The Dimer model and 2D fermions: combinatorics from spin TQFT's to Bosonization

UMD Math RIT Fall 2020

Sri Tata, Feb 27, 2020

Overview for this Talk

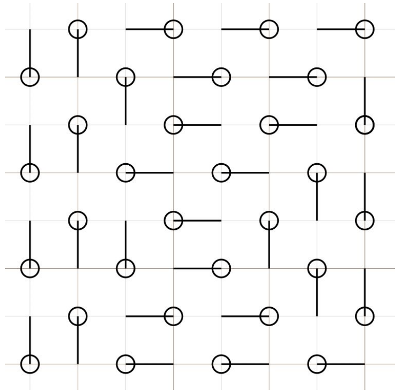
- In line with this reading group, we will discuss a TQFT for spin and pin$^-$ structures on 2D manifolds. The TQFT is known as the Arf-Brown-Kervaire (ABK), and gives values in $\mathbb{Z}_8 = \{\exp(2\pi i\, k/8) \mid k=0,...,7\}$
  - Physically, the ABK invariant is thought to be equivalent to the low-energy effective TQFT for the Kitaev chain protected by a time reversal symmetry $T^2 = 1$. (More on this later)
- But, we'll discuss it through the lens of a combinatorics problem, known as the 'dimer model', where it arises naturally. We'll get a neat perspective on spinors and fermions in (2+0)-D: a (discrete) Dirac operator, Grassmann integration, spin and pin$^-$ structures, ABK invariants, all have a natural role to play.
- We'll start by describing the dimer model, on the plane, then on more general surface, and build up more language as we progress.
  - We'll sprinkle in some words about SPT phases, and
- The second half of the talk will describe some other relationships of the dimer model to 2D fermions (and 2D bosons!)

PART 1:

(Double)-Dimers

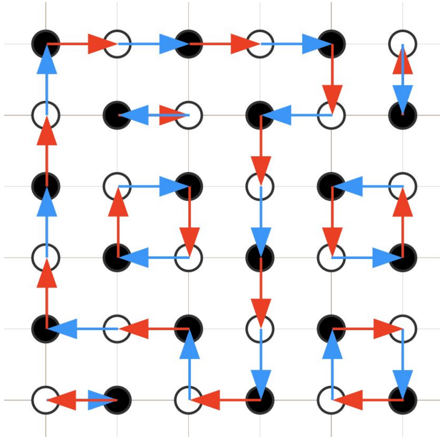
The (double-) Dimer model

- Consider some finite, simply connected subset U of $\mathbb{Z}^2$, considered as a graph. Our question is: “how many perfect matchings are there on U?” [Kasteleyn, Temperley, Fisher (at UMD!), ~1961]
  - A <u>perfect matching</u> (<u>dimer matching</u>) is a subset of the edges of the graph such that every vertex is part of exactly one edge.
  - Denote by $Z_d[U]$ = # of perfect matchings on U.



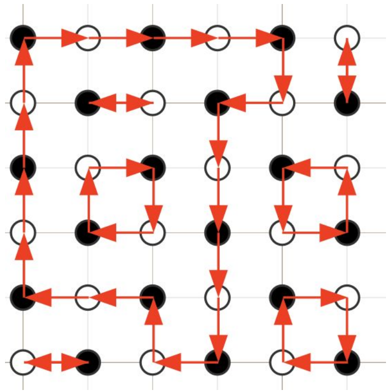
A perfect matching

- Note that U is bipartite.
  - Call the vertices {(x,y) | x+y is even} black, and {(x,y) | x+y is odd} white.
  - To any pair of matchings, we can superimpose them. For the first one, draw arrows going towards the black vertices (the blue matching on the right). For the other matching, draw an arrow pointing to the white vertex (the red on the right).
  - So, any *pair* of matches gives a <u>loop configuration</u> on the graph, a collection of directed edges such that every vertex has exactly one ingoing/outgoing edge.
  - Denote by $Z_{dd}[U]$ = # of loop configurations on U. (dd = double-dimer)
- We’ll have that $Z_{dd}[U] = (Z_d[U])^2$: since U is bipartite, there’s a bijection between loop configurations and pairs of perfect matchings.
  - We’ll focus on the double-dimer model on bipartite graphs, but the dimer story generalizes to more kinds of graphs. We’ll indicate when this is the case.



A pair of perfect matchings gives a loop configuration

Solving the Double-Dimer model



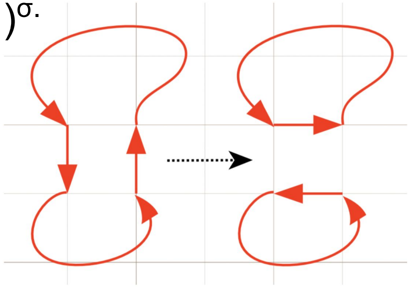
A collection of loops gives the cycle decomposition of a permutation on the set of vertices

- This directed loop configuration picture for $Z_{dd}[U]$ smells like a determinant.
  - For any matrix A:
    - $$\det(A) = \sum_\sigma (-1)^\sigma A_{1\sigma(1)} A_{2\sigma(2)} \dots A_{n\sigma(n)}$$ $$= \sum_\sigma (-1)^\sigma \prod_{\text{cycles } C \in \sigma} A_{C(1)C(2)} A_{C(2)C(3)} \dots A_{C(L_C)C(1)}$$
    - A permutation σ of {1,...,n} can be viewed as a collection of directed edges on the complete graph on n vertices, with loops giving the cycle decomposition of σ. The factor ∏ $(A_{C(1)C(2)} \dots A_{C(Lc)C(1)})$ gives a weight to each permutation.

- Suppose we choose A to be the adjacency matrix of the grid.
  - A(v,w) = 1 if v~w is an edge, A(v,w) = 0 otherwise.
  - Then $\prod(A_{C(1)C(2)} \dots A_{C(Lc)C(1)})$ = 1 if σ is a loop configuration of the graph, is zero otherwise.
- So, $\det(A) = \sum_{\text{loop configs}} (-1)^{\#\text{ of loops}}$, almost what we want! Except for that $(-1)^\sigma$ factor
  - Note that $(-1)^\sigma = (-1)^{\#\text{ of even-length loops}}$, and every loop here is even-length, since it's bipartite

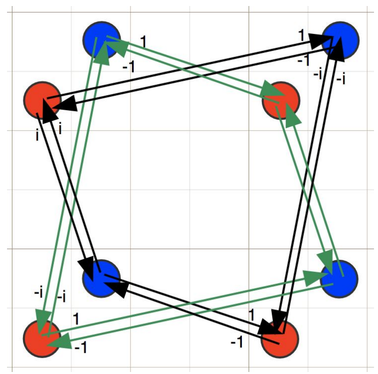
Solving the Double-Dimer model (cont.)

