Operator Theory:
Advances and Applications, Vol.17

SOME RESULTS ON COHOMOLOGY WITH BOREL COCHAINS, WITH APPLICATIONS TO GROUP ACTIONS ON OPERATOR ALGEBRAS

Jonathan Rosenberg *)

0. INTRODUCTION

This paper is an outgrowth of joint work with Richard Herman [5] and Iain Raeburn [13]. In both of these projects, questions concerning group actions on operator algebras naturally led to a study of obstruction classes in $H^2(G,U(A))$, where U(A) is the (suitably topologized) unitary group of an abelian operator algebra A. The appropriate cohomology theory here is the "Borel cochain" theory of C.C.Moore, as developed and systematized in [8]. In case A is a von Neumann algebra, U(A) is essentially what Moore calls $U(X,\mathbf{T})$ (X here is some standard measure space), and machinery for computing the relevant cohomology groups is developed and applied in [8] and [9].

We were interested, however, in problems concerning separable C^*-algebras. In this case, U(A) becomes $C(X,\mathbf{T})$, the continuous functions into the circle group **T** on a second-countable topological space X. The study of $H^n(G,C(X,\mathbf{T}))$ now becomes more a matter of topology than of measure theory, and techniques different from those of [8] are called for. The purpose of this paper is to compute the Moore cohomology groups in certain cases relevant to operator algebraists, and then to translate some of these calculations back into statements about operator algebras.

I wish to thank my coworkers Richard Herman and Iain Raeburn for their help in getting me started on the work described here. It will be obvious that my results depend heavily on the machines developed in [8] and [17]. Finally, I wish to thank the Mathematical Sciences Research Institute for its congenial and stimulating environment.

1. NOTATION AND REVIEW OF KNOWN FACTS

In this section we shall establish notation and review some known facts about "cohomology with Borel cochains" and its relation to problems concerning group actions on C^*-algebras. If G is a second-countable locally compact group and A a Polish G-

*) Research supported in part by NSF Grants DMS-8400900 and 8120790.

-module (that is, a metrizable topological G-module complete in its two-sided uniformity – see [8], § 2), then $H^n(G,A)$ $(n \geq 0)$ will always denote the cohomology groups defined by C.C.Moore in [8]. As noted there, these are the cohomology groups of either of the complexes $\{C^n_{Borel}(G,A), \delta\}$ or $\{\underline{C}^n(G,A), \delta\}$, where

$$C^n_{Borel}(G,A) = \{\text{Borel functions } G^n \to A\},$$

δ is the usual coboundary operator, and $\underline{C}^n(G,A)$ denotes the quotient of $C^n_{Borel}(G,A)$ by the equivalence relation $\sim$, where $f_1 \sim f_2$ if $f_1 = f_2$ almost everywhere (with respect to Haar measure on G^n).

We shall sometimes also refer to the "continuous cohomology" groups $H^n_{cont}(G,A)$, in other words, the cohomology groups of the complex $\{C^n_{cont}(G,A), \delta\}$ of continuous cochains. Continuous cohomology does not in general have good functorial properties, because of the fact that the functor $A \mapsto C^n_{cont}(G,A)$ is only left exact. Thus a short exact sequence of Polish G-modules need not give a long exact sequence in continuous cohomology, unless one considers only short exact sequences which are topologically split (see [4] for a systematic development). Nevertheless, continuous cohomology is sometimes more computable than Borel cochain cohomology. This makes it appropriate to study the relationship between the two theories. One always has $H^n(G,A) = H^n_{cont}(G,A)$ for $n \leq 1$ (this is essentially [8], Theorem 3 and corollaries); $H^2(G,A)$ classifies topological group extensions of G by A ([8], Theorem 10), whereas $H^2_{cont}(G,A)$ classifies extensions which split topologically.

If X is a paracompact topological space and G a topological group, $H^n(X,\underline{G})$ will denote the sheaf cohomology, which may be computed by the Čech process, of X with coefficients in the sheaf $\underline{G}$ of germs of continuous functions with values in G. This makes sense for all n if G is abelian, and only for n = 0 or 1 if G is non-commutative. Recall that via the correspondence between bundles and systems of transition functions, $H^1(X,\underline{G})$ classifies equivalence classes of locally trivial principal G-bundles over X. If G is discrete, $\underline{G}$ is the constant sheaf G and we have usual Čech cohomology $H^n(X,G)$.

Now suppose A is a separable C^*-algebra, not necessarily unital. We denote by M(A) the multiplier algebra of A (see [11], § 3.12), which is separable and metrizable in the *strict* topology defined by the semi-norms

$$x \mapsto \|xa\| + \|ax\|, \quad a \in A.$$

Aut(A) will denote the group of $*$-automorphisms of A; this is a Polish group in the topology of pointwise convergence. Inn(A) denotes the *inner* automorphisms, i.e., those of the form

$$a \mapsto (\mathrm{Ad}\,u)(a) = uau^*,$$

where $u \in U(M(A))$, the unitary group of $M(A)$. Note that the map

$$\mathrm{Ad} : U(M(A)) \to \mathrm{Aut}(A)$$

is a continuous homomorphism for the appropriate Polish topologies, with image $\mathrm{Inn}(A)$ and kernel

$$Z(U(M(A))) = U(Z(M(A))) \simeq C(\mathrm{Prim}\,A, \mathbf{T})$$

(by the Dauns-Hofmann Theorem, [11], Corollary 4.4.8). Here $\mathbf{T}$ is the circle group and $C(\mathrm{Prim}\,A,\mathbf{T})$ is the group of continuous functions from the primitive ideal space of A (or what is the same, its maximal Hausdorff quotient X) into $\mathbf{T}$. The strict topology on $U(M(A))$ restricts to the compact-open topology on $C(X,\mathbf{T})$. In general, $\mathrm{Inn}(A)$ is not closed in $\mathrm{Aut}(A)$. However, the following results from [13] will be useful.

THEOREM 1.1. ([13], §0). a) *If* A *is any separable* C^**-algebra,* $\mathrm{Inn}(A)$ *is a Borel subset of* $\mathrm{Aut}(A)$*, and any continuous homomorphism or crossed homomorphism*

$$\phi : G \to \mathrm{Aut}(A)$$

from a Polish group G *which takes its values in* $\mathrm{Inn}(A)$ *is automatically continuous for the Polish topology on* $\mathrm{Inn}(A)$ *coming from its identification with the quotient* $U(M(A))/C(\mathrm{Prim}\,A,\mathbf{T})$.

b) *If* X *is a second-countable locally compact space with* $H^2(X,\mathbf{Z})$ *countable (in particular, if* X *is compact or has a compact deformation retract) and if* A *is a separable continuous-trace algebra with spectrum* X*, then* $\mathrm{Inn}(A)$ *is closed in* $\mathrm{Aut}(A)$*. Furthermore,* $\mathrm{Inn}(A)$ *is open in the subgroup* $\mathrm{Aut}_{C_0(X)}A$ *of automorphisms of* A *that leave* X *pointwise fixed.* ■

Now let G be a second-countable locally compact group, A a separable C^*-algebra. An *action* of G on A (sometimes called a C^**-dynamical system* or *locally compact automorphism group*) means a continuous homomorphism $\alpha: G \to \mathrm{Aut}(A)$, where $\mathrm{Aut}(A)$ as usual has the topology of pointwise convergence. From such an action one can construct a crossed product $A \rtimes_\alpha G$, and when G is abelian, a dual action of $\hat{G}$. The action α is said to be *inner* or *unitary* if there is a (continuous) homomorphism $u : G \to U(M(A))$ such that $\alpha_g = \mathrm{Ad}\,u_g$ for all $g \in G$. In this case $A \rtimes_\alpha G \simeq A \otimes C^*(G)$ (the crossed product for the trivial action of G), and the isomorphism is equivariant for the dual action of $\hat{G}$ in case G is abelian. More generally, two actions α and β of G are said to be *exterior equivalent* if $\alpha_g \beta_g^{-1} = \mathrm{Ad}\, u_g$ for some 1-cocycle $u : G \to U(M(A))$ (with respect to the action of G on A given by β), or equivalently if α and β may be realized as opposite "corners" of an action of G on $M_2(A)$ ([11], Lemma 8.11.2). Thus α is unitary if and only if α is exterior equivalent to the trivial action. A weaker equivalence relation, which still implies isomorphism of $A \rtimes_\alpha G$ with $A \rtimes_\beta G$ (equivariant for $\hat{G}$ when G is abelian), is exterior equivalence composed with conjugacy, i.e., exterior equivalence of α with $g \mapsto \gamma \beta_g \gamma^{-1}$, for some $\gamma \in \mathrm{Aut}\, A$. Further refinements of this will be studied in § 4 below.

REMARK 1.2. Given an action α of G on A, there are two obstructions to its being unitary. First of all, $\alpha(G)$ must be contained in Inn(A), and in particular must act trivially on $\hat{A}$. (Note that if A has continuous trace and $H^2(\hat{A}, \mathbf{Z})$ is countable, then this is automatic by Theorem 1.1 (b) if G is connected, once $\alpha(G) \subseteq \mathrm{Aut}_{C_0(\hat{A})} A$.) Then by Theorem 1.1 (a), α may be viewed as a homomorphism into $U(M(A))/C(\mathrm{Prim}\, A, \mathbf{T})$, and one must be able to lift this homomorphism to some

$$u : G \to U(M(A)) .$$

By the cohomology exact sequence for

$$1 \to C(\mathrm{Prim}\, A, \mathbf{T}) \to U(M(A)) \to U(M(A))/C(\mathrm{Prim}\, A, \mathbf{T}) \to 1,$$

which is valid as far as $H^2(G, C(\mathrm{Prim}\, A, \mathbf{T}))$ despite non-commutativity of U(M(A)), the obstruction to this is precisely a class in $H^2(G, C(\mathrm{Prim}\, A, \mathbf{T}))$ (where the G-module has trivial G-action). Similarly, given two actions α and β of G on A, there are two obstructions to their being exterior equivalent. First, we must have $\alpha_g \beta_g^{-1} \in \mathrm{Inn}(A)$ for all $g \in G$, and secondly, once this is the case, there is an additional obstruction in $H^2(G, C(\mathrm{Prim}\, A, \mathbf{T}))$ (this time the G-action on the module is non-trivial and comes from the action of either of α or β on Prim A). In the special case $A = \mathcal{K}$, the algebra of compact operators on a separable infinite-dimensional Hilbert space, every automorphism of A is inner, and an action α of G on A just amounts to a projective unitary representation of G. In this case the obstruction class of α in $H^2(G, \mathbf{T})$ is the usual Mackey obstruction to lifting a projective representation to an ordinary representation, and the obstruction to exterior equivalence of α and β is the difference between their Mackey obstructions. ∎

Remark 1.2 was implicitly used in [5], where we noted that if α and β are two actions of a connected group G on a separable C^*-algebra A, all of whose derivations are inner (this applies both to simple C^*-algebras and to continuous-trace algebras), such that $\|\alpha_g - \beta_g\| < 2$ for all g in a neighborhood of the identity element e in G, then

$\alpha_g \beta_g^{-1}$ ε Inn(A) for all g and the obstruction to exterior equivalence of α and β lies in $H^2(G,C(\text{Prim } A,\mathbf{T}))$. This is somewhat relevant to certain problems of stability in quantum field theory and quantum statistical mechanics; for a discussion of physical implications, see the references quoted in [5].

