5.2.3 by Hettmansperger (1984), i.e.,

$$\hat{\underset{\sim}{u}}_n = \underset{\sim}{B}^o_n(\hat{\underset{\sim}{\beta}}_M - \underset{\sim}{\beta}) = \tilde{\underset{\sim}{u}}_n + o_p(1), \quad \text{as } n\to\infty,$$

where $\hat{\underset{\sim}{u}}_n$ minimizes $D(\underset{\sim}{Y} - \underset{\sim}{D}_n \underset{\sim}{u})$. □

THEOREM 4.3. Assume (A1), (A2), (B), (C1) and (C2). Then, for any $K>0$,

(4.8) $$\sup_{|\underset{\sim}{u}|\le K} \left| \underset{\sim}{E}_n(\underset{\sim}{u}) - \underset{\sim}{E}_n(\underset{\sim}{0}) + \underset{\sim}{Q}_n \underset{\sim}{u} \gamma_\psi \right| \xrightarrow{P} 0, \quad \text{as } n\to\infty,$$

where $\gamma_\psi = \int F' d(\psi\circ F) > 0$. Therefore, R-estimator $\hat{\underset{\sim}{\beta}}_R$ is asymptotically normal, i.e.,

(4.9) $$\underset{\sim}{C}^o_n(\hat{\underset{\sim}{\beta}}_R - \underset{\sim}{\beta}) \xrightarrow{D} N_p(\underset{\sim}{0},\ \gamma_\psi^{-2} A^2 \underset{\sim}{Q}^{-1}), \quad \text{as } n\to\infty.$$

Proof. It is easy to show that

(4.10) $$E_{nj}(\underset{\sim}{u}) = -\int_0^1 S^*_{nj\text{-}}(F_n^{*-1}(t,\underset{\sim}{u}),\underset{\sim}{u})\, d\psi(t),$$

where $S^*_{nj\text{-}}(t,\underset{\sim}{u}) = \sum_{i=1}^n c_{nij} I\{Y_i < F^{-1}(t) + \underset{\sim}{c}^T_{ni} \underset{\sim}{u}\}$. Let $S^*_{nj=}(t,\underset{\sim}{u}) = S^*_{nj}(t,\underset{\sim}{u}) - S^*_{nj\text{-}}(t,\underset{\sim}{u}) =$

$\sum_{i=1}^{n} c_{nij} I\{Y_i = F^{-1}(t) + \underset{\sim}{c}_{ni}^{T} \underset{\sim}{u}\}$. By Lemma 3.4, we have

(4.11) $$\sup\Big\{|S^{*}_{nj=}(t,\underset{\sim}{u})|;\ t\in[0,1],\ |\underset{\sim}{u}|\le K\Big\} \xrightarrow{P} 0, \qquad \text{as } n\to\infty.$$

Hence, we have

(4.12) $$\sup_{|\underset{\sim}{u}|\le K} \Big| E_{nj}(\underset{\sim}{u}) + \int_0^1 S^{*}_{nj}(F_n^{*-1}(t,\underset{\sim}{u}),\underset{\sim}{u})\ d\psi(t) \Big| \xrightarrow{P} 0, \qquad \text{as } n\to\infty.$$

Note that (4.12) indicates that $\underset{\sim}{E}_n(\underset{\sim}{v})$ is equivalent to the functional Ψ of $\big(\underset{\sim}{S}_n^{*}(\cdot,\underset{\sim}{v}),$ $F_n^{*}(\cdot,\underset{\sim}{v})\big)$. Since

$$\int_0^1 S^{*}_{nj}(F_n^{*-1}(t,\underset{\sim}{u}),\underset{\sim}{u})\ d\psi(t) = \int_0^1 S^{*+}_{nj}(F_n^{*-1}(t,\underset{\sim}{u}),\underset{\sim}{u})\ d\psi(t) - \int_0^1 S^{*-}_{nj}(F_n^{*-1}(t,\underset{\sim}{u}),\underset{\sim}{u})\ d\psi(t)$$
$$= \sum_{i=1}^{n} c^{+}_{nij}\Psi\Big(\frac{S^{*+}_{nj}(\cdot,\underset{\sim}{u})}{\sum_{i=1}^{n} c^{+}_{nij}}, F_n^{*}(\cdot,\underset{\sim}{u})\Big) - \sum_{i=1}^{n} c^{-}_{nij}\Psi\Big(\frac{S^{*-}_{nj}(\cdot,\underset{\sim}{u})}{\sum_{i=1}^{n} c^{-}_{nij}}, F_n^{*}(\cdot,\underset{\sim}{u})\Big),$$

by Lemma 4.1, Corollary 3.2 and (4.12) we have

(4.13) $$\sup_{|\underset{\sim}{u}|\le K} \Big| E_{nj}(\underset{\sim}{u}) + \int_0^1 S^{*}_{nj}(t,\underset{\sim}{u})\ d\psi(t) \Big| \xrightarrow{P} 0, \qquad \text{as } n\to\infty.$$