- How to fix the $(-1)^{\#\text{ of loops}}$? Consider “local flips”: flipping pairs of edges on opposite ends of a square.
  - This will change # of loops by exactly ±1.
  - It will leave $\prod(A_{C(1)C(2)}\ldots A_{C(Lc)C(1)})$ the same.
  - So, two loop configurations separated by one ‘flip move’ will have different $(-1)^{\sigma}$.
- <u>Lemma:</u> Any two loop configurations of a simply-connected squared grid can be connected by such flip moves. (‘flip connectivity’)
  - Proof: Fun exercise. (HINT: consider nested loops, induct on size of outermost loop in nesting: or c.f. [4] for more general statements)
- Given the lemma, we have a strategy to solve the model. We need to find a matrix K for whom the product of weights changes by -1 wrt a flip move, the ‘flip condition’.
  - Then, changes in $\prod(K_{C(1)C(2)}\ldots K_{C(Lc)C(1)})$ cancel out the changes in $(-1)^{\sigma}$.
  - So, the flip-connectivity lemma tells us that ***det K = ±Z_{dd}[U]***
- So, now we need to find such an operator K…
  - Turns out that a discrete version of the Dirac operator works!
  - We’ll construct it explicitly on the next slide

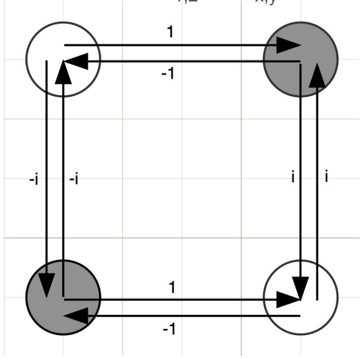


Solving the Double-Dimer model (cont.)

- Step 1: double the grid (replace every vertex with a pair at the same location)
- Step 2: Let $K' = \sigma_1 \otimes \partial_1 + \sigma_2 \otimes \partial_2$ = " $\not{\partial}$"
  - $\sigma_{1,2}$ are any two distinct choices of the Pauli matrices below. They are internal to the doubled grid.
  - $\partial_{1,2}$ are discretized derivative operators.
    - $\partial_1(v_a,v_b) = \pm 1$ iff $(v_a=v_b\pm(1,0))$ && $\partial_2(v_a,v_b) = \pm 1$ iff $(v_a=v_b\pm(0,1))$
- Step 3: Note that K' decouples as two operators on two disjoint copies lattices (see figure to the right). Pick one and call it K.
  - K's product of weights changes by -1 under a flip move.
  - One can verify this for each case. But there is a nice proof using just the Pauli matrix (Clifford) algebra $\sigma_i\sigma_j + \sigma_j\sigma_i = 2\,\delta_{ij}$
  - So $\det(K) = Z_{dd}[U]$!
- There are many different choices of K that compute Z
  - E.g. $\partial_1 + i\,\partial_2$ works as well.
  - Matrices giving same weights are "gauge equivalent"



Doubled Dirac operator and its reduced version, choosing $\sigma_{1,2}$ as $\sigma_{x,y}$



$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},\quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

The solved Double-dimer model (see lectures [3])

- For a square grid, size m x n, we can explicitly evaluate the determinants using Fourier analysis. This is easier to do for $\partial_1 + i\partial_2$.
  - Basis of eigenfunctions (j=1,...,m, k=1,...,n): $f_{j,k} = (-1)^{x+y} \sin\left(\frac{\pi j x}{m+1}\right) \sin\left(\frac{\pi k y}{n+1}\right)$
  - Eigenvalues: $\lambda_{j,k} = 2\cos\left(\frac{\pi j}{m+1}\right) + 2i\cos\left(\frac{\pi k}{n+1}\right)$
  - $Z_{dd}[U] = \det(\partial_1 + i\,\partial_2) = \prod \lambda_{jk}$
  - $Z_d[U] = \det^{1/2}(\partial_1 + i\,\partial_2) = \prod \lambda^{1/2}_{jk}$
- NOTE: This determinant can be expressed in more familiar terms as the Grassmann Integral.

$$Z_{dd}[U] = \int [D\bar{\psi}][D\psi] e^{\text{“} \int \bar{\psi} \not{\partial} \psi \text{”}} = \det \not{\partial}$$

  - Where the quotations means the discretized version of the integral and Dirac operator.
  - So, it seems that the dimer model is a discrete version of a 2D Dirac fermionic path integral

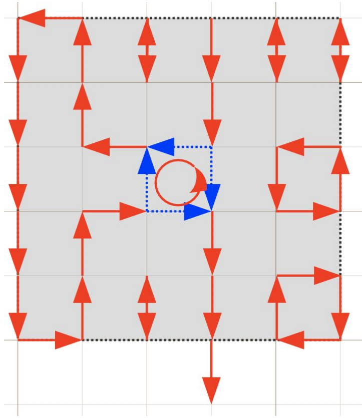
- It turns out that <u>any planar graph</u> can have its dimer matchings computed as the determinant of some matrix K. Such a matrix is known as a ‘Kasteleyn’ matrix. Non-planar graphs (more specifically ones that don’t have $K_{3,3}$ as a minor) can’t have a Kasteleyn matrix.
- Now, let’s consider the dimer model on higher genus surfaces, where the ABK invariants and (s)pin structures will come into play.

PART 2:

Dimers and 2D spin topology

The Double-dimer model on a Torus

- Now, let's consider a square grid on a torus (for ease, let's focus on m x n torus where m,n are even, so that the graph remains bipartite) and try to count its loop configurations.
- The same kind of matrix that satisfies the flip condition can be defined on a torus.
  - BUT, there's an issue. Loop configurations on a torus aren't flip connected.
  - A flip move cannot change the total homology class of the loops (i.e. sums of homology classes of loops), since a flip move changes the total homology class by a trivial cycle!
- From [4], for any square gridded 2D manifold, possibly with holes, higher genus, or a non-orientable surface, any two loop configurations that share the same total homology class in $H_1(M,\mathbb{Z})$ are flip connected.
  - So, this implies that if K satisfies the flip condition, any loop configurations in the same homology class give the same weight in det(K)
  - The flip condition isn't true for general graphs, but a morally similar statement is [1,2]:
    - There exist matrices K s.t. any two dimer matchings whose difference is trivial in $H_1(M,\mathbb{Z}_2)$ (mod $\mathbb{Z}_2$, since graph may not be bipartite, and general edges aren't directed) giving the same weight in det(K).

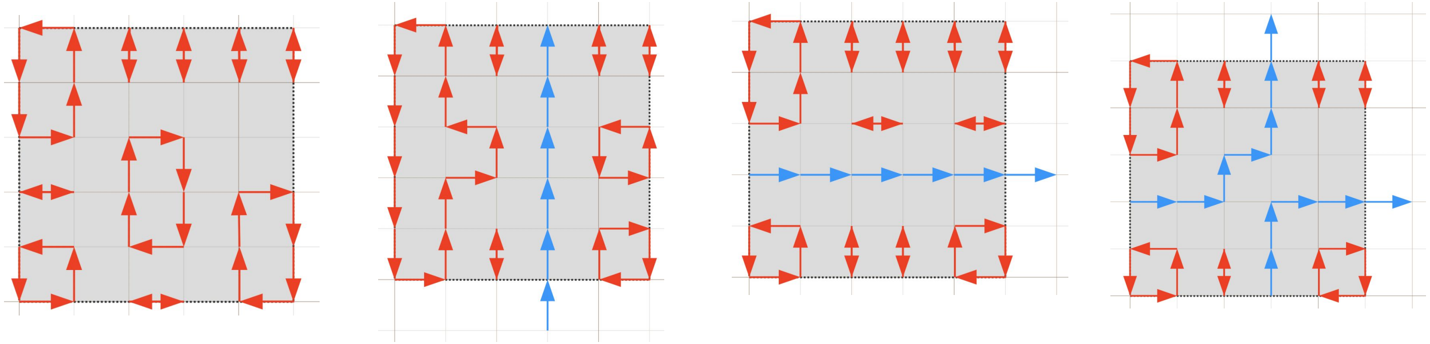


A loop configuration on a square grid on a torus (fundamental domain in gray) with one nontrivial loop. A possible flip move in blue, changes the total homology class by a trivial cycle (red circle).