A final notion we shall need is that of a *locally unitary* group action. Given a type **I** C^*-algebra A, an action α of G on A is said to be *pointwise unitary* if for each $\pi \in \hat{A}$ there exists a strongly continuous unitary representation u of G on H_π such that

$$\pi(\alpha_g(a)) = u_g \pi(a) u_g^* \qquad \text{for all } g \in G, \alpha \in A .$$

Equivalently, α fixes $\hat{A}$ pointwise, and all Mackey obstructions in $H^2(G,\mathbf{T})$ vanish. Then α is said to be *locally unitary* if there is a covering $\{U_i\}$ of $\hat{A}$ by open sets (locally closed sets will also do if their interiors cover $\hat{A}$) and if the restrictions of α to the corresponding ideals (or subquotients) of A are unitary. Locally unitary actions were studied in great detail in [12] (using a slightly stronger definition: the implementing map u_i : : $G \to U(M(A_i))$, where A_i is the subquotient of A with spectrum U_i, was supposed to come from a map $G \to M(A)$. This was never used in an essential way, and anyway we shall only be interested in the case where A is separable and $\hat{A}$ Hausdorff, in which case the two definitions are equivalent. The reason is that then we may refine our original covering to a covering by compact, hence *closed* sets, so that the A_i's are all *quotients* of A. Then the natural maps $M(A) \to M(A_i)$ are surjective ([11], Proposition 3.12.10) continuous maps of Fréchet spaces, and so have continuous sections by a selection theorem of E.Michael ([6], Corollary 7.3)).

2. COHOMOLOGY WITH COEFFICIENTS IN A TRIVIAL MODULE AND OBSTRUCTIONS TO AN ACTION BEING UNITARY

The elegant description of locally unitary group actions given in [12] makes it natural to ask when a pointwise unitary action is locally unitary. We shall give a substantial improvement of [12], Proposition 1.1, showing that this is automatic under very mild conditions. The tool needed is a generalization of Theorem 2.6 of [5], which in view of Example 1.2 of [12] is best possible if G is abelian.

THEOREM 2.1. *Let* X *be a second-countable locally compact space and let* G *be a second-countable locally compact group with* $H^2(G,\mathbf{T})$ *Hausdorff and with the abelianization* $G_{ab} = G/[G,G]^-$ *compactly generated. (It suffices, but is not necessary, for* G *to satisfy one of the following:* a) G *is abelian and compactly generated, or* b) G = [G,G], *or* c) G_{ab} *compactly generated and* $H^2(G,\mathbf{T})$ *countable, or* d) G *a connect-*

-ed, simply connected Lie group.) Let $C(X,\mathbf{T})$ *be given the compact-open topology and the trivial action of* G, *and let* $[\alpha] \in H^2(G,C(X,\mathbf{T}))$ *be pointwise trivial. (In other words, for each* $x \in X$, *we assume that* $(e_x)_*[\alpha] = 0$ *in* $H^2(G,\mathbf{T})$, *where* $e_x : C(X,\mathbf{T}) \to \mathbf{T}$ *is evaluation at* x.*) Then* $[\alpha]$ *is locally trivial,* i.e., *there is an open covering* $\{V_i\}$ *of* X *such that the image of* $[\alpha]$ *is zero in each* $H^2(G,C(V_i,\mathbf{T}))$.

PROOF. Let $\alpha : G \times G \to C(X,\mathbf{T})$ be a Borel 2-cocycle representing the class $[\alpha]$. Proceeding as in the proof of [5], Theorem 2.6, we may view α as a continuous map

$$f_\alpha : X \to \underline{B}^2(G,\mathbf{T}) \simeq \underline{C}^1(G,\mathbf{T})/\mathrm{Hom}(G,\mathbf{T}).$$

Now *a priori*, f_α is continuous (using [8], Proposition 6) only for the relative topology on $\underline{B}^2(G,\mathbf{T})$ as a subset of $\underline{C}^2(G,\mathbf{T})$, but this coincides with the quotient topology if $H^2(G,\mathbf{T})$ is Hausdorff. So to prove the theorem, it is enough to show that the quotient map

$$q : \underline{C}^1(G,\mathbf{T}) \to \underline{C}^1(G,\mathbf{T})/\mathrm{Hom}(G,\mathbf{T})$$

is locally trivial (topologically), for given U open in X and sufficiently small, it will follow that $f_\alpha|U$ can be lifted to a continuous map $U \to \underline{C}^1(G,\mathbf{T})$ which will provide (for reasons we will discuss below) an element of $\underline{C}^1(G,C(U,\mathbf{T}))$ of which $\alpha|U$ is the coboundary. By the Palais local cross-section theorem ([10], §4.1), local triviality of q is automatic provided $\mathrm{Hom}(G,\mathbf{T})$ is a Lie group. But $\mathrm{Hom}(G,\mathbf{T}) \simeq \mathrm{Hom}(G_{ab},\mathbf{T})$ is just the Pontryagin dual of G_{ab}, which will be a Lie group if and only if G_{ab} is compactly generated.

There is a point still to be settled, which is to see why a continuous map $\bar{\psi} : U \to \underline{C}^1(G,\mathbf{T})$ with coboundary $f_\alpha|U \in C(U,\underline{C}^2(G,\mathbf{T}))$ gives an element of $\underline{C}^1(G,C(U,\mathbf{T}))$ with coboundary $\alpha|U \in \underline{C}^2(G,C(U,\mathbf{T}))$. (There is no obvious reason why $\underline{C}^n(G,C(U,\mathbf{T}))$ and $C(U,\underline{C}^n(G,\mathbf{T}))$ should be isomorphic, though the first includes in the second.) The point is that by [8], $\{C_{\mathrm{Borel}}(G,\cdot)\}$ and $\{\underline{C}(G,\cdot)\}$ give the same cohomology groups, so we may represent $\bar{\psi}$ by a Borel function $\psi : G \times U \to \mathbf{T}$ with $\delta(\psi) = \alpha|U$ (as a Borel function on $G \times G \times U$) everywhere, not just almost everywhere. The problem is then to show that if $x_n \to x$ in U, then $\psi(g,x_n) \to (g,x)$ for all g. By definition of the topology of $C(U,\underline{C}^1(G,\mathbf{T}))$, we know that $\psi(\cdot,x_n) \to \psi(\cdot,x)$ in measure (as functions on G). Furthermore, since $\delta(\psi) = \alpha|U$ is a Borel function from $G \times G$ to $C(U,\mathbf{T})$, we have for $g_1, g_2 \in G$,

$$\delta(\psi)(g_1,g_2,x_n) \to \delta(\psi)(g_1,g_2,x),$$

i.e.,

$$(*) \qquad \psi(g_1,x_n)\psi(g_2,x_n)\psi(g_1g_2,x_n)^{-1} \to \psi(g_1,x)\psi(g_2,x)\psi(g_1g_2,x)^{-1}.$$

Now if there is a $g \in G$ for which $\psi(g,x_n) \not\to \psi(g,x)$, we may pass to a subsequence and assume $|\psi(g,x_n) - \psi(g,x)| \geq \varepsilon > 0$ for all n. By [8], Proposition 6, however, after passage to a further subsequence there is a null set N in G such that $\psi(g_1,x_n) \to \psi(g_1,x)$ for $g_1 \notin N$. Since $\psi(g,x_n) \not\to \psi(g,x)$, it follows from (*) that also $\psi(gg_1,x_n) \not\to \psi(gg_1,x)$ for $g_1 \notin N$. Since $(G\setminus N) \cap g(G\setminus N) \neq \emptyset$, this is a contradiction, and so ψ has the necessary continuity.

Finally, we comment on sufficiency of conditions (a) – (d). If G is abelian, $H^2(G,\mathbf{T})$ is Hausdorff by [9], Theorem 7. If $G = [G,G]$, $H^2(G,\mathbf{T})$ is Hausdorff by [9], Theorem 13. If $H^2(G,\mathbf{T})$ is countable, it is automatically Hausdorff by [9], Proposition 6. And if G is a connected, simply connected Lie group, $H^2(G,\mathbf{T})$ is a (Hausdorff) vector group by [7], Theorem A and subsequent remarks. ∎

COROLLARY 2.2. *Let* G *be as in Theorem* 2.1. *Then any pointwise unitary action* α *of* G *on a separable continuous-trace algebra* A *is automatically locally unitary.*

PROOF. Since $\hat{A}$ is locally compact and the problem is local, we may assume without loss of generality that $\hat{A}$ is compact. Then $H^2(\hat{A},\mathbf{Z})$ is countable and Inn(A) is open in $\mathrm{Aut}_{C_0(\hat{A})}A$ by Theorem 1.1; in fact

$$\mathrm{Aut}_{C_0(\hat{A})}A/\mathrm{Inn}\,A \hookrightarrow H^2(\hat{A},\mathbf{Z})$$

topologically by [12], Corollary 3.12 together with [13], §0. Since α is pointwise unitary, in particular $\alpha(G) \subseteq \mathrm{Aut}_{C_0(\hat{A})}A$, and by passage to the quotient we get a continuous map

$$\bar{\alpha} : G \to H^2(\hat{A},\mathbf{Z}).$$

This map must factor through G_{ab}, which by assumption is compactly generated, so since $H^2(\hat{A},\mathbf{Z})$ is countable and discrete, the image of $\bar{\alpha}$ is *finitely* generated. Choose a finite set of generators for $\bar{\alpha}(G)$ and an open covering $\{U_i\}$ of $\hat{A}$ which simultaneously trivializes all these cocycles. (Just choose a covering for each generator and take appropriate intersections.) Then $\bar{\alpha}$ is trivial over each U_i, i.e., α takes G into $\mathrm{Inn}(A\,|\,U_i)$ for each i. By Remark 1.2, the only obstruction to α being unitary over each U_i is a class $[\alpha\,|\,U_i]$ in $H^2(G,C(U_i,\mathbf{T}))$, and this class is pointwise trivial since α was assumed pointwise unitary. The conclusion now follows from Theorem 2.1. ∎

The most interesting case of the above occurs when G is abelian and compactly generated, in which case the above proof suggests a link between the Moore cohomology group $H^2(G,C(X,\mathbf{T}))$ and the Phillips-Raeburn classification of locally unitary G-actions on continuous-trace algebras with spectrum X in terms of locally trivial $\hat{G}$-bundles. The key to a more precise analysis is the appearance in the proof of the quotient map q. The significance of this may be seen in the following results, due essentially to D.Wigner, which will also be used in § 3 below.

LEMMA 2.3. *Let* (Y,μ) *be a standard measure space without atoms and let* A *be a Polish group. Then* $U(Y,A)$ *as defined on* p.5 *of* [8], *that is, the set of equivalence classes (modulo agreement* μ-a.e.) *of* A*-valued Borel functions on* Y, *with the topology of convergence in measure, is contractible.*

PROOF. We may assume Y is the unit interval $[0,1]$ and μ is Lebesgue measure. Then as pointed out on pp.86-87 of [17],

$$h_t(f)(x) = \begin{cases} f(x) & \text{if } x \geq t, \\ e, \quad \text{the identity of } A, & \text{if } x < t, \end{cases}$$

gives an explicit homotopy from the identity map h_0 on $U([0,1],A)$ to the map h_1 collapsing $U([0,1],A)$ to a point. ■

PROPOSITION 2.4. a) *If* G *is a second-countable locally compact abelian group which is compactly generated and non-discrete, then*

$$q : \underline{C}^1(G,\mathbf{T}) \to \underline{C}^1(G,\mathbf{T})/\mathrm{Hom}(G,\mathbf{T}) \simeq \underline{B}^2(G,\mathbf{T})$$

is a universal $\hat{G}$*-bundle, and* $\underline{B}^2(G,\mathbf{T})$ *is a classifying space for* $\hat{G}$.

b) *Let* G *be any Lie group with countably many components, or more generally, any second-countable Banach Lie group (e.g., the unitary group of a separable unital* C^**-algebra). Then*

$$U([0,1],G) \to U([0,1],G)/G$$

is a universal G*-bundle and* $U([0,1],G)/G$ *is a classifying space for* G.

c) *If* A *is a locally arcwise connected Polish abelian group, then so is* $BA =$ $= U([0,1],A)/A$, *and* BA *is a weak classifying space for* A. *(This means* $H^1(X,A) \simeq [X,BA]$ *for* X *a* CW*-complex, where* $[X,BA]$ *denotes homotopy classes of maps from* X *into* BA. *Equivalently,* BA *has the weak homotopy type of a classifying space for* A.*)*

