

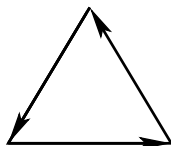
# Math 734, Assignment #2

## Relative Homology, Chain Complexes, and Exact Sequences

Jonathan Rosenberg

due Monday, February 11, 2008

1. (a) Show that  $\Delta^n$  (the standard  $n$ -simplex) is homeomorphic to  $D^n$  (the unit disk in  $\mathbb{R}^n$ ) and that its boundary is homeomorphic to  $S^{n-1}$ .
- (b) Show that the identity map  $\Delta^n \rightarrow \Delta^n$  defines a class in  $H_n(D^n, S^{n-1})$  which under the boundary map of the long exact homology sequence maps to the class in  $H_{n-1}(S^{n-1})$  represented by the sum of the faces of  $\Delta^n$ , with appropriate signs, like this:



2. Consider short exact sequences of  $\mathbb{Z}$ -modules

$$0 \rightarrow \mathbb{Z}/4 \rightarrow A \rightarrow \mathbb{Z}/4 \rightarrow 0.$$

- (a) What are the possibilities for  $A$  (up to isomorphism)?
- (b) Define an equivalence relation on such short exact sequences by saying two are equivalent if there is a commutative diagram

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \mathbb{Z}/4 & \longrightarrow & A & \longrightarrow & \mathbb{Z}/4 \longrightarrow 0 \\
 & & \parallel & & \downarrow \cong & & \parallel \\
 0 & \longrightarrow & \mathbb{Z}/4 & \longrightarrow & B & \longrightarrow & \mathbb{Z}/4 \longrightarrow 0.
 \end{array}$$

Show that there are exactly four equivalence classes.

- (c) Show that if we require  $A$  to be a  $\mathbb{Z}/4$ -module (and not just a  $\mathbb{Z}$ -module), then the short exact sequence must split.
3. In this problem, let  $C_*$  be a chain complex with  $C_* = 0$  for  $* < 0$ , and let  $H_*$  be the associated homology groups.

- (a) Show that  $H_*$  itself becomes a chain complex if we define all its boundary maps to be 0, and that this chain complex has the same homology as the original complex.
- (b) Show by examples (of chain complexes of  $\mathbb{Z}$ -modules) that there is not necessarily a chain map  $C_* \rightarrow H_*$  or  $H_* \rightarrow C_*$  inducing an isomorphism on homology.
- (c) Show that in the case of chain complexes over a *field* (not over  $\mathbb{Z}$ !) that one can always construct a splitting  $C_* \cong E_* \oplus H_*$ , where  $E_*$  is exact (has no homology) and where the projection  $C_* \rightarrow H_*$  and the inclusion  $H_* \rightarrow C_*$  induce isomorphisms on homology. (You will need the fact, which you can assume, that any linearly independent set in a vector space can be extended to a basis.)