The Double-dimer model on a Torus (cont.)

- An additional condition we want our matrices K to satisfy is loop-reversal invariance, i.e. that a directed loop and its reversed partner must have the same weight. If we can arrange for this (we can [1,2]), then only the total mod 2 homology classes, in $H_1(M,\mathbb{Z}_2)$, can have different weights in det(K)
- So, on a torus have four different possible homology classes of loops that may have different weights.
  - We'll end up needing four different Kasteleyn matrices K to compute Z_{dd}
    - (Spoiler: the four matrices correspond to the Dirac operators on the four spin structures on a torus)

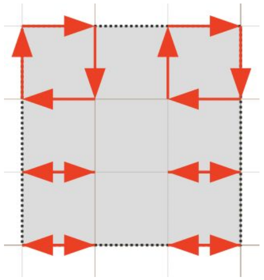
Representatives of the possible total Mod 2 homology classes of loops.



(0,0) (0,1) (1,0) (1,1)

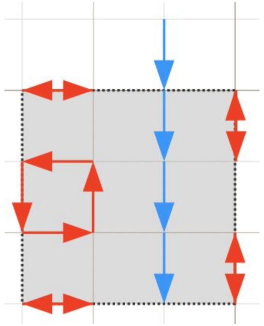
The Double-dimer model on a Torus (cont.)

- For the original matrix K constructed from $\sigma_1 \otimes \partial_1 + \sigma_2 \otimes \partial_2$, let's figure out the weights in det K for each class in $H_1(M,\mathbb{Z}_2)$. (m,n are even)

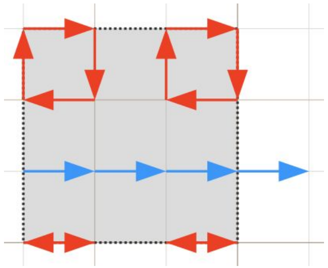


$(0,0) \longrightarrow 1$

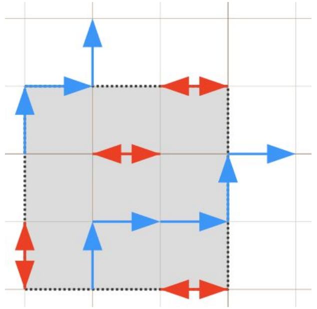
This class has weights all equal to 1. This is because it can be embedded on the plane, where all weights were equal 1



$(0,1) \longrightarrow -1$



$(1,0) \longrightarrow -1$



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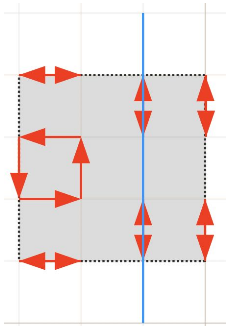
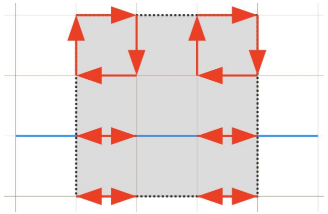
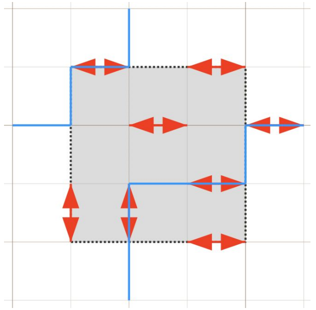






<u>For nontrivial configs, replace every nontrivial loop with doubled edges along the nontrivial cycle, length L.</u>

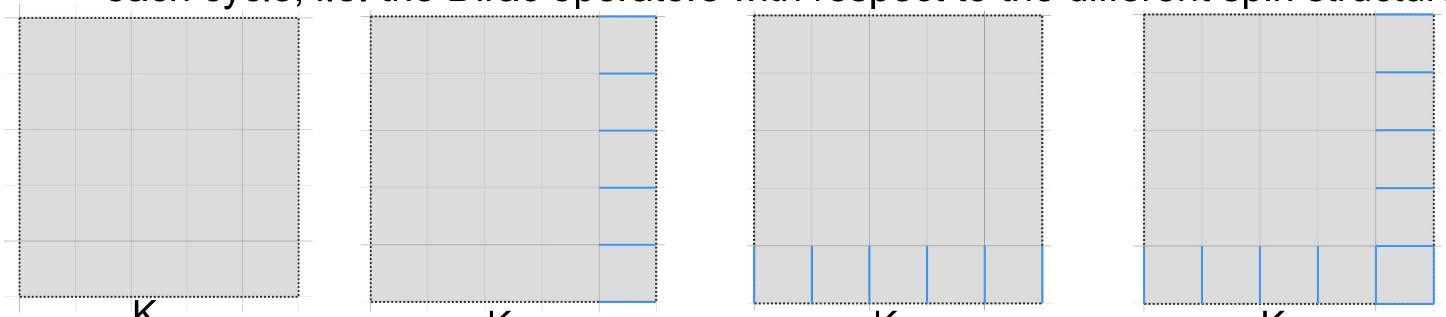
- Total signs of matrix weights of edges changes by $(-1)^{L/2}$ (easy check)
- Number of total loops goes from 1 to (L/2), so $(-1)^{\#\text{ of loops}}$ changes by $(-1)^{L/2-1}$
- So, total sign differs for all trivial configurations by -1

<u>In total,</u>

- (1,0), (0,1), (1,1) all have weights -1 in det(K)

The Double-dimer model on a Torus (cont.)

- So, we have det(K) = N_{00} - N_{10} - N_{01} - N_{11}, not enough to find total number of matchings.
  - N_{ij} = # of loop configs in a homology class.
  - (actually det(K) = 0 due to zero eigenvalue, ‘constant spinor’)
- But, we have 3 more choices of K that we can choose. Start with a given matrix K, and change matrix signs of edges along the cycles (1,0), (0,1), (1,1)
  - Will not change total signs of trivial loops, since every trivial loop will cross them twice: just signs of nontrivial ones change, i.e. flip condition holds.
  - (Analogous to changing between ground states of toric code)
- It turns out that any matrix satisfying the flip condition is equivalent to one of these four matrices [1,2].
- This is the Dirac operator $\sigma_1 \otimes \partial_1 + \sigma_2 \otimes \partial_2$, just with periodic or antiperiodic boundary conditions on each cycle, i.e. the Dirac operators with respect to the different spin structures on a torus!



K_{00} (Original K) K_{01} K_{10} K_{11}

Blue lines mean that an edge on a blue line differs in weight from K_{00} by factor of -1.

The double-dimer model on a Torus (cont.)