PROOF. a) Since G is non-discrete, G is non-atomic with respect to Haar measure. But as a space $\underline{C}^1(G,\mathbf{T})$ is the same as $U(G,\mathbf{T})$, which is contractible by Lemma 2.3. The map q is a locally trivial principal $\hat{G}$-bundle by [10], § 4.1, and of course $\underline{B}^2(G,\mathbf{T})$ is metrizable, hence paracompact. Thus all conditions for a universal $\hat{G}$-bundle are satisfied ([3], Theorem 7.5).

b) works exactly the same way, except that if G is not a Lie group, local triviality of the bundle can still be proved by [6], Corollary 7.3. (In case G is the unitary group of a unital separable C^*-algebra, G is locally isomorphic to the real Banach space of skew-adjoint elements, via the exponential map.)

c) is similar except that the hypothesis is not strong enough to guarantee that the quotient map to BA is locally trivial. It is, however, a Serre fibration by [17], Proposition 3. As is well known, this is sufficient to make BA a weak classifying space, but for completeness we give the argument here. Let $EA = U([0,1],A)$. Then

$$0 \to A \to EA \to BA \to 0$$

is a short exact sequence of Polish abelian groups. If X is a CW-complex (it's enough to consider the case of a finite complex), we obtain an exact sequence

$$0 \to \underline{A} \to \underline{EA} \to \underline{BA}$$

of sheaves over X. The map $\underline{EA} \to \underline{BA}$ is surjective as a map of sheaves since $EA \to BA$ is a Serre fibration so that a continuous map $X \to BA$ can be lifted (by the HLP, homotopy lifting property) in a neighborhood of any point in X. Thus we have a long exact sequence in sheaf cohomology

$$H^0(X,\underline{EA}) \to H^0(X,\underline{BA}) \to H^1(X,\underline{A}) \to H^1(X,\underline{EA}).$$

Here $H^1(X,\underline{EA}) = 0$ since EA is contractible (either by the theory of [3] or else by Lemma 4 of [2]), so $H^1(X,\underline{A})$ is a quotient of

$$H^0(X,\underline{BA}) = C(X,BA).$$

But by contractibility of EA, a continuous map $X \to BA$ with a lifting $X \to EA$ must be null-homotopic, and conversely, any null-homotopic map $X \to BA$ has a lifting by the HLP. Thus

$$H^1(X,A) \cong C(X,BA)/\{\text{null-homotopic maps}\} \cong [X,BA]. \qquad \blacksquare$$

Parts (b) and (c) of the above proposition were only included here for completeness, since they logically belong with Lemma 2.3. However, part (a) can be used immediately to prove the following.

THEOREM 2.5. *Let* G *be a connected, second-countable, locally compact abelian group, and let* X *be a second-countable locally compact space with* $H^2(X,\mathbf{Z})$ *countable (for instance, a compact metric space). Let* A *be any separable continuous-trace alge-*

bra with spectrum X *and* $\alpha : G \to \mathrm{Aut}_{C_0(X)}A$ *any action of* G *on* A *inducing the trivial action on* X. *Then:*

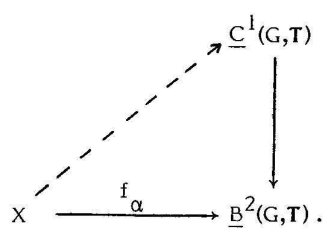
a) $\alpha(G)$ *consists of inner automorphisms, and* α *is locally unitary if and only if it is pointwise unitary (which is automatic if* G *is compact),*

b) *the pointwise trivial part of* $H^2(G,C(X,\mathbf{T}))$ *is naturally isomorphic to* $H^1(X,\underline{\hat{G}})$, *and*

c) *when* α *is pointwise unitary, the obstruction in* $H^2(G,C(X,\mathbf{T}))$ *to* α *being unitary may be identified under the isomorphism of* (b) *to the Phillips-Raeburn obstruction* $\zeta(\alpha) \in H^1(X,\underline{\hat{G}})$.

PROOF. (a) Since $\alpha(G)$ is a connected subgroup of $\mathrm{Aut}_{C_0(X)}A$, it must lie in Inn(A) by Theorem 1.1 (b). Thus an obstruction $[\alpha] \in H^2(G,C(X,\mathbf{T}))$ is defined by Remark 1.2, and this is the only obstruction to α being unitary. We know α will be pointwise unitary if and only if the Mackey obstructions $(e_x)_*[\alpha] \in H^2(G,\mathbf{T})$, $x \in X$, all vanish. Thus it is useful to note that by [7], Proposition 2.1 and Theorem 2.1, $H^2(G,\mathbf{T}) = 0$ if G is compact, since then G is an inverse limit of tori. If α is pointwise unitary, it is then locally unitary by Corollary 2.2.

(b) The proof of Theorem 2.1 shows that if α is pointwise unitary, the obstruction $[\alpha]$ may be identified with the obstruction to finding a lifting in the diagram



By Proposition 2.4 (a), this is the same as determining the homotopy class of f_α in $[X,B\hat{G}] \simeq H^1(X,\underline{\hat{G}})$. In this way one obtains an injection from the pointwise trivial part of $H^2(G,C(X,\mathbf{T}))$ into $H^1(X,\underline{\hat{G}})$, and the map is obviously surjective since any continuous $f : X \to \underline{B}^2(G,\mathbf{T})$ gives rise to a pointwise trivial element of $\underline{Z}^2(G,C(X,\mathbf{T}))$. (This part of the argument doesn't use connectedness of G except to imply G is non-discrete.)

(c) Note by the structure theory of locally compact abelian groups that if K is the maximal compact subgroup of G, then $G \simeq K \times \mathbf{R}^n$ for some n, so that actually $H^1(X,\underline{\hat{G}}) \simeq H^1(X,\hat{K})$. Thus it is only the compact part of G that really matters, although we shall not use this.

Let's compare $[\alpha] \in H^2(G,C(X,\mathbf{T}))$ with $\zeta(\alpha) \in H^1(X,\underline{\hat{G}})$. The latter is computed by covering X by sets X_j over which α is unitary, so that one has maps

$$v^j : G \to U(M(A \mid X_j))$$

implementing α locally. On the overlap sets $X_{jk} = X_j \cap X_k$, v^j and v^k must implement the same automorphism group, hence they differ by some

$$v^{jk} \in \mathrm{Hom}(G,C(X_{jk},\mathbf{T})) \simeq C(X_{jk},\hat{G}).$$

Then $\{v^{jk}\}$ is a Čech cocycle in $Z^1(X,\underline{\hat{G}})$ whose cohomology class is the obstruction $\zeta(\alpha)$ to patching the v^j's to a global homomorphism $G \to U(M(A))$. But the X_j's also trivialize $[\alpha] \in H^2(G,C(X,\mathbf{T}))$, which under the isomorphism of (b) goes to the "patching data" for local liftings $X_j \to \underline{C}^1(G,\mathbf{T})$ of f_α. This is clearly the same cohomology class obtained from the $\{v^{jk}\}$. ∎

3. CONTINUOUS VS. BOREL COHOMOLOGY AND INDUCTION

In this section we first show that $H^n(G,A)$ often coincides with $H^n_{cont}(G,A)$ if G is a vector group. Then we discuss "continuous induction" and the expected form of a "Shapiro's Lemma" for continuously induced modules. This is applied to the computation of the cohomology groups of certain **R**-modules that sometimes arise in applications. We also mention some results on cohomology of modules for compact Lie groups, complementing some results in [5] and [13], and explain how to generalize to general Lie groups.

Our first result is an interesting curiosity that is perhaps known to topologists, but which we haven't seen in the literature. The hypothesis on A could be weakened provided BA is an infinite loop space.

PROPOSITION 3.1. *Let* X *be a* CW *complex,* A *a locally arcwise connected Polish abelian group. Then for* $q \geq 1$, *there is a natural isomorphism*

$$H^q(X,\underline{A}) \simeq [X,B^qA],$$

where the iterated (weak) classifying spaces BA, $B^2A = B(BA), \ldots, B^qA = B(B^{q-1}A)$ *may be constructed to satisfy the same conditions as* A *by the procedure of Proposition* 2.4 (c). *In particular,* $H^q(X,\underline{A}) = 0$ *for all* $q \geq 1$ *provided either* X *or* A *is contractible.*

PROOF. Let us show, using Proposition 2.4 (c), that for all $q > 1$,

$$H^q(X,\underline{A}) \simeq H^{q-1}(X,\underline{BA}).$$

Since we already know that $H^1(X,\underline{A}) = [X,BA]$ and that BA can be constructed to have all the same properties as A, the result then follows by iteration. But recall from the proof of Proposition 2.4 (c) that we have a short exact sequence of sheaves over X

$$0 \to \underline{A} \to \underline{EA} \to \underline{BA} \to 0.$$

Furthermore, since EA is contractible, the sheaf $\underline{EA}$ is acyclic by [2], Lemma 4. So we get the result immediately from the long exact sequence

$$0 = H^{q-1}(X,\underline{EA}) \to H^{q-1}(X,\underline{BA}) \to H^q(X,\underline{A}) \to H^q(X,\underline{EA}) = 0. \qquad \blacksquare$$

PROPOSITION 3.2. *Suppose* G *is a vector group* (i.e., $\mathbf{R}^n$ *for some* n) *and* A *is a Polish* G*-module with "property F" of* [17]. *(It suffices for* A *to be locally arcwise connected.) Then the natural map* $C^*_{cont}(G,A) \to C^*_{Borel}(G,A)$ *induces isomorphisms*

$$H^n_{cont}(G,A) \to H^n(G,A)$$

for all n. *In particular, every extension of* G *by* A *splits topologically.*

PROOF. By [17], Theorem 2, there is a natural spectral sequence converging to $H^*(G,A)$ with

$$E_1^{p,q} \simeq H^q_{sheaf}(G^p,\underline{A})$$

and

$$d_1 : E_1^{p,0} = C^p_{cont}(G,A) \to E_1^{p+1,0} = C^{p+1}_{cont}(G,A)$$

the usual coboundary operator δ. (Caution to the reader: there is a misprint on p.91, l.6 of [17]; p and q are reversed there.) Since G, hence G^p is contractible, we have $E_1^{p,q} = 0$ for $q > 0$ by Proposition 3.1. Thus the spectral sequence degenerates at E_2 and gives the desired result. Of course, the statement about topological splitting of extensions, which may be read off from the equality $H^2_{cont}(G,A) \simeq H^2(G,A)$, also follows directly from the HLP built into the definition of property F, except for the problem that the definition deals only with abelian extensions. $\blacksquare$

Now if G is a locally compact group, among the most commonly encountered G--modules are modules induced from a closed subgroup. When the induction process is the "Borel induction" of [8], p.14, the cohomology of the induced module is given by Moore's version of "Shapiro's Lemma", [8], Theorem 6. Often, however, one is interested in *continuous* induction, so we outline here the analogous theory.

Suppose H is a closed subgroup of a second-countable locally compact group G, and A is a Polish G-module. We define the *continuously induced* G-module to be

$$\mathrm{Ind}_{H\uparrow G}A = \{f \in C(G,A) \mid f(gh) = h^{-1}\cdot f(g) \text{ for all } h \in H\} ,$$

equipped with the topology of uniform convergence on compacta and with the action of G by left translation. When G/H is discrete, $\mathrm{Ind}_{H\uparrow G}A$ may be identified with Moore's $I_H^G(A)$ (which is defined similarly using measurable functions in place of continuous functions). Usually, however, the two will differ. For instance, if $A = \mathbf{T}$ with trivial H-action, $I_H^G(\mathbf{T})$ is the unitary group (with the weak topology) of the von Neumann algebra $L^\infty(G/H)$, whereas $\mathrm{Ind}_{H\uparrow G}(\mathbf{T}) = C(G/H,\mathbf{T}) = U(M(C_0(G/H)))$, equipped with the strict topology. Thus continuous induction is related to problems in C^*-algebra theory in the same way Borel induction is related to problems in von Neumann algebra theory.

