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Topologie différentielle/*Differential Topology*

Equivariant topological rigidity phenomena

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Abstract — We announce proofs of certain cases of equivariant analogues of the well-known conjectures of Novikov and of Borel, thus of equivariant topological rigidity phenomena for actions of a compact Lie group on aspherical manifolds.

Phénomènes de rigidité topologique équivariante

Résumé — Nous annonçons des preuves de certains cas des analogues équivariants des conjectures célèbres de Novikov et de Borel, donc de phénomènes de rigidité topologique équivariante pour les actions d'un groupe de Lie compact sur les variétés asphériques.

Version française abrégée — On connaît déjà beaucoup d'exemples de rigidité topologique pour les variétés non simplement connexes et surtout pour les variétés asphériques. Nous en citons deux : la conjecture de Novikov et celle de Borel. Si M est une variété compacte connexe orientée (sans bord), la formule de Hirzebruch exprime la signature de M, clairement invariante par équivalences d'homotopie préservant l'orientation, comme $\langle L(M), [M] \rangle$, où $L(M)$ est une classe caractéristique du fibré tangent. Si π est un groupe discret et $f: M \to B\pi$, $\alpha \in H^*(B\pi, \mathbf{Q})$, on peut définir

$$\operatorname{sign}_\alpha(M) = \langle f^*(\alpha) \cup L(M), [M] \rangle \in \mathbf{Q}.$$

La conjecture célèbre de Novikov est que ces nombres sont des invariants d'homotopie orientée. La conjecture est vérifiée pour beaucoup de groupes (*voir* par exemple [15], [3], [6], [5]). Le cas le plus important pour nous est celui d'un sous-groupe discret d'un groupe de Lie réel connexe, traité par Kasparov [13].

S'il existe une variété $K(\pi, 1)$, la conjecture de Novikov est liée à une conjecture plus précise de rigidité topologique, attribuée souvent à A. Borel :

CONJECTURE. — Soit $h: M_1 \to M_2$ une équivalence d'homotopie de variétés asphériques (c'est-à-dire, de variétés dont les revêtements universels sont contractiles). Supposons que h est un homéomorphisme en dehors d'un compact. Alors h est homotope (rel. au complément d'un compact plus grand) à un homéomorphisme.

Cette conjecture est aussi vérifiée dans certains cas; *voir* en particulier [27], [7], [3], [10]. (Pour les conséquences en K-théorie algébrique, *voir* [25], [8].) La version stable, où l'on fait un produit avec un espace euclidien, a été vérifiée plus souvent.

Dans cette Note, nous considérons quelques versions équivariantes de ces conjectures, pour une variété sur laquelle un groupe de Lie G agit de façon lisse (ou, plus généralement, de façon lipschitzienne — *voir* [18], [20]). Soit $\Delta^G(M) \in K_*^G(M)$ la classe de l'opérateur signature équivariant. Par la méthode de Kasparov, nous prouvons :

THÉORÈME [19]. — *Supposons que* G *agit par isométries, préservant une orientation, sur une variété complète* W *de courbure non positive. Si* $f: M \to W$ *est* G-*équivariant, alors*

$$f_*(\Delta^G(M)) \in K_*^G(W)$$

est un invariant d'homotopie équivariante orientée. De plus, si $h: M' \to M$ *est équivariant et est une équivalence* (*non-équivariante*) *d'homotopie, alors* $f_*(\Delta^G(M)) = f_* h_*(\Delta^G(M'))$.

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Dans un grand nombre de cas, on peut vérifier l'hypothèse d'existence d'une application équivariante f: $M \to W$ en utilisant la théorie des applications harmoniques, comme dans [21].

Ce résultat sur la conjecture de Novikov équivariante a pour analogue pour la conjecture de Borel le résultat suivant.

THÉORÈME [12]. — *Soit* f: $M \to W$ *une équivalence d'homotopie équivariante, où* M *est une variété complète de courbure non positive sur laquelle* G *agit par isométries. Si* f *est un homéomorphisme en dehors d'un compact, alors* M *et* W *sont stablement* G-*homéomorphes.*

La même méthode topologique s'applique [12] à la démonstration d'injectivité de certaines « applications d'assemblage » dans la K-théorie algébrique, la L-théorie (théorie de chirurgie), et la A-théorie (K-théorie des espaces) de Waldhausen.