- A loop configuration $K_{\alpha\beta}$ will have an additional sign, $(-1)^{\#\text{ of blue edges hit}}$, as compared to its value in K_{00}. I.e. for every blue edge a loop configuration hits, its value shifts by -1
- Let's write for some η in {0,1}x{0,1}, $\det(K_\eta) = \Sigma_{\xi \text{ in } \{0,1\}\times\{0,1\}}(-1)^{q_\eta(\xi)}N_\xi$
  - ξ is in $H_1(M,\mathbb{Z}_2)$, represented by a pair in {0,1}x{0,1}
  - η is a spin structure, also represented by a pair in {0,1}x{0,1}
  - Not hard to see that $(-1)^{q_\eta(\xi)} = (-1)^{q_{00}(\xi)}(-1)^{\eta.\xi}$

Table of $(-1)^{q_\eta(\xi)}$

η\ξ	(0,0)	(1,0)	(0,1)	(1,1)
(0,0)	1	-1	-1	-1
(1,0)	1	1	-1	1
(0,1)	1	-1	1	1
(1,1)	1	1	1	-1

- $(-1)^{q_\eta(\xi)}$ is twice an orthogonal matrix, so its inverse is half its transpose
- So we can write $Z_{dd} = N_{00} + N_{10} + N_{01} + N_{11}$ as:

$$Z_{dd} = ½ (- Z_{00} + Z_{01} + Z_{10} + Z_{11})$$

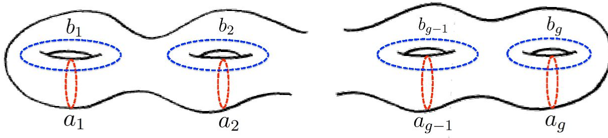
Where $Z_{\alpha\beta}=\det(K_{\alpha\beta})$

- Here is our friend Arf/ABK. (explanation soon):

$$Z_{dd} = ½ \,\Sigma_\eta(-1)^{\mathrm{Arf}(\eta)}Z_\eta$$

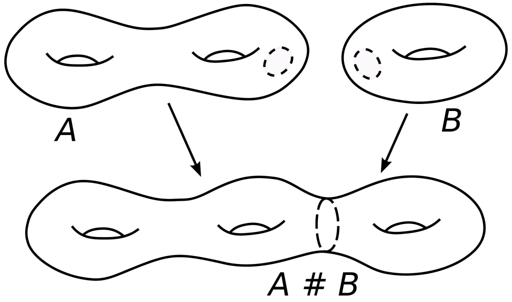
Spin structures, quadratic refinements, and Arf

- (Low-brow) "definition" of a spin structure η on genus g surface: take the Dirac operator, considered on the space all functions with periodic or antiperiodic boundary conditions on each nontrivial cycle.
  - To each spin structure is a "quadratic form" $q_\eta : H_1(M,\mathbb{Z}_2) \to \mathbb{Z}_2$ defined as follows
  - For basis of cycles $\{a_i, b_i\}$, $q_\eta(a_i/b_i) = 1$ if periodic around a_i/b_i, 0 if antiperiodic around a_i/b_i (Atiyah 1971, [5])
    - Set $q_\eta(0) = 0$
    - Extend to rest of $H_1(M,\mathbb{Z}_2)$ by linearity
    - More 'invariant' definitions in Johnson [8]. Explained well in [17].



  - $q_\eta(\xi)$ satisfies "quadratic refinement" (compare to previous slides)
    - $q_\eta(\xi_1+\xi_2) = q_\eta(\xi_1)+q_\eta(\xi_2) + \xi_1 \cap \xi_2$, or $(-1)^{q_\eta(\xi_1+\xi_2)} = (-1)^{\xi_1 \cap \xi_2}(-1)^{q_\eta(\xi_1)}(-1)^{q_\eta(\xi_2)}$
      - "$\cap$" is the intersection number on $H_1(M,\mathbb{Z}_2)$ (i.e. number of times mod 2 that $\xi_1 \cap \xi_2$ intersect each other)
- Given a spin structure, can act on it by an element α in $H^1(M,\mathbb{Z}_2)$, by making a cycle ξ differ in periodicity by factor of $\alpha(\xi)$ (flipping signs on a cycle), creating a new spin structure $\alpha.\eta$
  - Spin structures are a '$H^1(M,\mathbb{Z}_2)$ torsor' (i.e. there's a free and transitive action of $H^1(M,\mathbb{Z}_2)$ on spin structures. So spin structures are in 1-1 correspondence with $H^1(M,\mathbb{Z}_2)$)
  - $q_{\alpha.\eta}(\xi) := q_\eta(\xi) + \alpha(\xi)$, satisfies the quadratic refinement property iff $q_\eta(\xi)$ does
- The Arf invariant of η is defined as: $(-1)^{Arf(\eta)} = \frac{1}{\sqrt{|H_1(X,\mathbb{Z}_2)|}} \sum_\xi (-1)^{q_\eta(\xi)}$
  - I.e. proportional to sum of rows of matrix on the previous slide

Arf is a (spin) Cobordism Invariant



Proof sketch

- Need to check that for spin manifolds (M_1,η_1) (M_2,η_2)
  - $\mathrm{Arf}[(M_1\#M_2,\eta_1\#\eta_2)] = \mathrm{Arf}[(M_1,\eta_1)] + \mathrm{Arf}[(M_2,\eta_2)]$
  - "#" = connected sum
  - $\eta_1\#\eta_2$ has same periodicity conditions as $\eta_{1,2}$ on the respective parts of $M_{1,2}$
- It's a fact that $H_1(M_1\#M_2,\mathbb{Z}_2) = H_1(M_1,\mathbb{Z}_2)\oplus H_1(M_2,\mathbb{Z}_2)$
  - Mayer-Vietoris sequence. In 2D, not true mod $\mathbb{Z}$ because of torsion, need to modify argument.
  - The $\xi_{1,2}$ in $M_{1,2}$ map (ξ_1,ξ_2) to the corresponding homology class in $M_1\#M_2$.
  - Since periodicity/antiperiodicity around loops are preserved in the cobordism,
    - $q_{\eta1\#\eta2}(\xi_1,\xi_2) = q_{\eta1}(\xi_1) + q_{\eta2}(\xi_2)$, from 'definitions' based on periodicity and extending by linearity
- $$\begin{aligned}(-1)^{Arf(M_1\#M_2,\eta_1\#\eta_2)} &= \frac{1}{\sqrt{|H_1(M_1\#M_2,\mathbb{Z}_2)|}}\sum_{\xi\in H_1(M_1\#M_2,\mathbb{Z}_2)}(-1)^{q_{\eta_1\#\eta_2}(\xi)}\\ &= \frac{1}{\sqrt{|H_1(M_1,\mathbb{Z}_2)|}}\frac{1}{\sqrt{|H_1(M_2,\mathbb{Z}_2)|}}\sum_{(\xi_1,\xi_2)\in H_1(M_1,\mathbb{Z}_2)\times H_1(M_2,\mathbb{Z}_2)}(-1)^{q_{\eta_1}(\xi_1)+q_{\eta_2}(\xi_2)}\\ &= \left(\frac{1}{\sqrt{|H_1(M_1,\mathbb{Z}_2)|}}\sum_{\xi_1\in H_1(M_1,\mathbb{Z}_2)}(-1)^{q_{\eta_1}(\xi_1)}\right)\left(\frac{1}{\sqrt{|H_1(M_2,\mathbb{Z}_2)|}}\sum_{\xi_2\in H_1(M_2,\mathbb{Z}_2)}(-1)^{q_{\eta_2}(\xi_2)}\right)\\ &= (-1)^{Arf(M_1,\eta_1)}(-1)^{Arf(M_2,\eta_2)}\end{aligned}$$

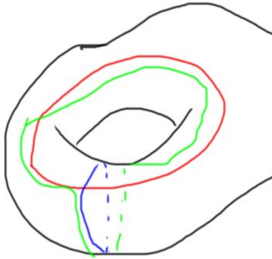
A bit more on Arf

- <u>Thm:</u> $\Omega^{Spin}_2 = \mathbb{Z}_2$ (c.f. Kirby, Taylor [6])
- The Arf invariant is a $\mathbb{Z}_2$, invariant, only takes values in ±1.
  - Turns out to completely characterize the bordism class of a spin structure.
- Arf[η] = (# of zero modes of Dirac operator on η) mod 2 [5,18]
- Also, it is thought that Ω^{Spin}_2 gives the group of interacting fermionic gapped states in 1+1-D, with the only symmetry being $(-1)^F$ [13]
  - It's thought to correspond to the TQFT of the Kitaev chain, which is also $\mathbb{Z}_2$.