We would like to prove a variant of Shapiro's Lemma for continuous induction; however, it is *not* true in general that $H^n(G,\mathrm{Ind}_{H\uparrow G}A) \simeq H^n(H,A)$. The problem is that the functor $A \mapsto \mathrm{Ind}_{H\uparrow G}A$ (from Polish H-modules to Polish G-modules) is left exact but not usually right exact. For in the simplest case where

$$0 \to A \to B \to C \to 0$$

is a short exact sequence of Polish abelian groups with trivial H-action, exactness of $\mathrm{Ind}_{H\uparrow G}$ would mean that every continuous map $G/H \to C$ can be lifted to a continuous map $G/H \to B$. This will rarely be the case, even if $B \to C$ is a fibration, since G/H may not be contractible.

The correct formalism is suggested by homological algebra. If $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ are abelian categories with enough injectives and $T : \mathcal{A} \to \mathcal{B}$, $S : \mathcal{B} \to \mathcal{C}$ are left-exact functors, recall that one can define right-derived functors $R^nT : \mathcal{A} \to \mathcal{B}$, $R^nS : \mathcal{B} \to \mathcal{C}$, and $R^n(S\circ T) : \mathcal{A} \to \mathcal{C}$. Under a mild technical condition, one has a "composition of functors" spectral sequence

$$R^pS(R^qT(A)) \Longrightarrow R^{p+q}(S\circ T)(A),$$

and in fact most of the familiar spectral sequences of homological algebra are of this type. Here is a standard example: the Hochschild-Serre spectral sequence in group cohomology. Suppose $\Delta \triangleleft \Gamma$ are abstract groups, and take

$\mathcal{A}$ = category of Γ-modules,

$\mathcal{B}$ = category of Γ/Δ-modules,

$\mathcal{C}$ = category of abelian groups,

$T : \mathcal{A} \to \mathcal{B}$ the functor $A \mapsto A^{\Delta}$, "fixed points under Δ",

$S : \mathcal{B} \to \mathcal{C}$ the functor $B \mapsto B^{\Gamma/\Delta}$, "fixed points under Γ/Δ".

Then $S \circ T : A \mapsto A^{\Gamma}$, $\mathcal{R}^n T = H^n(\Delta, \_\_)$, $\mathcal{R}^n S = H^n(\Gamma/\Delta, \_\_)$, and the spectral sequence is the usual one

$$H^p(\Gamma/\Delta, H^q(\Delta, A)) \Longrightarrow H^{p+q}(\Gamma, A).$$

In our situation we do not have abelian categories with enough injectives, but for G a Lie group, $\mathcal{A}$ the quasi-abelian S-category of Polish H-modules with property F, $\mathcal{B}$ the category of Polish G-modules with property F, $\mathcal{C}$ the category of abelian groups, $T = \mathrm{Ind}_{H\uparrow G}$, and $S : A \mapsto A^G$, the same formalism suggests a spectral sequence of the sort

$$H^p(G, \mathcal{R}^q \mathrm{Ind}_{H\uparrow G} A) \Longrightarrow H^{p+q}(H, A),$$

which should enable one to compute $H^*(G, \mathrm{Ind}_{H\uparrow G} A)$ in terms of $H^*(H, A)$, provided one can obtain enough information about the "derived functors" of $\mathrm{Ind}_{H\uparrow G}$. (Compare the situation for Zuckerman's "cohomological induction functor" in [15], Theorem 6.2.14.)

For lack of compelling applications, we refrain from trying to work out the theory in this generality, and content ourselves with a few special cases. First we use Proposition 3.2 to prove "Shapiro's Lemma" in the one case one would expect it in the usual form.

PROPOSITION 3.3. *Let* A *be a locally arcwise connected Polish abelian group,* G *a vector group. Then* $H^n(G, \mathrm{Ind}_{1\uparrow G} A) = 0$ *for* $n > 0$. *(Since, clearly,* $H^0(G, \mathrm{Ind}_{1\uparrow G} A) \simeq$ $\simeq$ {constant functions $G \to A$} $\simeq A$, *this is the same as saying* $H^n(G, \mathrm{Ind}_{1\uparrow G} A) \simeq H^n(1, A)$ *for all* n.*)*

PROOF. Observe that if A is locally arcwise connected and A_0 is the path component of 0 in A, then A_0 is open in A and $C(G, A_0) = \mathrm{Ind}_{1\uparrow G} A_0$ is open in $\mathrm{Ind}_{1\uparrow G} A$. Since G is contractible, every continuous function $G \to A_0$ is homotopic to a constant function, and so $C(G, A_0)$ is path connected. Thus $\mathrm{Ind}_{1\uparrow G} A$ is locally arcwise connected and so has property F by [17], Proposition 3. Applying Proposition 3.2, we obtain $H^n(G, \mathrm{Ind}_{1\uparrow G} A) \simeq H^n_{cont}(G, \mathrm{Ind}_{1\uparrow G} A)$. The vanishing is now implicit in [17] and much easier than Theorem 4 in [8]: given $f : G^n \to \mathrm{Ind}_{1\uparrow G} A$ a continuous n-cocycle, we define a continuous (n - 1)-cochain with values in Ind A by

$$h(g_1, \ldots, g_{n-1})(x) = f(g_1, \ldots, g_{n-1}, g_{n-1}^{-1} \cdots g_j^{-1} \cdots g_1^{-1} x)(x),$$

and a direct calculation with the cocycle identity shows

$$(\delta h)(g_1,\ldots,g_n) = \pm f(g_1,\ldots,g_n). \qquad \blacksquare$$

We specialize now to the case $H = \mathbf{Z}$, $G = \mathbf{R}$, A a Polish abelian group with trivial H-action, so that $\mathrm{Ind}_{\mathbf{Z}\uparrow\mathbf{R}}A$ is just $C(\mathbf{R}/\mathbf{Z},A)$. Now if A has property F and we have a short exct sequence

$$0 \to A \to B \to C \to 0$$

of Polish groups, this is a Serre fibration and we obtain the long exact homotopy sequence

$$\ldots \to \pi_1(B) \to \pi_1(C) \to \pi_0(A) \to \pi_0(B) \to \pi_0(C) \to 0.$$

This means that we should regard $\pi_0(A) = A/A_0$ as $R^1\mathrm{Ind}_{\mathbf{Z}\uparrow\mathbf{R}}A$, since this is what we must add on the right to continue the exact sequence

$$0 \to \mathrm{Ind}_{\mathbf{Z}\uparrow\mathbf{R}}A \to \mathrm{Ind}_{\mathbf{Z}\uparrow\mathbf{R}}B \to \mathrm{Ind}_{\mathbf{Z}\uparrow\mathbf{R}}C.$$

In other words, we should expect a degenerate spectral sequence

$$H^p(\mathbf{R},R^q\mathrm{Ind}_{\mathbf{Z}\uparrow\mathbf{R}}A) \Longrightarrow H^*(\mathbf{Z},A),$$

with $E_2^{p,0} = H^p(\mathbf{R},\mathrm{Ind}_{\mathbf{Z}\uparrow\mathbf{R}}A)$, $E_2^{p,1} = H^p(\mathbf{R},A/A_0)$, and $E_2^{p,q} = 0$ for $q > 1$. We proceed to establish this by mimicing the usual process of resolving A, applying $\mathrm{Ind}_{\mathbf{Z}\uparrow\mathbf{R}}$, and resolving again to get a double complex. Actually carrying out the process and indentifying explicitly the modules obtained leads to the following lemma. We make the technical hypothesis that the path component of 0 in A should be closed, since it is not clear if this is always the case in a Polish group.

LEMMA 3.4. *Let* A *be a Polish abelian group, viewed as a* **Z***-module with trivial action. Assume the path component* A_0 *of* 0 *in* A *is closed, hence Polish in the relative topology. Then the following sequence of Polish* **R***-modules is exact (topologically)*

$$0 \to \mathrm{Ind}_{\mathbf{Z}\uparrow\mathbf{R}}A \to \mathrm{Ind}_{1\uparrow\mathbf{R}}A \xrightarrow{\beta} \mathrm{Ind}_{1\uparrow\mathbf{R}}A_0 \to 0,$$

where β *is given by* $(\beta f)(s) = f(s+1) - f(s)$, *for* $f \in C(\mathbf{R},A)$.

PROOF. Identifying periodic functions $\mathbf{R} \to A$ with a subset of the continuous functions gives an obvious embedding of $\mathrm{Ind}_{\mathbf{Z}\uparrow\mathbf{R}}A$ as a closed submodule of $\mathrm{Ind}_{1\uparrow\mathbf{R}}A$. Furthermore, the indicated formula for β gives an R-module map from $\mathrm{Ind}_{1\uparrow\mathbf{R}}A$ to itself whose kernel consists precisely of functions $f : \mathbf{R} \to A$ satisfying $f(s+1) = f(s)$ for all s, i.e., $f \in C(\mathbf{R}/\mathbf{Z}, A) = \mathrm{Ind}_{\mathbf{Z}\uparrow\mathbf{R}}A$. Thus it's enough to show that the image of β is precisely $\mathrm{Ind}_{1\uparrow\mathbf{R}}A_0$.

First of all note that if $f : \mathbf{R} \to A$ is continuous, its image must lie in a single path-component of A; hence the difference of two values of f lies in A_0. Thus β maps *into* $\mathrm{Ind}_{1\uparrow\mathbf{R}}A_0$. To show β maps *onto* this module, choose any continuous $\phi : \mathbf{R} \to A_0$. We shall construct a continuous $f : \mathbf{R} \to A$ with $\phi = \beta(f)$. To do this, we begin by setting $f(0) = 0$, $f(1) = \phi(0)$. Since $\phi(0) \in A_0$, there exists a continuous path from $f(0)$ to $f(1)$; choose one and let it define $f(s)$ for $0 \le s \le 1$. It's then easy to see that the functional equation

$$f(s+1) - f(s) = \phi(s)$$

has a unique continuous solution defined for all real s and extending f as already defined on $[0,1]$. This shows β maps onto $\mathrm{Ind}_{1\uparrow\mathbf{R}}A_0$, and of course β is continuous. Since all groups involved are Polish, the open mapping theorem says β is open, i.e., the exact sequence is topological. ∎

THEOREM 3.5. *Let* A *be a Polish abelian group viewed as a* **Z***-module with trivial action, and let* A_0 *be the path-component of the identity in* A. *There are natural isomorphisms*

$$H^n(\mathbf{R}, \mathrm{Ind}_{\mathbf{Z}\uparrow\mathbf{R}}A) \simeq \begin{cases} A, & \text{if } n = 0, \\ A_0, & \text{if } n = 1 \text{ and } A_0 \text{ is closed in } A, \\ 0, & \text{if } n > 1 \text{ and } A_0 \text{ is open in } A\ . \end{cases}$$

PROOF. Consider the long exact cohomology sequence coming from the short exact sequence of Lemma 3.4. Recall that $H^1(\mathbf{R}, \mathrm{Ind}_{1\uparrow\mathbf{R}}A) = H^1_{cont}(\mathbf{R}, \mathrm{Ind}_{1\uparrow\mathbf{R}}A) = 0$ (by Proposition 3.3). Thus we have the exact sequence

$$0 \to H^0(\mathbf{R}, \mathrm{Ind}_{\mathbf{Z}\uparrow\mathbf{R}}A) \to A \xrightarrow{\beta_*} A_0 \to H^1(\mathbf{R}, \mathrm{Ind}_{\mathbf{Z}\uparrow\mathbf{R}}A) \to 0.$$

It is clear that $H^0(\mathbf{R}, \mathrm{Ind}_{\mathbf{Z}\uparrow\mathbf{R}}A) \to A$ is an isomorphism and that $\beta_* = 0$, which gives us the result $H^1(\mathbf{R}, \mathrm{Ind}_{\mathbf{Z}\uparrow\mathbf{R}}A) \simeq A_0$ (when A_0 is closed in A).