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1. Hirzebruch's well known formula relates an *a priori* homotopy invariant to characteristic classes. That is, there is a stable characteristic class, L(M), of the tangent bundle of a smooth manifold M such that if M is oriented and closed, $\langle L(M), [M] \rangle$ is the signature of M, that is the signature of the quadratic form on middle-dimensional cohomology. In the simply connected case this is only homotopy-invariant characteristic number. If f: $M \to B\pi$ classifies the fundamental group of a nonsimply connected manifold and $\alpha \in H^*(B\pi; \mathbf{Q})$, then one can form

$$\operatorname{sign}_\alpha(M) = \langle f^*(\alpha) \cup L(M), [M] \rangle \in \mathbf{Q}.$$

The Novikov conjecture is that all of these numbers are oriented homotopy invariants. This has been verified in many cases by a wide variety of techniques (*see* [15], [3], [6], [5] for a sample). Of most relevance to this announcement is Kasparov's verification for discrete subgroups of connected real Lie groups [13].

2. One can obtain more precise information by viewing the signature of M as the index of the signature operator on M defined by Atiyah-Singer [1] in the smooth case, and by Sullivan-Teleman [24] for topological manifolds (using techniques of Lipschitz analysis). One gets a class $\Delta(M) \in K_*(M)$ (K-homology), and one can study

$$f_*(\Delta(M)) \in K_*(B\pi).$$

Certain topological considerations make this less natural to study at 2. Away from 2, however, it is important to understand when it is a homotopy invariant. For π finite it needn't be, but it often is for torsion-free groups.

3. If a compact Lie group G acts smoothly on M (or more generally if M has a Lipschitz G-action, *see* [18], [20]), preverving the orientation, one can form an equivariant signature class $\Delta^G(M) \in K_*^G(M)$. We have the following generalization of Kasparov's theorem:

THEOREM [19]. — *Suppose* G *acts by orientation-preserving isometries on a complete nonpositively curved manifold* W *and suppose* f: $M \to W$ *is equivariant; then*

$$f_*(\Delta^G(M)) \in K_*^G(W)$$

is an equivariant oriented homotopy invariant. In fact, if h: $M' \to M$ *is equivariant and an oriented homotopy equivalence then* $f_*(\Delta^G(M)) = f_* h_*(\Delta^G(M'))$.

For more general equivariant $K(\pi, 1)$'s, one has counterexamples.

Kasparov's method depends on looking at the image of $f_*(\Delta(M))$ under a certain natural map β: $K_*(B\pi) \to K_*(C_r^*(\pi))$, where $C_r^*(\pi)$ is the reduced group C*-algebra, a certain completion of $C\pi$. In the equivariant case, however, it may happen that $W^G=\varnothing$, so that G doesn't act naturally on either $\pi=\pi_1(W)$ or on $C_r^*(\pi)$. Accordingly, we are forced to consider the fundamental groupoids of the manifolds concerned. G acts on these groupoids and on their C*-algebras. An important technical tool is:

THEOREM [19]. — *If* Y *and* Y′ *are connected* G-*manifolds and* f: Y → Y′ *is a* 1-*equivalence and is* G-*equivariant, then* f *induces an isomorphism* $K_*^G(C_r^*(\pi(Y))) \to K_*^G(C_r^*(\pi(Y')))$, *where* $\pi(Y)$ *denotes the fundamental groupoid of* Y.

4. In general, it is quite difficult to tell when one can find an equivariant map f: M → W. However, in special cases one can establish this, often by using existence and uniqueness theory for harmonic maps, essentially as in [21]. As a sample one has:

PROPOSITION [21]. — *If* W^k *is a hyperbolic manifold of finite volume,* $k>2$, *and if* $\pi_1 M \to \pi_1 W$ *is an isomorphism, then for any* G-*action on* M *there is a unique* G-*action on* W *by isometries and an equivariant map* M → W (*which is unique up to equivariant homotopy*) *inducing the given homomorphism.*

For $k=2$ a more complicated statement (involving changing the hyperbolic structure) holds.

5. The Novikov conjecture is intimately related to a topological rigidity conjecture, often attributed to A. Borel. We state it, somewhat more generally than is common, as follows:

CONJECTURE. — Let h: $M_1 \to M_2$ be a homotopy equivalence between aspherical manifolds (i. e., manifolds whose universal covers are contractible) which is a homeomorphism outside of some compact region. Then h is homotopic, relative to the complement of a larger compact set, to a homeomorphism.

This too has been verified in certain cases; most notably *see* [27], [7], [3], [10]. (For the K-theoretic implications *see* [25], [8].) It implies the (more refined) Novikov conjecture for $\pi_1 M_1$. The stable version, where one crosses with a Euclidean space, has been verified more often.

One can formulate (as has Quinn [16] for compact lattice quotients with action by isometries) an equivariant version of this conjecture for group actions all of whose fixed sets are disjoint unions of aspherical manifolds. (This is "the" equivariant analogue of aspherical.) The conjecture is false in the original form due to an equivariant topological Whitehead torsion obstruction which is related to Nil of the isotropy groups. Assuming equivariant topological simplicity makes the conjecture more reasonable. F. Connolly and T. Kosiewiski [4] have verified this for odd-order affine group actions on tori (satisfying certain mild conditions). Nonetheless, there are further obstructions and rigidity fails (*see* [28]) for linear involutions on tori. These involutions are stably equivalent.