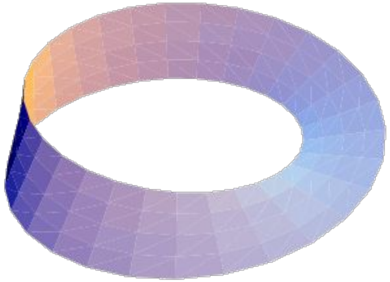
In Turkey, Arf = $$

(Double)-dimers on higher genus surfaces

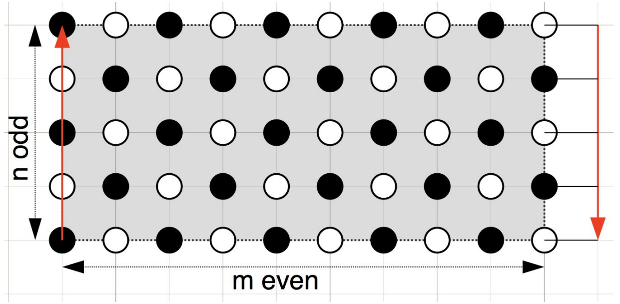
- Given any Kasteleyn matrix K, we can construct 2^{2g} others by flipping signs along cycles as we did before, giving matrices K_η indexed by the 2^{2g} spin structures, with determinants $\det(K_\eta)$
- The quadratic refinement property is the main thing that holds on higher genus surface: $q_\eta(\xi_1+\xi_2) = q_\eta(\xi_1)+q_\eta(\xi_2) + \xi_1 \cap \xi_2$.
- Intuitively, if there's two configurations with nontrivial loop classes ξ_1,ξ_2, with matrix weights $w(\xi_1),w(\xi_2)$ then the total matrix weights of a configuration with loop class $\xi_1+\xi_2$ will be $w(\xi_1)w(\xi_2)$.
  - BUT, the $(-1)^{\#\text{ of loops}}$ factor will differ by how many intersections are between ξ_1,ξ_2.
  - Gives $(-1)^{q_\eta(\xi_1+\xi_2)}=(-1)^{q_\eta(\xi_1)}(-1)^{q_\eta(\xi_2)}(-1)^{\xi_1\cap\xi_2}$
- $Z_{dd}= 1/2^g \; \Sigma_\eta(-1)^{\mathrm{Arf}(\eta)}\det(K_\eta)$
- This kind of intuition literally holds for certain classes of graphs, but generally there are some hairy combinatorics to confirm the picture (see [1,2]). It's more hairy for dimers than double-dimers: we need some extra minus signs.
  - [1,2] also give a more explicit relationship with spin structures.
    - "Kasteleyn Orientations" are a way to encode the K_η



Double-Dimers on the Möbius band (or RP^2) (originally in [7,2])

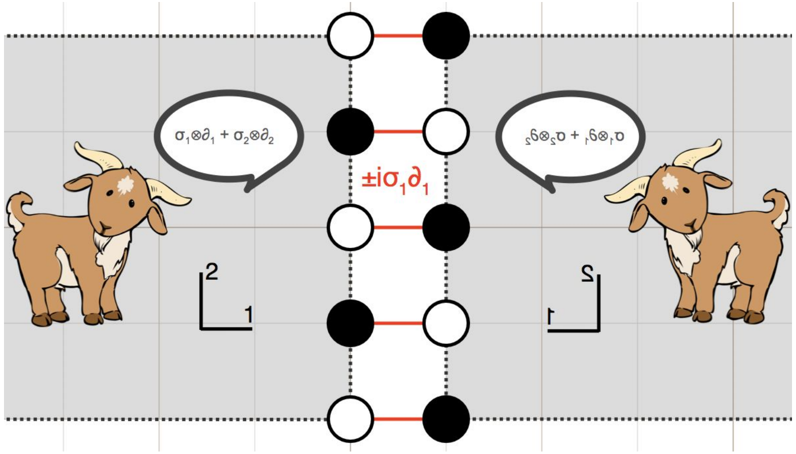


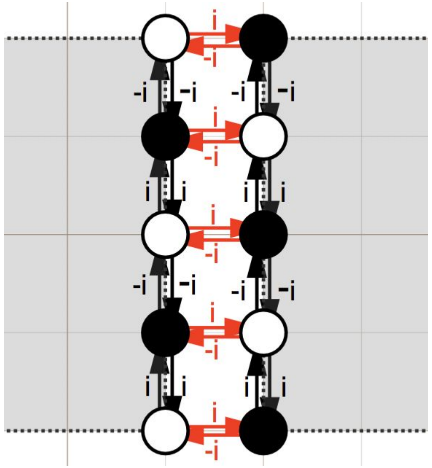
- For now, take m x n fundamental domain. Take m even & n odd (so that the graph remains bipartite).
  - In [4], they show that loop configs are connected by flips and loop reversals iff they have the same class in $H_1(M,\mathbb{Z}_2)=\mathbb{Z}_2$
- For the (simply connected) fundamental domain, we use a $K' = \sigma_1 \otimes \partial_1 + \sigma_2 \otimes \partial_2$ as before.
  - Use this weight in the gray region
- Now we need to figure out how to patch the connecting edges together to make a weighting invariant under flips AND under reversing the loop going exactly once around the strip.
  - i.e. need the flip condition to hold on squares in the connecting region



Double-Dimers on the Möbius band (or RP^2) (cont.)

- Prescription:
  - Given $K' = \sigma_1 \otimes \partial_1 + \sigma_2 \otimes \partial_2$ on the interior region, add $\pm i\sigma_1\partial_1$ on the boundary.
    - ∂_1 doesn't strictly make sense on bdy due to orientation reversal, but the $\pm$ gives both choices
  - The matrix will decompose into two disjoint matrices:
    - Pick one of them and call it $K_\pm$.
  - <u>NOTE:</u> The two choices, $\pm$, are different Kasteleyn matrices. (more on this later)
- One can check that the flip condition holds on the connecting squares. And that there is loop reversal invariance for the non-trivial loops.
  - These are two Kasteleyn weightings!
  - We can use them to find the number of loop configs

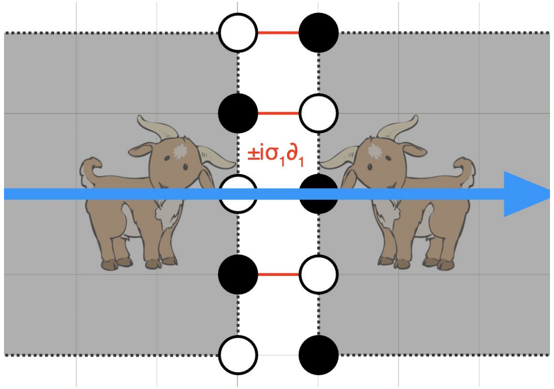




Weights on the connecting region if we choose $\sigma_{1,2}$ as $\sigma_{x,y}$

Double-Dimers on the Möbius band (or RP^2) (cont.)