When also A_0 is open in A, then A and A_0 are locally path-connected and we may apply Proposition 3.3 to conclude that all higher cohomology groups of $\mathrm{Ind}_{1\uparrow\mathbf{R}}A$ and of $\mathrm{Ind}_{1\uparrow\mathbf{R}}A_0$ vanish. Plugging this back into the exact cohomology sequence coming from Lemma 3.4, we conclude that $H^n(\mathbf{R}, \mathrm{Ind}_{\mathbf{Z}\uparrow\mathbf{R}}A) = 0$ for $n \ge 2$. ∎

COROLLARY 3.6. *Let* A *be a locally path-connected Polish abelian group, with* A_0 *the path-component of the identity in* A. *Then there are natural isomorphisms*

$$H^n(\mathbf{T},\mathrm{Ind}_{1\uparrow\mathbf{T}}A) \simeq \begin{cases} A & \text{if } n = 0, \\ A/A_0 & \text{if } n = 2,4,6,\ldots, \\ 0 & \text{if } n \text{ odd.} \end{cases}$$

PROOF. If we view $\mathrm{Ind}_{1\uparrow\mathbf{T}}A$ as the R-module $\mathrm{Ind}_{\mathbf{Z}\uparrow\mathbf{R}}A$, then $\mathbf{Z}$ acts trivially and so

$$H^q(\mathbf{Z},\mathrm{Ind}_{\mathbf{Z}\uparrow\mathbf{R}}A) \simeq \begin{cases} \mathrm{Ind}_{1\uparrow\mathbf{T}}A & \text{if } q = 0 \text{ or } 1, \\ 0 & \text{if } q > 1. \end{cases}$$

Since these groups are always Hausdorff, we may apply the spectral sequence of [8], Theorem 9, p.29. We obtain a spectral sequence with E_2-terms

$$E_2^{p,q} = \begin{cases} H^p(\mathbf{T},\mathrm{Ind}_{1\uparrow\mathbf{T}}A) & \text{if } q = 0 \text{ or } 1, \\ 0 & \text{if } q > 1 \end{cases}$$

converging to $H^{p+q}(\mathbf{R},\mathrm{Ind}_{\mathbf{Z}\uparrow\mathbf{R}}A)$. The only differential which is possibly nonzero is

$$d_2 : E_2^{p,1} \to E_2^{p+2,0}.$$

But by Theorem 3.5, we must have $E_\infty^{0,0} = A$, $E_\infty^{p,q} = 0$ for $p > 1$ or $p = 1$, $q = 1$. Finally, we must have an exact sequence

$$0 \to E_\infty^{1,0} \to A_0 \to E_\infty^{0,1} \to 0.$$

But

$$H^1(\mathbf{T},\mathrm{Ind}_{1\uparrow\mathbf{T}}A) \simeq H^1_{\mathrm{cont}}(\mathbf{T},\mathrm{Ind}_{1\uparrow\mathbf{T}}A) \simeq 0,$$

so $E_\infty^{1,0} = 0$ and $E_\infty^{0,1} \simeq A_0$. This gives the exact sequence

$$0 \to A_0 \to E_2^{0,1} \simeq A \xrightarrow{d_2} E_2^{2,0} \to 0,$$

so $E_2^{2,0} = H^2(\mathbf{T},\mathrm{Ind}_{1\uparrow\mathbf{T}}A) \simeq A/A_0$. Since

$$d_2 : E_2^{2,1} \to E_2^{4,0}$$

must be an isomorphism to give $E_\infty^{2,1} = 0$ and $E_\infty^{4,0} = 0$, we have

$$E_2^{4,0} = H^4(\mathbf{T},\mathrm{Ind}_{1\uparrow\mathbf{T}}A) \simeq E_2^{2,1} \simeq A/A_0.$$

Continuing this way by induction we obtain the result. ■

REMARK 3.7. Recall that if

$$0 \to A \to B \to C \to 0$$

is a Serre fibration, then the sequence

$$\pi_1(A) \to \pi_1(B) \to \pi_1(C) \to \pi_0(A) \to \pi_0(B) \to \pi_0(C) \to 0$$

is exact. This suggests that for A locally path connected, one should have

$$\begin{cases} R^1 \mathrm{Ind}_{1\uparrow \mathbf{T}} A = A/A_0 \,(= \pi_0(A)), \\ R^q \mathrm{Ind}_{1\uparrow \mathbf{T}} A = 0 \quad \text{for } q > 1. \end{cases}$$

Then one should have the "composition of functors" spectral sequence

$$H^p(\mathbf{T}, R^q \mathrm{Ind}_{1\uparrow \mathbf{T}} A) \Longrightarrow H^{p+q}(1, A),$$

and once again one could obtain the same result as before for $E_2^{p,0} = H^p(\mathbf{T}, \mathrm{Ind}_{1\uparrow \mathbf{T}} A)$, knowing that it must cancel against

$$E_2^{p,1} = H^p(\mathbf{T}, A/A_0).$$

However, A/A_0 is discrete and carries trivial $\mathbf{T}$-action, so by [17], Theorem 4,

$$H^p(\mathbf{T}, A/A_0) \cong H^p_{top}(B\mathbf{T}, A/A_0),$$

which gives A/A_0 for $p = 2, 4, \ldots$ by the universal coefficient theorem. ($H^*_{top}(B\mathbf{T}, \mathbf{Z})$ is a polynomial ring on a single generator in degree 2.) ■

REMARK 3.8. One can also compute $H^1(\mathbf{R}, \mathrm{Ind}_{\mathbf{Z}\uparrow\mathbf{R}} A)$ directly from the definition, using the fact ([8], Theorem 3) that any Borel 1-cocycle is automatically continuous. An element of $Z^1(\mathbf{R}, \mathrm{Ind}_{\mathbf{Z}\uparrow\mathbf{R}} A)$ may be viewed as a function $\phi : \mathbf{R} \times \mathbf{R}/\mathbf{Z} \to A$ which is (jointly) continuous and satisfies the cocycle identity $\phi(s + t, \dot{r}) = \phi(s, \dot{r}) + \phi(t, \dot{r} - \dot{s})$. From this it follows that $\phi(0, \dot{r}) = 0$ for all $\dot{r} \in \mathbf{R}/\mathbf{Z}$ and that $\phi(n, \dot{r})$ is independent of $\dot{r} \in \mathbf{R}/\mathbf{Z}$ for fixed $n \in \mathbf{Z}$. The isomorphism $H^1(\mathbf{R}, \mathrm{Ind}_{\mathbf{Z}\uparrow\mathbf{R}} A) \to A_0$ of Theorem 3.5 is then given by

$$[\phi] \mapsto \phi(1, \dot{r}).$$

This is independent of $\dot{r}$ and lies in A_0 since it is path-connected to $\phi(0, \dot{r}) = 0$. ■

Using Proposition 3.3 and Theorem 3.5 in the case where A is of the form $C(Y, \mathbf{T})$ (where $H^1(Y, \mathbf{Z})$ should be countable so that A is locally path-connected) it is easy to deduce the vanishing of $H^n(\mathbf{R},C(X,\mathbf{T}))$, $n \geq 2$, for suitable $\mathbf{R}$-spaces X on which the action of $\mathbf{R}$ is proper. In [13], Theorem 4.1, we gave a stronger vanishing theorem that allows arbitrary $\mathbf{R}$-action on X. We conclude the present section with an analogous vanishing theorem for actions of compact semisimple Lie groups (e.g., SU(N)).

THEOREM 3.9. *Let G be a connected, simply connected compact Lie group, and let X be any second-countable locally compact G-space with $H^0(X,\mathbf{Z})$ and $H^1(X,\mathbf{Z})$ countable (it suffices for X to be compact). Give $C(X,\mathbf{T})$ the topology of uniform convergence on compacta and the action of G coming from the G-action on X. Then the cohomology groups $H^n(G,C(X,\mathbf{T}))$ are countable for $n \geq 1$ and vanish for $n = 1,2$.*

PROOF. As in the proof of [13], Theorem 4.1, we use the short exact sequences of G-modules

$$1 \to C(X,\mathbf{T})_0 \to C(X,\mathbf{T}) \to H^1(X,\mathbf{Z}) \to 1$$

and

$$1 \to H^0(X,\mathbf{Z}) \to C(X,\mathbf{R}) \to C(X,\mathbf{T})_0 \to 1.$$

These are topological short exact sequences of Polish groups by the countability assumptions on $H^0(X,\mathbf{Z})$ and $H^1(X,\mathbf{Z})$, together with Proposition 6 of [9]. Now since $C(X,\mathbf{R})$ is a topological vector space and G is compact, $H^n(G,C(X,\mathbf{R})) = 0$ for $n \geq 1$ by a generalization of the "averaging argument" of [7], p.60, or else by reduction to continuous cochains using [17], Theorem 3, followed by [4], Corollaire III.2.1. On the other hand, since $H^0(X,\mathbf{Z})$ and $H^1(X,\mathbf{Z})$ are discrete and G is connected, the G-action on them is trivial. But for A a countable group with trivial G-action, Theorem 4 of [17] gives $H^n(G,A) \simeq H^n(BG,A)$, which is countable for all n. By the long exact cohomology sequences, we now get an exact sequence

$$H^{n+1}(BG,H^0(X,\mathbf{Z})) \to H^n(G,C(X,\mathbf{T})) \to H^n(BG,H^1(X,\mathbf{Z}))$$

for any $n \geq 1$. This proves the countability. Furthermore, the assumptions on G guarantee that G is 2-connected ([16], p.198), hence BG is 3-connected and we deduce vanishing of $H^n(G,C(X,\mathbf{T}))$ for $n = 1$ or 2. ■

COROLLARY 3.10. *If G and X are as in Theorem 3.9 and α and β are actions of G on a separable continuous-trace algebra A, such that $\hat{A} = X$ and α and β induce the same action of G on X, then α and β are exterior equivalent.*

PROOF. This follows from Theorem 3.9 together with Remark 1.2. ■

COROLLARY 3.11. *If* G *and* X *are as in Theorem* 3.9 *and* A *is a separable* C^*-*-algebra with* $\hat{A} = X$ *and with all derivations inner, then any two norm-close actions of* G *on* A *are exterior equivalent.*

PROOF. As we noted in § 1, this follows from Theorem 3.9 together with the analysis in [5]. ■

REMARK 3.12. In fact, compactness of G was only used at one point in the proof of Theorem 3.9. Since any connected, simply connected Lie group is 2-connected even if non-compact, the same argument together with Theorem 3 of [17] and Corollaire III.7.5 of [4] (a form of Van Est's Theorem) shows that for any such G (and X as above)

$$H^2(G,C(X,\mathbf{T})) \cong H^2(G,C(X,\mathbf{T})_0) \cong H^2(\underline{g},\underline{k};C(X,\mathbf{R})_\infty),$$

the relative Lie algebra cohomology for the Lie algebra of G relative to the Lie algebra of a maximal compact subgroup, with coefficients in the Fréchet space of continuous real-valued functions on X which are smooth along orbits of G. There are some cases in which this will be computable. For instance, if X = G/H is a homogeneous space of G, we get

$$H^2(G,C(G/H,\mathbf{T})) \cong H^2(\underline{g},\underline{k};C^\infty(G/H)),$$

which may be computed by the methods of [1]. For instance, if H is a lattice subgroup and G is semisimple of real rank ≥ 3, the cohomology must vanish by [1], Theorem V.3.3. This has implications such as those of Corollaries 3.10 and 3.11 for G-actions on C^*-algebras with spectrum G/H (as a G-space). Of course, vanishing of the cohomology when G acts trivially on X was already proved in [5], Theorem 2.6. ■

4. BUNDLES AND PULL-BACKS

This section is based on ideas of [13] and a suggestion of Iain Raeburn, for which I am grateful. It concerns the situation of a principal G-bundle $p : \Omega \to X$, where G is a suitable group (the most interesting case being $G = \mathbf{T}$), and a comparative analysis of $H^*(G,C(\Omega,\mathbf{T}))$, where G acts via the free G-action on Ω, and of $H^*(G,C(X,\mathbf{T}))$, with G acting trivially on $C(X,\mathbf{T})$. Then we apply this to study certain group actions on C^*-algebras. Though it would probably be possible to work with a somewhat larger class of groups (using some of the techniques discussed elsewhere in this paper), we limit the discussion to tori and "solenoids". The finite-dimensionality hypothesis on G is only needed so that we can apply Theorem 4 of [17], and is probably unnecessary.