Since

$$-\int_0^1 S^{*}_{nj-}(t,\underset{\sim}{u})\ d\psi(t) = -\sum_{i=1}^{n} c_{nij} \int_{F(Y_i-\underset{\sim}{c}_{ni}^{T}\underset{\sim}{u})}^{1} d\psi(t)$$

(4.14) $$= -\sum_{i=1}^{n} c_{nij}\Big(\psi(1)-\psi(F(Y_i-\underset{\sim}{c}_{ni}^{T}\underset{\sim}{u}))\Big) = \sum_{i=1}^{n} c_{nij}\psi(F(Y_i-\underset{\sim}{c}_{ni}^{T}\underset{\sim}{u})) = N_{nj}(\underset{\sim}{u}),$$

therefore, by (4.11) and (4.13), we have

(4.15) $$\sup_{|\underset{\sim}{u}|\le K} \Big|[E_{nj}(\underset{\sim}{u})-E_{nj}(\underset{\sim}{0})] - [N_{nj}(\underset{\sim}{u})-N_{nj}(\underset{\sim}{0})]\Big| \xrightarrow{P} 0, \qquad \text{as } n\to\infty.$$

By Theorem 4.2, for $\underset{\sim}{\alpha}_{ni}=\underset{\sim}{\beta}_{ni}=\underset{\sim}{c}_{ni}$ and $\varphi=\psi\circ F$, we have as $n\to\infty$

$$\sup_{|\underset{\sim}{u}|\le K} \Big|\underset{\sim}{N}_n(\underset{\sim}{u}) - \underset{\sim}{N}_n(\underset{\sim}{0}) + \sum_{i=1}^{n} \underset{\sim}{c}_{ni}\underset{\sim}{c}_{ni}^{T}\underset{\sim}{u}\gamma_{\psi}\Big| \xrightarrow{P} 0.$$

Therefore, (4.8) follows from $\underset{\sim}{Q}_n=\sum_{i=1}^{n}\underset{\sim}{c}_{ni}\underset{\sim}{c}_{ni}^{T}$ and (4.15).

From (4.8) and a similar proof of Lemma 1 by Jaeckel (1972), we have for any $K>0$,

(4.16) $$\sup_{|\underset{\sim}{u}|\le K} \Big|D(\underset{\sim}{Y}-\underset{\sim}{D}_n\underset{\sim}{u})-Q(\underset{\sim}{Y}-\underset{\sim}{D}_n\underset{\sim}{u})\Big| \xrightarrow{P} 0, \qquad \text{as } n\to\infty,$$

where

$$D(\underset{\sim}{Y}-\underset{\sim}{D}_n\underset{\sim}{u}) = \sum_{i=1}^{n} a(R(Y_i-\underset{\sim}{c}_{ni}^{T}\underset{\sim}{u}))(Y_i-\underset{\sim}{c}_{ni}^{T}\underset{\sim}{u})$$
$$Q(\underset{\sim}{Y}-\underset{\sim}{D}_n\underset{\sim}{u}) = D(\underset{\sim}{Y})-\underset{\sim}{u}^{T}\underset{\sim}{E}_n(\underset{\sim}{0})+2^{-1}\gamma_{\psi}\underset{\sim}{u}^{T}\underset{\sim}{E}_n(\underset{\sim}{0})\underset{\sim}{u}.$$

Note that, by Theorem V.1.5 of Hájek and Šidák (1967), $\tilde{\underset{\sim}{u}}_n=\gamma_{\varphi}^{-1}\underset{\sim}{Q}_n^{-1}\underset{\sim}{E}_n(\underset{\sim}{0})$ is asymptotically normal: $N_p(\underset{\sim}{0},\ A^2\gamma_\varphi^{-2}\underset{\sim}{Q}_n^{-1})$, and $\tilde{\underset{\sim}{u}}_n$ uniquely minimizes $Q(\underset{\sim}{Y}-\underset{\sim}{D}_n\underset{\sim}{u})$, and that $D(\underset{\sim}{Y}-\underset{\sim}{D}_n\underset{\sim}{u})$ is convex in $\underset{\sim}{u}$ (Jaeckel, 1972). Hence, using (4.16) and a similar proof of Theorem 5.2.3 by Hettmansperger (1984), we have

$$\hat{\underset{\sim}{v}}_n = \tilde{\underset{\sim}{u}}_n + o_p(1), \qquad \text{as } n\to\infty, \tag{4.17}$$

where $\hat{\underset{\sim}{v}}_n=\underset{\sim}{C}_n^o(\hat{\underset{\sim}{\beta}}_R-\underset{\sim}{\beta})$ minimizes $D(\underset{\sim}{Y}-\underset{\sim}{D}_n\underset{\sim}{u})$. Therefore, (4.9) follows from (A2) and (4.17). □