- A loop config with a nontrivial loop in $H_1(M,\mathbb{Z}_2)=\mathbb{Z}_2$ will receive a weight $\pm i$ in $\det(K_\pm)$.
  - $\det(K_\pm) = N_{trivial} \pm i\, N_{nontrivial}$
- So, $Z_{dd} = N_{trivial} + N_{nontrivial} = ½\,(1-i)\det(K_+) + ½\,(1+i)\det(K_-)$



- $$\boxed{Z_{dd} = 1/\sqrt{2}\,[e^{-2\pi i/8}\det(K_+) + e^{2\pi i/8}\det(K_-)]}$$
- What does this have to do with ABK?
  - $e^{\pm 2\pi i/8}$ are the ABK invariants of the pin$^-$ structures on the Möbius strip and RP^2!

Pin structures

- A spin$^{\pm}$ structure is "An equivariant lift of the principal SO(n) orthonormal frame bundle to a principal Spin$^{\pm}$(n) bundle".
  - Spin(n) is a double-cover of SO(n) [spin structs only make sense if manifold is oriented]
  - Pin$^{\pm}$(n) are two different double-covers of O(n), each with different ways of lifting of the orientation-reversing elements of O(n)
  - They may not exist: w_2 is obstruction for spin or pin^{+} structure, $w_2+w_1^2$ is obstruction for Pin^{-}
- In local coordinates, we can define Pin structures them in terms of the Dirac operator associated to them (see e.g. Appendix of [22]):
  - Boundary conditions around the 'orientation reversing wall' are to multiply the spinor by:
    - For Pin^{-} : $\pm i\gamma_n$, where γ_n is the Clifford algebra element associated to the normal vector 'n' of the wall
    - For Pin^{+} : $\pm \gamma_n$, where γ_n is the Clifford algebra element associated to the normal vector 'n' of the wall
  - Boundary conditions are ±1 on the other cycles.
  - Structures (if they exist) are in bijection with $H^1(M,\mathbb{Z}_2)$, by flipping signs around cycles.
- Pin^{-} is the group that shows up in the Dimer model

Quadratic refinements, ABK

- Quadratic forms work in basically the same way as with pin, with modification due to orientation reversing wall, EXCEPT they take values in $\mathbb{Z}_4$
  - q_η: $H_1(M,\mathbb{Z}_2) \to \mathbb{Z}_4$
  - $q_\eta(a_i)$ = 2 if periodic around a_i, 0 if antiperiodic around a_i, ±1 if goes around orientation reversing wall: a_i is a basis of generating cycles
  - Quadratic refinement: $q_\eta(\xi_1+\xi_2) = q_\eta(\xi_1)+q_\eta(\xi_2) + 2\, \xi_1 \cap \xi_2$ OR $i^{q_\eta(\xi_1+\xi_2)} = (-1)^{\xi_1 \cap \xi_2}\, i^{q_\eta(\xi_1)} i^{q_\eta(\xi_2)}$
  - $q_{\alpha.\eta}(\xi) := q_\eta(\xi) + 2\,\alpha(\xi)$ is the $H^1(M,\mathbb{Z}_2)$ torsor's action, for α in $H^1(M,\mathbb{Z}_2)$
- $Z_{dd}(M) = \frac{1}{\sqrt{|H_1(M,\mathbb{Z}_2)|}} \sum_{\eta \in Pin^-(M)} e^{\frac{2\pi i}{8} ABK(\eta)} Z_\eta$ is the dimer count on a general 2D manifold.
- <u>Thm:</u> $\Omega^{Pin-}_2 = \mathbb{Z}_8$ (c.f. Kirby, Taylor [6])
- The ABK is a Pin^- cobordism invariant and completely characterizes the Pin^- bordism classes, which is needed on nonorientable manifolds
  - The generator of 1 in $\mathbb{Z}_8$ is given by a pin^- structure on RP^2
- It is thought that Ω^{Pin-}_2 gives the group of interacting fermionic gapped states in 1+1-D, with an additional time-reversal symmetry with T^2=1.
  - The Kitaev chain with interactions with such time-reversal symmetry imposed does in fact have a $\mathbb{Z}_8$ classification

More on Dimers and Fermions

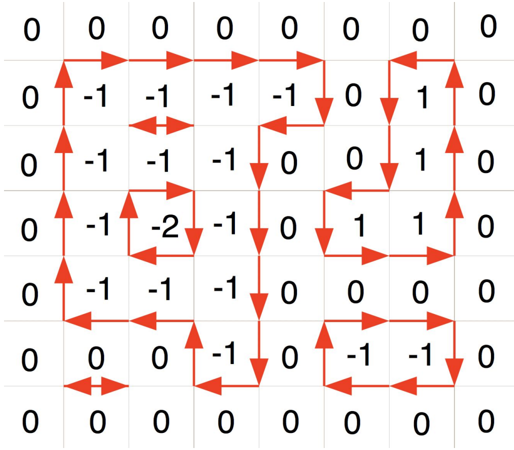
- Turns out the 2D Ising model in zero magnetic field can be written as a dimer model.
    - I.e. its partition function on the plane is the determinant of a dimer model's Kasteleyn matrix.
    - On surfaces of higher genus, can use the same Arf techniques for Ising [19]
- Bosonization
    - The formula $Z = (-Z_{00} + Z_{01} + Z_{10} + Z_{11})/2$ is one that occurs in Conformal Field theory when 'bosonizing' a fermionic theory on a torus, e.g. the Dirac/Boson correspondence or the Majorana/Ising correspondence. (Chapters 10,12 of [11])
    - In fact the formula $Z = 1/2^g \ \Sigma_\eta (-1)^{Arf(\eta)} Z_\eta$ was proposed by string theorists in the 80's as a way to bosonize on higher genus surfaces [8], and was used in [1] as inspiration to figure out how to count dimers on more general surfaces
        - Although even in the 60's the torus formula for dimers was known.
- The "bosonic dual" is a so-called height function on the graph obtained from a dimer matching. [9,10].
    - Let's explore this more.
    - We'll see the dimer model's height function correlations (not the double-dimer!) match exactly with those of a bosonized fermion. And the partition functions in a background gauge field match exactly with the single dimer model. [10]

PART 3:

Dimers and Bosons

Height functions on Double-dimers (planar case)

- The loop configuration can be thought of as a bunch of contour lines defining a ‘height function’ on the plane.
  - The “height” on a plaquette is defined by:
    - Defining the height at the boundary as zero (‘Dirichlet conditions’)
    - The height on a neighboring plaquette would change by -1 or 1 every time one crosses a clockwise or counterclockwise loop
  - $h(p)$:= ‘height’ at a plaquette p
    - $h(p)$ = (# CCW loops around p) - (# CW loops around p)
- Natural questions...
  - What are the ‘correlation functions’ or ‘expectation values’ of the height field? (averaged over the loop ensemble), e.g. $\langle h(p_1)\, h(p_2)\rangle$
  - What about in the continuum limit?
    - i.e. when the lattice spacing goes to zero, for plaquettes a fixed distance apart
  - <u>NOTE:</u> $\langle h(p)\rangle$ will be zero: reversing all loops sends $h \to -h$, and is an involution