THEOREM 4.1. *Let G be a compact finite-dimensional connected metrizable abelian group, and let X be a locally compact second-countable space with $H^0(X,\mathbf{Z})$ and $H^1(X,\mathbf{Z})$ countable (for instance, a compact metrizable space). Let $p : \Omega \to X$ be a principal G-bundle, and let G act trivially on $C(X,T)$ and via the action on Ω on $C(\Omega,T)$. Then $H^0(G,C(X,T)) \simeq H^0(G,C(\Omega,T)) \simeq C(X,T)$, and for all $n > 0$, the groups $H^n(G,C(X,T))$ and $H^n(G,C(\Omega,T))$ are countable. In particular, $H^1(G,C(X,T)) \simeq H^0(X,\hat{G})$ maps via p^* onto*

$$H^1(G,C(\Omega,T)) \simeq H^0(X,\hat{G})/\text{image of } H^1(\Omega,\mathbf{Z}),$$

and

$$H^1(X,\hat{G}) \simeq H^2(G,C(X,T)) \hookrightarrow H^2(G,C(\Omega,T)) \text{ via } p^* .$$

PROOF. Note that the assumption on G means $\hat{G}$ is a countable torsion-free abelian group of finite rank. Since G is an inverse limit of tori and Čech cohomology commutes with inverse limits, $H^2(G,\mathbf{Z}) \simeq \hat{G} \simeq H^2(BG,\mathbf{Z})$. In fact, since the cohomology ring of BT is a polynomial ring on a 2-dimensional generator, $H^*(BG,\mathbf{Z})$ is all concentrated in even degrees and is isomorphic to the symmetric algebra on $\hat{G}$. Note, too, that $H^0(\Omega,\mathbf{Z}) \simeq H^0(X,\mathbf{Z})$ and that $H^1(\Omega,\mathbf{Z})$ is related to $H^1(X,\mathbf{Z})$ by the following exact Gysin sequence (the sequence of edge terms of the Leray-Serre spectral sequence for p):

$$0 \to H^1(X,\mathbf{Z}) \xrightarrow{p^*} H^1(\Omega,\mathbf{Z}) \xrightarrow{p!} H^0(X,\hat{G}) \to H^2(X,\mathbf{Z}).$$

Now consider the following commutative diagrams of short exact sequences:

$$(A)\quad \left|\begin{array}{ccccccccc} 0 & \to & C(X,T)_0 & \to & C(X,T) & \to & H^1(X,T) & \to & 0, \\ & & \downarrow p^* & & \downarrow p^* & & \downarrow p^* & & \\ 0 & \to & C(\Omega,T)_0 & \to & C(\Omega,T) & \to & H^1(\Omega,T) & \to & 0, \end{array}\right.$$

$$(B)\quad \left|\begin{array}{ccccccccc} 0 & \to & H^0(X,\mathbf{Z}) & \to & C(X,R) & \to & C(X,T)_0 & \to & 0 \\ & & \simeq\downarrow p^* & & \downarrow p^* & & \downarrow p^* & & \\ 0 & \to & H^0(\Omega,\mathbf{Z}) & \to & C(\Omega,R) & \to & C(\Omega,T)_0 & \to & 0. \end{array}\right.$$

As all maps are equivariant for the action of G and all groups are Polish by the countability assumptions, we can consider the associated diagrams of long exact sequences in G-cohomology. As in the proof of Theorem 3.9, all higher cohomology vanishes for $C(X,R)$ and $C(\Omega,R)$. Furthermore, since $H^0(X,\mathbf{Z})$, $H^1(X,\mathbf{Z})$ and $H^1(\Omega,\mathbf{Z})$ are all trivial discrete G-modules, we may use the fact that by [17], Theorem 4, $H^*(G,M) \simeq \simeq H^*(BG,M) \simeq H^*(BG,\mathbf{Z})\otimes M$ for such a module. (The last equality follows from the universal coefficient theorem.) Of course the calculation of H^0 is obvious. From (B) we obtain the commutative diagram

$$\begin{array}{ccc} H^n(G,C(X,\mathbf{T})_0) & \xrightarrow{\simeq} & H^{n+1}(BG,\mathbf{Z})\otimes H^0(X,\mathbf{Z}) \\ \simeq\downarrow p^* & & \simeq\downarrow p^* \\ H^n(G,C(\Omega,\mathbf{T})_0) & \xrightarrow{\simeq} & H^{n+1}(BG,\mathbf{Z})\otimes H^0(X,\mathbf{Z}) \end{array}$$

for any $n \geq 1$; note in particular that $H^n(G,C(\Omega,\mathbf{T})_0) = 0$ for n even. Then from (A) we obtain commutative diagrams of exact sequences

$$\begin{array}{ccccccccccc} 0 \to & C(X,\mathbf{T})_0 & \to & C(X,\mathbf{T}) & \to & H^1(X,\mathbf{Z}) & \xrightarrow{0} & H^1(G,C(X,\mathbf{T})_0) & \to & H^1(G,C(X,\mathbf{T})) & \to 0 \\ & \downarrow\simeq & & \downarrow\simeq & & \downarrow p^* & & \simeq\downarrow p^* & & \downarrow p^* & \\ 0 \to & C(X,\mathbf{T})_0 & \to & C(X,\mathbf{T}) & \to & H^1(\Omega,\mathbf{Z}) & \to & H^1(G,C(\Omega,\mathbf{T})_0) & \to & H^1(G,C(\Omega,\mathbf{T})) & \to 0 \end{array}$$

and (for $n \geq 1$)

$$\begin{array}{ccccccccc} 0 \to & H^{2n}(G,C(X,\mathbf{T})) & \to & H^{2n}(BG,\mathbf{Z})\otimes H^1(X,\mathbf{Z}) & \to & H^{2n+1}(G,C(X,\mathbf{T})_0) & \to & H^{2n+1}(G,C(X,\mathbf{T})) & \to 0 \\ & \downarrow p^* & & \downarrow \mathrm{id}\otimes p^* & & \simeq\downarrow p^* & & \downarrow p^* & \\ 0 \to & H^{2n}(G,C(\Omega,\mathbf{T})) & \to & H^{2n}(BG,\mathbf{Z})\otimes H^1(\Omega,\mathbf{Z}) & \to & H^{2n+1}(G,C(\Omega,\mathbf{T})_0) & \to & H^{2n+1}(G,C(\Omega,\mathbf{T})) & \to 0 \end{array}$$

from which we can read off almost all we want. In particular,

$$p^* : H^{2n}(G,C(X,\mathbf{T})) \to H^{2n}(G,C(\Omega,\mathbf{T}))$$

is injective and

$$p^* : H^{2n+1}(G,C(X,\mathbf{T})) \to H^{2n+1}(G,C(\Omega,\mathbf{T}))$$

is surjective. Though this would suffice for most of our purposes, we would like to know that $p^* : H^2(G,C(X,\mathbf{T})) \to H^2(G,C(\Omega,\mathbf{T}))$ is actually an isomorphism, when $G = \mathbf{T}$, and then say something about p^* for general G.

When $G = \mathbf{T}$, first note that the exact sequences up to H^1 reduce to

$$\begin{array}{ccccccccc} & & & & H^0(X,\mathbf{Z}) & \xrightarrow{\simeq} & H^1(G,C(X,\mathbf{T})) & & \\ & & & & \downarrow{\scriptstyle\simeq} & & \downarrow & & \\ 0 \to H^1(X,\mathbf{Z}) & \xrightarrow{p^*} & H^1(\Omega,\mathbf{Z}) & \to & H^0(X,\mathbf{Z}) & \to & H^1(G,C(\Omega,\mathbf{T})) & \to & 0. \end{array}$$

We would like to identify the map $H^1(\Omega,\mathbf{Z}) \to H^0(X,\mathbf{Z})$ as the Gysin map $p_!$. This is not hard to do, modulo a sign which depends on a choice of orientation conventions, since the map must be natural for all $\mathbf{T}$-bundles and have the correct kernel, hence must be of the form $p_!$ followed by multiplication by an integer. To check that the integer is $(\pm)1$, it's enough to know that when p is a trivial bundle (so $p_!$ is surjective), $H^1(G,C(\Omega,\mathbf{T})) = 0$. But in this case, $C(\Omega,\mathbf{T}) \simeq \mathrm{Ind}_{1\uparrow\mathbf{T}}C(X,\mathbf{T})$, so the vanishing of H^1 follows from Corollary 3.6. In fact, 3.6 tells us that in this case, $H^{2n}(G,C(\Omega,\mathbf{T})) \simeq H^1(X,\mathbf{Z})$ and $H^{2n+1}(G,C(\Omega,\mathbf{T})) = 0$ for $n \geq 1$.

Now (going back to the case of general p, but still with $G = \mathbf{T}$), once we've computed $H^1(G,C(\Omega,\mathbf{T}))$, everything else follows by "periodicity". For we have (by [8], Theorem 9) spectral sequences

$$\begin{array}{ccc} H^p(\mathbf{T},H^q(\mathbf{Z},C(X,\mathbf{T}))) & \Longrightarrow & H^{p+q}(\mathbf{R},C(X,\mathbf{T})) \\ \downarrow{\scriptstyle p^*} & & \downarrow{\scriptstyle p^*} \\ H^p(\mathbf{T},H^q(\mathbf{Z},C(\Omega,\mathbf{T}))) & \Longrightarrow & H^{p+q}(\mathbf{R},C(\Omega,\mathbf{T})), \end{array}$$

and since the $\mathbf{R}$-cohomology vanishes for $p + q \geq 2$ (by [13], Theorem 4.1), we must have a commutative diagram of periodicity isomorphisms

$$\begin{array}{ccc} H^p(\mathbf{T},C(X,\mathbf{T})) & \xrightarrow{\simeq} & H^{p+2}(\mathbf{T},C(X,\mathbf{T})) \\ \downarrow{\scriptstyle p^*} & & \downarrow{\scriptstyle p^*} \\ H^p(\mathbf{T},C(\Omega,\mathbf{T})) & \xrightarrow{\simeq} & H^{p+2}(\mathbf{T},C(\Omega,\mathbf{T})) \end{array}$$

for $p \geq 1$. This means the maps

$$H^{2n}(BG,\mathbf{Z})\otimes H^1(\Omega,\mathbf{Z}) \to H^{2n+1}(G,C(\Omega,\mathbf{T})_0) \to H^{2n+1}(G,C(\Omega,\mathbf{T}))$$

must reduce to

$$H^1(\Omega,\mathbf{Z}) \xrightarrow{p_!} H^0(X,\mathbf{Z}) \to H^{2n+1}(G,C(\Omega,\mathbf{T})) \to 0$$

for all $n \geq 1$, and since the kernel of $p_!$ is $p^*(H^1(X,\mathbf{Z}))$,

$$H^1(X,\mathbf{Z}) \simeq H^{2n}(G,C(X,\mathbf{T})) \xrightarrow{p^*} H^{2n}(G,C(\Omega,\mathbf{T}))$$

is in fact an isomorphism, while $H^{2n+1}(G,C(\Omega,\mathbf{T})) \simeq \mathrm{coker}(p_!)$.