Remark. From the discussion Remark 1-3 in Section 2, we can see that, in Theorem 4.3, our assumptions on the design matrix $\underset{\sim}{D}_n$ are weaker than Jurečková's (1971) and Aubuchon's (1982). We require essentially the same conditions on the score-generating function ψ as Jurečková (1971), Heiler and Willers (1988), which are more general requirements than Aubuchon's (1982). But we do not require finite Fisher's information condition on F as Jurečková (1971), Aubuchon (1982), Heiler and Willers (1988). Also, note that, in our proof of (4.8), we do not need (A2). This means that we can achieve the results by Heiler and Willers (1988) with weaker condtions on F by using our method if we only require (A1) along with some additional condition.

THEOREM 4.4. In addition to (A1), (A2), (B), (C1), (C2) and (D), assume $\bar{\underset{\sim}{c}} = \underset{\sim}{0}$. Then, we have as $n\to\infty$

$$\underset{\sim}{C}_n^o(\hat{\underset{\sim}{\beta}}_R - \hat{\underset{\sim}{\beta}}_M) \xrightarrow{D} N_p(\underset{\sim}{0},\ \gamma\underset{\sim}{Q}^{-1}), \tag{4.18}$$

where $\gamma=\int_0^1\left\{\gamma_\varphi^{-1}\varphi(F^{-1}(t))-\gamma_\psi^{-1}\psi(t)\right\}^2dt$. If we further assume that $\varphi=\psi\circ F$, then for $K>0$,

$$\sup_{|\underset{\sim}{u}|\le K}\left|\underset{\sim}{E}_n(\underset{\sim}{u})-\underset{\sim}{M}_n(\underset{\sim}{u})\right| \xrightarrow{P} 0, \qquad \text{as } n\to\infty, \tag{4.19}$$

therefore,

$$|\underset{\sim}{C}_n^o(\hat{\underset{\sim}{\beta}}_R - \hat{\underset{\sim}{\beta}}_M)| \xrightarrow{P} 0, \qquad \text{as } n\to\infty. \tag{4.20}$$

Proof. First we notice that $\bar{\underset{\sim}{c}}=\underset{\sim}{0}$ implies $\underset{\sim}{B}_n^o = \underset{\sim}{C}_n^o$ and $\underset{\sim}{Q}_n = \underset{\sim}{Q}_n$. Therefore, by Theorem 4.2, we have

$$\underset{\sim}{C}_n^o(\hat{\underset{\sim}{\beta}}_M-\underset{\sim}{\beta}) = \gamma_\varphi^{-1}\underset{\sim}{Q}_n^{-1}\underset{\sim}{M}_n(\underset{\sim}{0}) + o_p(1), \tag{4.21}$$

and by (4.17), we have

$$\underset{\sim}{C}_n^o(\hat{\underset{\sim}{\beta}}_R-\underset{\sim}{\beta}) = \gamma_\psi^{-1}\underset{\sim}{Q}_n^{-1}\underset{\sim}{E}_n(\underset{\sim}{0}) + o_p(1). \tag{4.22}$$

Hence,

$$\underset{\sim}{C}_n^o(\hat{\underset{\sim}{\beta}}_M-\hat{\underset{\sim}{\beta}}_R) = \underset{\sim}{Q}_n^{-1}\left\{\gamma_\varphi^{-1}\underset{\sim}{M}_n(\underset{\sim}{0}) - \gamma_\psi^{-1}\underset{\sim}{E}_n(\underset{\sim}{0})\right\} + o_p(1). \tag{4.23}$$

By (4.13) and (4.14), we have

$$\underset{\sim}{C}_n^o(\hat{\underset{\sim}{\beta}}_M-\hat{\underset{\sim}{\beta}}_R) = \underset{\sim}{Q}_n^{-1}\left\{\gamma_\varphi^{-1}\underset{\sim}{M}_n(\underset{\sim}{0}) - \gamma_\psi^{-1}\underset{\sim}{N}_n(\underset{\sim}{0})\right\} + o_p(1). \tag{4.24}$$

Therefore (4.18) follows from Theorem V.1.2 of Hájek and Šidák (1967).

To complete our proof, we can easily see that (4.19) follows from (4.13) and (4.14), and that (4.20) follows from (4.24). □

Remark. In Theorem 4.4, without requiring any essentially stronger conditions on ψ and φ than Jurečková (1977), we have weakened the conditions on the design matrix and the underlying distribution function F.

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