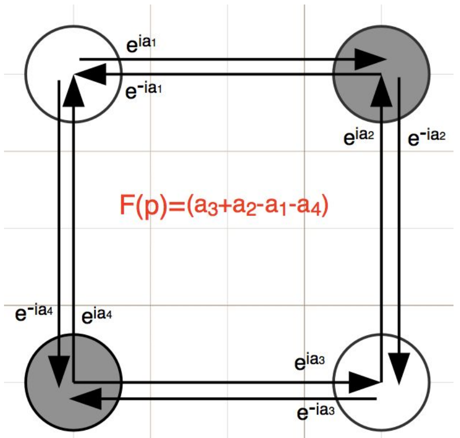


Height field of a loop configuration

Height functions on Double-dimers: Gauge fields

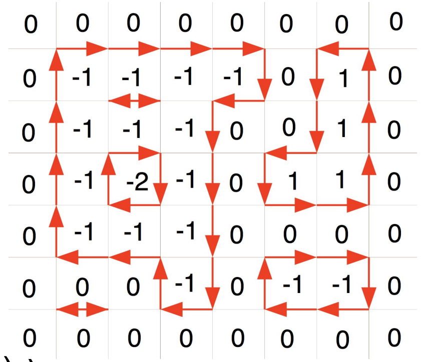
- Main Tool: introduce a “gauge field” on the graph!
  - Known as the ‘moment method’: from e.g. [10,20]
  - Let’s think of this idea on its own at first, then apply it to the height field.
- A “U(1) gauge field” on the graph is a matrix constructed from our original Kasteleyn matrix: call the (discrete) gauge field U (or A)
  - U is a function, U: {directed edges} $\to$ U(1) with some conditions...
  - For a directed edge e := v$\to$w and its reverse -e := w$\to$v:
    - <u>IMPOSE:</u> U(e) = 1/U(-e). If U(e) = exp(i A(e)), then U(-e) = exp(-i A(e))
    - Means: “parallel transport in a direction=inverse of the opposite direction’s”
    - “$\mathrm{Pexp}(i\int_C A) = \mathrm{Pexp}(i\int_{-C} A)^{-1}$ ”
  - For a closed curve C, $\mathrm{Hol}(C) := \prod_{v\to w \text{ in } C} U(v\to w)$ is ‘holonomy’ around C
  - A_1,A_2 are <u>gauge equivalent</u> iff $\mathrm{Hol}_1(C) = \mathrm{Hol}_2(C)$ for all C.
  - Define $e(i\,F(p)) := \mathrm{Hol}(C_p)$, C_p is the CCW curve going around plaquette p.
- Start with original matrix, K, construct a new matrix K[A]
  - For a directed edge v$\to$w
    - K[A](v,w) := K(v,w)A(v,w)
    - Analog of ‘coupling to a gauge field’, $\partial\to(\partial\text{-}iA)$

Discrete gauge field and holonomy



Height functions on Double-dimers: Gauge fields (cont)

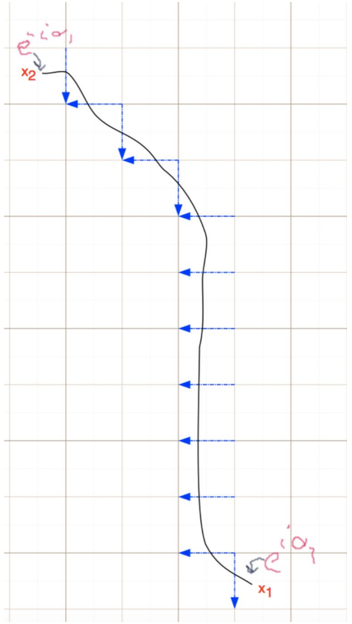
- <u>Lemma:</u> On the plane, A_1, A_2 are gauge equivalent iff $F_1(p) = F_2(p)$ for all p
    - *Proof:* On the plane, a closed curve C (WLOG counterclockwise) encloses plaquettes {q}.
        - So, $\mathrm{Hol}(C) = \prod_{q \text{ inside } C} \exp(i\, F(q)) = \exp(i\, \Sigma_{p \text{ in } C} F(p))$
            - discrete Stokes' theorem, all internal edges cancel leaving just the outer loop's holonomy
            - $\mathrm{Hol}(C) = \exp(-i\, \Sigma_p F(p))$ if C is clockwise
        - So, holonomies of closed loops only depend on F(p)



- Now, let's see what det(K[A]) can tell us.
    - Recall that $\det(K[A]) = \Sigma_{\text{loop configs, L}} \prod_{C \text{ in } L}(- K[A]_{C(1)C(2)} \ldots K[A]_{C(Lc)C(1)})$.
    - For K = K[0], we always had $\prod_{C \text{ in } L}(- K[0]_{C(1)C(2)} \ldots K[0]_{C(Lc)C(1)}) = 1$
    - So, since $K[A](v,w) := K(v,w)A(v,w)$, we'll have
        - $\det(K[A]) = \Sigma_{\text{loop configs, L}} \prod_{C \text{ in } L} \mathrm{Hol}(C) = \Sigma_{\text{loop configs, L}} \prod_{C \text{ in } L} \exp(\pm i\, \Sigma_{p \text{ in } C} F(p))$
            - $\pm$ means + if CCW, - if clockwise
        - $\Sigma_{\text{loop configs, L}} \prod_{\text{all plaquettes } p} \exp(i\, F(p)\, [(\text{\# of CCW loops around p}) - (\text{\# of CW loops around p})])$
- So, we've found that <u>Lemma:</u> $\det(K[A]) = \Sigma_{\text{loop configs, L}} \exp(i\, \Sigma_{\text{plaquettes } p} F(p)\, h(p))$
    - So, $\det(K[A])/\det(K[0]) = \langle \exp(i\, \Sigma_{\text{plaquettes } p} F(p)\, h(p)) \rangle$
        - Since det(K[0]) = # of loop configs, gives normalization for expectation value

Height functions on Double-dimers: Moment method

- This formula $\det(K[A])/\det(K[0]) = \langle \exp(i\Sigma_{\text{plaquettes } p} F(p)\, h(p))\rangle$ is remarkable, and will help us compute $\langle h(p_1)h(p_2)\rangle$.
  - Use a "singular gauge field". Arrange gauge field so that $F(p_1) = a_1$ and $F(p_2) = a_2$ and $F(p)=0$ for all other p.
- Then, we'll have $\det(K[A(a_1,a_2)])/\det(K[0]) = \langle e^{ia_1 h(p_1)} e^{ia_2 h(p_2)}\rangle$
  - Discrete version of vertex operator correlation functions!
  - We have that $\langle h(p_1)h(p_2)\rangle = -\partial_{a1}\partial_{a2} \langle \exp(ia_1 h(p_1))\exp(ia_2 h(p_2))\rangle|_{ai = 0}$.
  - In general, we can compute any correlation function in terms of the *moments* of $\det(K[A])$ with some judicious choices of A
- We want to take the 'continuum limit' of such a height correlator. It'll end up being that for real space coordinates $x_{1,2}$ w.r.t. Lattice spacing ε, $\langle h(x_1)h(x_2)\rangle \sim -\log(|x_1-x_2| / \varepsilon)$: diverges with ε.
  - Correlators like $\langle (h(x_1)-h(x_2))(h(x_3)-h(x_4))\rangle$ will be finite (and conformally invariant!)
  - We'll pick gauge fields such that $F(x_1) = a_1$, $F(x_2)= -a_1$, $F(x_3) = a_2$, $F(x_4)= -a_2$.
  - This can be accomplished by picking "zippers" along a path $x_1 \to x_2$ && $x_3 \to x_4$
    - A zipper is a path on the dual lattice on which we put edge weights $e^{\pm ia_i}$
    - Will lead to F(p) being zero everywhere except for the endpoints



Zipper for a path $x_1 \to x_2$: $U(e) = \exp(ia_1)$ in the direction of the arrows and $U(e) = \exp(-ia_1)$ in the opposite direction of the arrows.