The case of general G is similar, but for simplicity we concentrate only on the calculation of H^1 and H^2. (In fact, noting that $H^2(BG,\mathbf{Z}) \simeq \hat{G}$ and $H^3(G,C(\Omega,\mathbf{T})_0) \simeq$ $\simeq H^4(BG,\mathbf{Z})\otimes H^0(X,\mathbf{Z}) \simeq S^2(\hat{G})\otimes H^0(X,\mathbf{Z})$, one may identify the map

$$H^2(BG,\mathbf{Z})\otimes H^1(\Omega,\mathbf{Z}) \simeq \hat{G}\otimes H^1(\Omega,\mathbf{Z}) \to H^{2+1}(G,C(\Omega,\mathbf{T})_0) \simeq S^2(\hat{G})\otimes H^0(X,\mathbf{Z})$$

with the Gysin map

$$p_! : H^1(\Omega,\hat{G}) \to H^0(X,\hat{G}\otimes\hat{G}),$$

which has kernel $p^*(H^1(X,\hat{G}))$, followed by the map induced by the projection $\hat{G}\otimes\hat{G} \to$ $\to S^2(\hat{G})$.) By Theorem 2.5, $H^2(G,C(X,\mathbf{T})) \simeq H^1(X,\hat{G})$, and $H^1(G,C(X,\mathbf{T})) \simeq H^2(BG,\mathbf{Z})\otimes$ $\otimes H^0(X,\mathbf{Z}) \simeq H^0(X,\hat{G})$ by our exact sequences. The first step in computing $H^n(G,C(\Omega,\mathbf{T}))$ $(n = 1,2)$ is to compute them when $p : \Omega \to X$ is a trivial bundle. For this we need a substitute for Corollary 3.6. For any A and any G, $H^1(G,\mathrm{Ind}_{1\uparrow G}A) \simeq H^1_{cont}(G,\mathrm{Ind}_{1\uparrow G}A) = 0$ as in Proposition 3.3. Furthermore, if G is as in our theorem and A has property F, we may form $EA = U([0,1],A)$ as in [8], which is contractible by Lemma 2.3. Then $\mathrm{Ind}_{1\uparrow G}EA$ is also contractible, and by the argument of [17], p.91, $H^n(G,\mathrm{Ind}_{1\uparrow G}EA) = H^n_{cont}(G,\mathrm{Ind}_{1\uparrow G}EA) = 0$ for $n \geq 1$. As in the proof of Proposition 2.4 (c), one can see that

$$0 \to \mathrm{Ind}_{1\uparrow G}A \to \mathrm{Ind}_{1\uparrow G}EA \to \mathrm{Ind}_{1\uparrow G}BA \to [G,BA] \to 0$$

is exact (the non-trivial part being exactness on the right). Applying the long exact cohomology sequence together with the vanishing of $H^1(G,\mathrm{Ind}_{1\uparrow G}\text{--})$, we conclude that

$$H^2(G,\mathrm{Ind}_{1\uparrow G}A) \simeq H^1(G,(\mathrm{Ind}_{1\uparrow G}BA)_0) \simeq [G,BA].$$

Taking $A = C(X,\mathbf{T})$, this says $H^1(G,C(G\times X,\mathbf{T})) = 0$ and gives an explicit calculation of $H^2(G,C(G\times X,\mathbf{T}))$. In other words, when $p : \Omega \to X$ is trivial,

$p^* : H^1(G,C(X,\mathbf{T})) \to H^1(G,C(\Omega,\mathbf{T}))$ is zero

and

$p^* : H^2(G,C(X,\mathbf{T})) \to H^2(G,C(\Omega,\mathbf{T}))$ is an injection, but not always an isomorphism.

Then by naturality (in X) of the diagrams

$$\begin{array}{ccccccccc} & & & & H^0(X,\hat{G}) & \simeq & H^1(G,C(X,\mathbf{T})) & & \\ & & & & \downarrow \simeq & & \downarrow p^* & & \\ 0 \to & H^1(X,\mathbf{Z}) & \xrightarrow{p^*} & H^1(\Omega,\mathbf{Z}) \to & H^0(X,\hat{G}) & \to & H^1(G,C(\Omega,\mathbf{T})) & \to 0 & \end{array}$$

and

$$\begin{array}{ccccccc} & & H^2(G,C(X,\mathbf{T})) & \simeq & H^1(X,\hat{G}) & & \\ & & \downarrow p^* & & \downarrow p^* & & \\ 0 & \rightarrow & H^2(G,C(\Omega,\mathbf{T})) & \rightarrow & H^1(\Omega,\hat{G}) & \rightarrow & S^2(\hat{G})\otimes H^0(X,\mathbf{Z}) \end{array}$$

it is easy to see that in general,

$$H^1(G,C(\Omega,\mathbf{T})) \simeq \operatorname{coker}(p_! : H^1(\Omega,\mathbf{Z}) \rightarrow H^0(X,\hat{G}))$$

and

$$p^* : H^1(X,\hat{G}) \simeq H^2(G,C(X,\mathbf{T})) \hookrightarrow H^2(G,C(\Omega,\mathbf{T})). \qquad \blacksquare$$

Now we apply the theorem to the situation of [13], §1. Let $p : \Omega \rightarrow X$ be a principal G-bundle as in the theorem, and B a stable separable continuous-trace algebra with spectrum Ω. Then if $\beta : G \rightarrow \mathrm{Aut}(B)$ is an action of G on B inducing the G-action on Ω (free with quotient X), by Theorem 1.1 of [13], there is an isomorphism of B with p^*A, where A is a continuous-trace algebra with spectrum A and p^*A means the pull-back of A in the sense of [14], i.e., $C_0(\Omega)\otimes_{C_0(X)}A$. Furthermore, by Corollary 1.3 (loc.cit.), one may arrange this isomorphism so that β is exterior equivalent to $p^*\mathrm{id}$, the action coming from the tensor product of the translation action τ of G on $C_0(\Omega)$ and the trivial action of G on A. Using Theorem 4.1, we can obtain an interesting complement to the results of [14] and [13] concerning pull-back actions.

PROPOSITION 4.2. *Let* G *be a compact, connected, finite-dimensional, metrizable, abelian group, and let* $p : \Omega \rightarrow X$ *be a principal* G*-bundle with* $H^n(X,\mathbf{Z})$ *countable for* $n \leq 2$. *Let* A *be any separable continuous-trace algebra with spectrum* X, *and let* $B = p^*A$. *Then if* α_1 *and* α_2 *are two locally unitary actions of* G *on* A *(recall that by Theorem 2.5, this is automatic if* α_1 *and* α_2 *are any actions fixing* X *pointwise), then* $p^*\alpha_1$ *and* $p^*\alpha_2$ *are exterior equivalent as actions of* G *on* B *if and only if* α_1 *and* α_2 *are exterior equivalent as actions of* G *on* A.

PROOF. By Remark 1.2 and Theorem 2.5, the obstruction to exterior equivalence of α_1 and α_2 is the class $\zeta(\alpha_1) - \zeta(\alpha_2) \in H^2(G,C(X,\mathbf{T})) \simeq H^1(X,\hat{G})$, and the obstruction to exterior equivalence of $p^*\alpha_1$ and $p^*\alpha_2$ is a class in $H^2(G,C(\Omega,\mathbf{T}))$, which a simple calculation shows is just $p^*(\zeta(\alpha_1) - \zeta(\alpha_2))$. So the result follows from injectivity of $p^* : H^2(G,C(X,\mathbf{T})) \rightarrow H^2(G,C(\Omega,\mathbf{T}))$. $\blacksquare$

REMARK 4.3. One might think, since we showed that $p^* : H^2(G,C(X,\mathbf{T})) \to H^2(G,C(\Omega,\mathbf{T}))$ is often surjective, that every G-action β on p^*A, inducing the given G-action on Ω, is exterior equivalent to some $p^*\alpha, \alpha$ a locally unitary action on A. However, as was shown in Example 1.9 of [13], this is false in general. The reason is the following. Since G acts trivially on X, there is a natural map ζ that to any G-action trivial on X associates a class in $H^2(G,C(X,\mathbf{T}))$. However, there is no such map sending a G-action on p^*A inducing the given G-action on Ω to a class in $H^2(G,C(\Omega,\mathbf{T}))$. Instead, an obstruction in $H^2(G,C(\Omega,\mathbf{T}))$ is only defined given a pair of such actions, and the obstruction associated to a pair (α,γ) is not necessarily the sum of the obstructions for (α,β) and (β,γ), because of non-commutativity of the unitaries that appear in the relevant formulae. While one might be tempted to think that the map

$$\beta \mapsto (\text{obstruction to exterior equivalence of } \beta \text{ and } p^*\,\mathrm{id})$$

would have good additivity properties, it does not. In fact, the whole notion of the "basepoint" $p^*\mathrm{id}$ for the G-actions on p^*A inducing the G-action on Ω can be non-canonical, since it might be that $p^*A = p^*C$, and yet $p^*(\mathrm{id}_A)$ and $p^*(\mathrm{id}_C)$ are not exterior equivalent. (This is what happens in the example given in [13].) ∎

As we mentioned in §1, conjugacy together with exterior equivalence defines an equivalence relation on the G-actions on B which is weaker than exterior equivalence but still implies $\hat{G}$-equivariant isomorphism of the crossed products. We studied this phenomenon to some extent in Theorem 1.5 of [13], where we saw that (in the situation of Proposition 4.2) the following conditions are equivalent (assuming A is stable):

(a) $G \ltimes_{p^*\alpha} B$ is $\hat{G}$- and $C_0(X)$-equivariantly isomorphic to $G \ltimes_{p^*\mathrm{id}} B$;

(b) $\langle \zeta(\alpha),[p] \rangle = 0$, where the pairing is the cup product between $H^1(X,\hat{G})$ and $H^1(X,\underline{G})$, with values in $H^2(X,\underline{T}) \cong H^3(X,\mathbf{Z})$;

(c) $p^*\alpha$ is exterior equivalent to $\gamma(p^*\mathrm{id})\gamma^{-1}$ for some $\gamma \in \mathrm{Aut}_{C_0(\Omega)}B$.

This suggests a natural question: how unique is the γ in (c)? For instance, when can one take it to be the identity? The following answer (when $G = \mathbf{T}$) was conjectured by Iain Raeburn. With more work (which we leave to the reader), it could probably be adapted to the generality of Proposition 4.2.

THEOREM 4.4. *Let $p : \Omega \to X$ be a principal $\mathbf{T}$-bundle, with Ω and X second-countable, locally compact, and with the homotopy type of finite CW-complexes. Let A be a stable separable continuous-trace algebra with spectrum X, and let $\alpha: \mathbf{T} \to \mathrm{Aut}\,A$*

be an action which fixes X *pointwise, such that* $\zeta(\alpha)\cup[p] = 0$. *Then* $p^*\alpha$ *(on* $p^*A = B$*) is exterior equivalent to* $\gamma(p^*\mathrm{id})\gamma^{-1}$ *for* $\gamma \in \mathrm{Aut}_{C_0(\Omega)}B$, *and the class* $[\gamma] \in H^2(\Omega, \mathbf{Z}) = \mathrm{Aut}_{C_0(\Omega)}B/\mathrm{Inn}\,B$ *satisfies* $p_![\gamma] = \zeta(\alpha) \in H^1(X, \mathbf{Z})$. *Here* $p_!$ *is the Gysin map. The class* $[\gamma]$ *is uniquely determined modulo* $p^*(H^2(X, \mathbf{Z}))$.

PROOF. As we said above, it was shown in [13] that if $\zeta(\alpha)\cup[p] = 0$ in $H^3(X, \mathbf{Z})$, then $p^*\alpha$ is exterior equivalent to some $\gamma(p^*\mathrm{id})\gamma^{-1}$. Conversely, for every $\gamma \in \mathrm{Aut}_{C_0(\Omega)}B$, $\gamma(p^*\mathrm{id})\gamma^{-1}$ is an action of $\mathbf{T}$ on B inducing the standard action of $\mathbf{T}$ on Ω, hence there is a well-defined class $\phi(\gamma) \in H^2(\mathbf{T},C(\Omega,\mathbf{T})) \cong H^1(X, \mathbf{Z})$ which measures the obstruction to exterior equivalence between $\gamma(p^*\mathrm{id})\gamma^{-1}$ and $p^*\mathrm{id}$. Since exterior equivalence classes don't change under conjugation by inner automorphisms, $\phi(\gamma)$ only depends on $[\gamma] \in H^2(\Omega, \mathbf{Z})$. We shall show ϕ induces a homomorphism $H^2(\Omega, \mathbf{Z}) \to H^1(X, \mathbf{Z})$ with the same formal properties as $p_!$, then deduce that the two coincide.

First we show that ϕ is a homomorphism. If γ is inner, then $\gamma(p^*\mathrm{id})\gamma^{-1}$ and $p^*\mathrm{id}$ are exterior equivalent, so ϕ sends $0 \in H^2(\Omega, \mathbf{Z})$ to $0 \in H^1(X, \mathbf{Z})$. We must check that $\phi(\gamma_1\gamma_2) = \phi(\gamma_1) + \phi(\gamma_2)$. To see this, recall $\phi(\gamma)$ is the coboundary in $H^2(\mathbf{T},C(\Omega,\mathbf{T}))$ of the 1-cocycle

$$t \mapsto \gamma(p^*\mathrm{id})_t\gamma^{-1}(p^*\mathrm{id})_{t^{-1}} : \mathbf{T} \to U(M(B))/C(\Omega,\mathbf{T}).$$

In other words, $\phi(\gamma)$ is defined by choosing (in a Borel fashion) unitaries v_t implementing $\gamma(p^*\mathrm{id})_t\gamma^{-1}(p^*\mathrm{id})_{t^{-1}}$, then defining a 2-cocycle by

$$(t,s) \mapsto v_{ts}(v_t(p^*\mathrm{id})_t(v_s))^{-1}$$

(which takes values in $U(Z(M(A))) \cong C(\Omega,\mathbf{T})$ since v_{ts} and $v_t(p^*\mathrm{id})_t(v_s)$ implement the same automorphism). Now given γ_1 and γ_2, choose in this way $\{v_t\}$ corresponding to γ_1 and $\{w_t\}$ corresponding to γ_2, and let $w'_t = \gamma_1(w_t)$. Then

$$\gamma_1\gamma_2(p^*\mathrm{id})_t\gamma_2^{-1}\gamma_1^{-1}(p^*\mathrm{id})_{t^{-1}} = \gamma_1(\gamma_2(p^*\mathrm{id})_t\gamma_2^{-1}(p^*\mathrm{id})_{t^{-1}}(p^*\mathrm{id})_t)\gamma_1^{-1}(p^*\mathrm{id})_{t^{-1}} =$$

$$= \gamma_1(\mathrm{Ad}\,w_t)(p^*\mathrm{id})_t\gamma_1^{-1}(p^*\mathrm{id})_{t^{-1}} = \mathrm{Ad}\,\gamma_1(w_t)\mathrm{Ad}\,v_t = \mathrm{Ad}(w'_tv_t).$$

Thus the 2-cocycle defining $\phi(\gamma_1\gamma_2)$ is

$$(t,s) \mapsto w'_{ts}v_{ts}(w'_tv_t(p^*\mathrm{id})_t(w'_sv_s))^{-1} =$$

$$= w'_{ts}[v_{ts}(p^*\mathrm{id})_t(v_s)^{-1}v_t^{-1}v_t](p^*\mathrm{id})_t(w'_s)^{-1}v_t^{-1}(w'_t)^{-1} =$$

$$= [v_{ts}(p^* id)_t(v_s^{-1})v_t^{-1}]w'_{ts}v_t(p^* id)_t(w'_s)^{-1}v_t^{-1}(w'_t)^{-1} =$$

$$= [v_{ts}(p^* id)_t(v_s^{-1})v_t^{-1}]\gamma_1(w_{ts})\gamma_1(p^* id)_t\gamma_1^{-1}(\gamma_1(w_s))^{-1}(\gamma(w_t))^{-1} =$$

$$= [v_{ts}(p^* id)_t(v_s^{-1})v_t^{-1}]\gamma_1[w_{ts}(p^* id)_t(w_s)^{-1}w_t^{-1}].$$

Since γ_1 was assumed to act trivially on the center of M(B), we see $\phi(\gamma_1\gamma_2) = \phi(\gamma_1) + \phi(\gamma_2)$, and ϕ is a homomorphism : $H^2(\Omega, \mathbf{Z}) \to H^1(X, \mathbf{Z})$. By [13] and our previous remarks, ϕ also has the property that $\phi(\gamma) \cup [p] = 0$ for all γ. Furthermore, ϕ vanishes on the image of $p^* : H^2(X, \mathbf{Z}) \to H^2(\Omega, \mathbf{Z})$, since if γ is pulled back from an automorphism $\delta \in \mathrm{Aut}_{C_0(X)}A$, i.e., γ comes from the automorphism $id \otimes \delta$ of $C_0(\Omega) \otimes_{C_0(X)} A$, then obviously γ commutes with $p^* id = \tau \otimes id$. Thus ϕ has all the formal properties of the Gysin map $p_! : H^2(\Omega, \mathbf{Z}) \to H^1(X, \mathbf{Z})$.

To check that $\phi = p_!$, we should note that ϕ does not depend on the choice of the continuous-trace algebra A, or equivalently, of its Dixmier-Douday invariant in $H^3(X, \mathbf{Z})$. This can be seen as follows. Let $C = C_0(X, \mathcal{K})$ be the stable, stably commutative continuous-trace algebra over X. Then $A \cong A \otimes_{C_0(X)} C$ and $B \cong B \otimes_{C_0(\Omega)} p^* C$. Since every class in $H^2(\Omega, \mathbf{Z})$ arises from a locally inner automorphism of $p^* C$, we may assume our automorphism is of the form $id \otimes \delta$, $\delta \in \mathrm{Aut}_{C_0(\Omega)} p^* C$, with respect to this decomposition, and then clearly $\phi(\gamma) = \phi(\delta)$. Thus, without loss of generality, we may assume A and B are stably commutative.

To show that ϕ and $p_!$ coincide, at least when X has the homotopy type of a finite complex, observe that both maps are natural, in the sense that if $(p : \Omega \to X, x \in H^2(\Omega, \mathbf{Z}))$ is the pull-back of $(p' : \Omega' \to X', y \in H^2(\Omega', \mathbf{Z}))$ under some map of $\mathbf{T}$-bundles

$$\begin{array}{ccc} \Omega & \longrightarrow & \Omega' \\ {\scriptstyle p = f^* p'}\downarrow & & \downarrow{\scriptstyle p'} \\ X & \xrightarrow{\ f\ } & X' \end{array},$$

then $\phi(x) = f^*(\phi'(y))$ and $p_!(x) = f^*(p'_!(y))$. Thus it is enough to show that $\phi = p_!$ on "universal examples". But given $p : \Omega \to X$ and $x \in H^2(X, \mathbf{Z})$, the pair $(p_!(x) \in H^1(X, \mathbf{Z}), [p] \in H^2(X, \mathbf{Z}))$ is pulled back from a map

$$X \to K(\mathbf{Z}, 1) \times K(\mathbf{Z}, 2) \simeq S^1 \times \mathbf{CP}^\infty,$$

and since $p_!(x) \cup [p] = 0$, the map can be lifted to a map $f : X \longrightarrow Y$, where Y is the homotopy fiber of the map

$$K(\mathbf{Z},1) \times K(\mathbf{Z},2) \longrightarrow K(\mathbf{Z},3)$$

corresponding to the cup product. The composite

$$Y \xrightarrow{\text{fiber inclusion}} K(\mathbf{Z},1) \times K(\mathbf{Z},2) \xrightarrow{\text{2nd projection}} K(\mathbf{Z},2)$$

induces a principal **T**-bundle $\pi : W \longrightarrow Y$, and from the Serre spectral sequences for the fibrations

$$\begin{array}{ccc} Y \longrightarrow K(\mathbf{Z},1) \times K(\mathbf{Z},2) & \qquad & S^1 \longrightarrow W \\ \downarrow & & \downarrow \pi \\ K(\mathbf{Z},3) & & Y \end{array}$$

we see that $H^n(Y, \mathbf{Z}) = \mathbf{Z}$ for $n = 0, 1, 2$ and that $H^2(W, \mathbf{Z})$ is infinite cyclic, say with generator w. Furthermore,

$$\pi^* : H^2(Y, \mathbf{Z}) \longrightarrow H^2(W, \mathbf{Z})$$

is the zero map, and $\pi_!(w)$ is a generator z of $H^1(Y, \mathbf{Z})$.

Now by our construction, $(p : \Omega \longrightarrow X) = f^*(\pi : W \longrightarrow Y)$, and $p_!(x) = f^*(z) = f^*(\pi_!(w)) = p_!(f^*(w))$. Thus $x - f^*(w) \in \ker p_! = p^*(H^2(X, \mathbf{Z}))$. Suppose we could ignore the difficulty that Y isn't locally compact, so that our definition of ϕ doesn't quite make sense for the bundle $\pi : W \longrightarrow Y$. If we could make sense of $\phi(w)$, it would have to be $\pm z$ (since ϕ is supposed to surject onto the kernel of $\cup[\pi]$). Adjusting if necessary the choice of sign in the definition of the Gysin map, we can suppose $\phi(w) = z = \pi_!(w)$, and then since ϕ vanishes on the image of p^*,

$$\phi(x - f^*(w)) = 0,$$

hence

$$\phi(x) = \phi(f^*(w)) = f^*(\phi(w)) = f^*(z) = p_!(x).$$

But if X has the homotopy type of a finite complex, then by cellular approximation the map $f : X \longrightarrow Y$ can be chosen to factor through some finite skeleton Y_n of Y (which is compact metric, so that $C(Y_n, \mathcal{K})$ makes sense). For n large enough, $Y_n \hookrightarrow Y$ induces isomorphisms on cohomology through degree 3, and the above argument works with Y_n in place of Y. ■

REFERENCES

1. **Borel, A. ; Wallach, N. :** *Continuous cohomology, discrete subgroups, and representations of reductive groups*, Annals of Math. Studies, Vol.**94**, Princeton University Press, Princeton, 1980.

2. **Dixmier, J. ; Douady, A. :** Champs continus d'espaces hilbertiens et de C^*-algèbres, *Bull. Soc. Math. France* **91**(1963), 227-284.

3. **Dold, A. :** Partitions of unity in the theory of fibrations, *Ann. of Math.* **78**(1963), 223-225.

4. **Guichardet, A. :** *Cohomologie des groupes topologiques et des algèbres de Lie*, Textes Mathématiques, vol.**2**, CEDIC/Nathan, Paris, 1980.

5. **Herman, R.H. ; Rosenberg, J. :** Norm-close group actions on C^*-algebras, *J. Operator Theory* **6**(1981), 25-37.

6. **Michael, E. :** Convex structures and continuous selections, *Canad. J. Math.* **11** (1959), 556-575.

7. **Moore, C.C. :** Extensions and low dimensional cohomology theory of locally compact groups. I, *Trans. Amer. Math. Soc.* **113**(1964), 40-63.

8. **Moore, C.C. :** Group extensions and cohomology for locally compact groups. III, *Trans. Amer. Math. Soc.* **221**(1976), 1-33.

9. **Moore, C.C. :** Group extentions and cohomology for locally compact groups. IV, *Trans. Amer. Math. Soc.* **221**(1976), 35-58.

10. **Palais, R. :** On the existence of slices for actions of non-compact Lie groups, *Ann. of Math.* **73**(1961), 295-323.

11. **Pedersen, G.K. :** *C^*-algebras and their automorphism groups*, London Math. Soc. Monographs, vol.**14**, Academic Press, London, 1979.

12. **Phillips, J. ; Raeburn, I. :** Crossed products by locally unitary automorphism groups and principal bundles, *J. Operator Theory* **11**(1984), 215-241.

13. **Raeburn, I. ; Rosenberg, J. :** Crossed products of continuous-trace C^*-algebras by smooth actions, Preprint 05711-85, MSRI, Berkeley, 1985.

14. **Raeburn, I. ; Williams, D.P. :** Pull-backs of C^*-algebras and crossed products by certain diagonal actions, *Trans. Amer. Math. Soc.*, **287**(1985), 755-777.

15. **Vogan Jr., D.A. :** *Representations of real reductive groups*, Progress in Math., vol. **15**, Birkhäuser, Boston, 1981.

16. **Whitehead, G.W. :** *Elements of homotopy theory*, Graduate Texts in Math., vol. **61**, Springer, New York, 1978.

17. **Wigner, D. :** Algebraic cohomology of topological groups, *Trans. Amer. Math. Soc.* **178**(1973), 83-93.

J.Rosenberg
Department of Mathematics
University of Maryland
College Park, MD 20742
U.S.A.