<u>Note:</u> $F(p)=0$ except for $F(x_1)=a_1$ && $F(x_2) = -a_1$

Height functions on Double-dimers: Moment method

- How do we compute $\langle(h(x_1)-h(x_2))(h(x_3)-h(x_4))\rangle = -\partial_{a1}\partial_{a2} \det(K[a_1,a_2])/\det(K[0])|_{ai=0}$?
  - First Taylor expand $K[a_1,a_2] = K[0] - i\, a_1 M_1 - i\, a_2 M_2 - a_1a_2M_{12} + \dots$
  - $\det(K[a_1,a_2])/\det(K[0]) = \det(K^{-1} \cdot K[a_1,a_2]) = \exp(\log \det(K^{-1} \cdot K[a_1,a_2]))$
  - $= \exp(\mathrm{tr} \ln(K^{-1} \cdot K[a_1,a_2])) = \exp(\mathrm{tr} \ln(1 - ia_1 K^{-1} M_1 - i\, a_2 K^{-1} M_2 - a_1a_2M_{12}+...))$
- We can then find after some calculus that
  - $-\partial_{a1}\partial_{a2} \det(K[a_1,a_2])/\det(K[0])|_{ai=0} = \mathrm{tr}(K^{-1}M_{12}) + \mathrm{tr}(K^{-1} M_1K^{-1} M_2) + \mathrm{tr}(K^{-1} M_1)\mathrm{tr}(K^{-1} M_2)$
  - K^{-1} is a subtle object to define, but it can be done
  - It turns out that $\mathrm{tr}(K^{-1} M_2) = 0$ due to some cancellations from $K[a_1,a_2]$ coming from a gauge field.
  - And, if we arrange the zippers to not intersect, there can never be any a_1a_2 cross terms, so $M_{12}=0$.
  - Leads to $\langle(h(x_1)-h(x_2))(h(x_3)-h(x_4))\rangle= \mathrm{tr}(K^{-1}M_1K^{-1}M_2)$
- <u>END RESULT:</u> as $\varepsilon\to0$, we'll have the following conformally invariant expression. It turns out that h becomes Gaussian: this is all we need to know to compute all correlation functions. h satisfies the Wick relation!

$$\langle(h(x_1)-h(x_2))(h(x_3)-h(x_4))\rangle = -\frac{1}{\pi^2}\log\frac{(x_3-x_1)(x_4-x_2)}{(x_3-x_2)(x_4-x_1)}$$

Bosonization

- The fact that the double-dimer model has 'bosonic' free-field correlation functions built into it is a sign of "bosonization".
  - In 2D, Bosons coupled to a background gauge field are the same as fermions in a background gauge field!
- The continuum bosonization can be derived by the exact same method.
  - Consider the Dirac fermion in 2D.
  - The moments of its partition function with respect to background (perhaps singular) gauge fields will give bosonic correlation functions!
- More precisely, we'll have:

DIRAC ACTION:

$$Z[A] = \int [\mathcal{D}\bar{\Psi}\mathcal{D}\Psi] e^{-S[\bar{\Psi},\Psi,A]}$$

$$S[\bar{\Psi}_L, \Psi_L, \bar{\Psi}_R, \Psi_R, A] = 2\int d^2x (\bar{\Psi}_L(i\partial_{\bar{z}} - A_{\bar{z}})\Psi_L + \bar{\Psi}_R(i\partial_z - A_z)\Psi_R)$$

BOSON ACTION:

$$\frac{Z[A]}{Z[0]} = \frac{1}{Z[0]} \int \mathcal{D}H e^{\int_{\mathbb{R}^2} d^2x \{\frac{1}{2}H(x)\pi\partial^2 H(x) + iF(x)H(x)\}}$$

- Note the exp(i∫F(x) H(x)). This is the same kind of term that came up before in the discrete version!
  - ~exp(i $\Sigma_{\text{plaquettes } p}$F(p) h(p))

Bosonization dictionary

- In the continuum case we have an operator dictionary (in the sense that all of their correlations match exactly) on both sides of the duality.

$$\bar{\Psi}\gamma_\mu\Psi \leftrightarrow \frac{1}{\sqrt{\pi}}\epsilon_{\mu\nu}(\partial\phi)^\nu = \frac{1}{\sqrt{\pi}}(\partial\phi)_\perp$$

(1) Dirac fermion current, j_μ

(2) 90° rotation of gradient of boson field, φ

- In QFT, the Dirac current j_μ is automatically divergence-free satisfies $\partial \cdot j = 0$ due to 'Ward Identities'. (This divergence is zero as an operator)
  - On the right hand side of this part of the dictionary, note that this is automatically true.

- For some φ a function, we'd have that $(\partial\varphi)_\perp$ gives a divergence free vector field whose integral curves are the contour lines of φ.
  - This is exactly the interpretation we started out with the height field, whose contour lines were the double-dimer contour lines!

More on dimers and bosons

- If we do the moment calculation for the continuum Dirac fermion, we end up getting that the analog of the bosonic height correlation functions :

$$\langle(\phi(z_2)-\phi(z_1))(\phi(z_4)-\phi(z_3))\rangle = -\frac{1}{2\pi^2}\log\left|\frac{(x_1-x_3)(x_2-x_4)}{(x_1-x_4)(x_2-x_3)}\right|$$

- This differs from the double-dimer heights by a factor of 2. (we had $1/\pi^2$, not $1/2\pi^2$)
- It turns out that the dimer model (i.e. just perfect matchings) have a height function associated to them as well (see [9]).
  - Their two-point height correlations are half the double-dimers' [10,21], intuitively. So the dimer height correlations match exactly with the free fermion!
  - Also, on a torus, the total (regularized) partition function of *single* dimer matchings exactly matches that of a Dirac fermion on a torus (see [14] for nice derivation).
- Natural notion of 'topological defects': monomers (= isolated vertices not in a dimer pair)
  - Turns out that monomers make the dimer height function ill-defined: going around the monomer creates a monodromy of the height function (see [9]). Related to "magnetic insertions" [12]
  - They're associated to vertex operators $\sim e^{i\varphi/\sqrt{2}}$ (of *T-dual* boson field) in the partition function (see [12]).
- These models are the simplest example of statistical mechanics models with 'discrete holomorphicity', which is a program/set of ideas: with it, can rigorously establish the conformal invariance of the Ising model at criticality. (c.f. [16]) (*Smirnov:* 2010 Fields Medal)

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