A (true) equivariant Borel conjecture is still possible for group actions with no even order isotropy.

6. One has the following result:

THEOREM [12]. — *Let* f: M → W *be an equivariant homotopy equivalence, where* G *acts by isometries on a complete nonpositively curved manifold* M *and* f *is a homeomorphism outside of compacts; then* M *and* W *are stably* G-*homeomorphic.*

We prove this theorem by showing that such an f is topologically tangential rel the given identification of topological tangent bundles near infinity. We use a parametrized relative equivariant version of the solution to Siebenmann's conjecture ([11], [23]). Here is an absolute, nonequivariant (not quite parametrized enough) version of a critical lemma.

LEMMA. — *Given k and n there exists $\varepsilon > 0$ such that if $p_1 \colon E_1 \to B$ and $p_2 \colon E_2 \to B$ are n-dimensional vector bundles over a k-dimensional polyhedron B and $f \colon E_1 \to E_2$ is a fiber-preserving map such that* $\operatorname{diam} f^{-1} f(x) < \varepsilon$ *for each x in the unit disk bundle of E_1 then E_1 and E_2 are equivalent as topological $\mathbf{R}^n$-bundles.*

A twisted product model for the tangent bundle (*see* [6] and [4]) allows us to apply this lemma to prove tangentiality for f as in the statement of the theorem. Stable homeomorphism then follows from standard techniques. This gives a new proof of the Novikov conjecture for the fundamental groups of these manifolds and is the first verification of stable homeomorphism in cases of infinite volume.

7. The same method implies stable homeomorphism if there is a uniform bound on the diameters of the point inverse images $f^{-1}(w)$. Such results are very metric-dependent and we do not know any plausible extension to general $K(\pi, 1)$'s.

8. The results of paragraphs 3 and 6 have implications for the computations of L-groups of groups with torsion. The results of 6 imply information at the prime 2, but both imply the following:

COROLLARY. — *Let $\Gamma \subset G$ be a discrete subgroup of a real connected semisimple Lie group. Let $\Gamma' \subset \Gamma$ be a torsion-free normal subgroup of finite index and let $\pi = \Gamma/\Gamma'$. If K denotes a maximal compact subgroup of G then there is an injection*

$$\mathrm{KO}^*(K\backslash G/\Gamma') \otimes \mathbf{Z}\left[\frac{1}{2}\right] \to L_*(\Gamma) \otimes \mathbf{Z}\left[\frac{1}{2}\right].$$

One conjectures that this is an isomorphism. We believe that our methods extend to all real Lie groups. Yamasaki has recently proven the $\otimes \mathbf{Z}[1/2]$ isomorphism for Γ cocompact in amenable G. (Amenable G can also be attacked by our methods).

9. The method of paragraph 6 can be used to study h-cobordisms, and concordances. (One uses the equivariant smoothing theory of [14] and the topological s-cobordism theorems of [22], [17].) This leads to

THEOREM [12]. — *Let Γ act properly discontinuously and isometrically on a simply connected manifold W of nonpositive sectional curvature; then one has an injection of a sheaf homology group*

$$\mathbf{H}_i(W/\Gamma;\, \mathbf{K}(\Gamma_w)) \to K_i(\mathbf{Z}\Gamma)$$

(where Γ_w is the isotropy at w, which is well-defined up to conjugacy) into an algebraic K-group, for $i \leq 1$.

This map is not in general an isomorphism; for $\Gamma = \mathbf{Z} \times G$, G finite, the cokernel is the Nil group. This is related to the phenomenon discussed in paragraph 5.

10. Many of the equivariant topology ideas extend to a polyhedral context. An analog of the results of the previous two paragraphs is:

THEOREM [12]. — *Let Γ be a torsion-free subgroup of a real connected semisimple Lie group, or more generally, a group that acts freely on a complete simply connected manifold* *of nonpositive sectional curvature, and let* π *be any group. Then one has injections*

$$\mathbf{H}_i(\mathrm{B}\,\Gamma;\ \mathbf{K}(\mathbf{Z}\,\pi)) \to \mathrm{K}_i(\mathbf{Z}[\Gamma\times\pi]),\qquad i\leqq 1$$

$$\mathbf{H}_i(\mathrm{B}\,\Gamma;\ \mathbf{L}^{-\infty}(\mathbf{Z}\,\pi)) \to \mathrm{L}_i^{-\infty}(\mathbf{Z}\,[\Gamma\times\pi]),\quad \textit{all } i.$$

Notice that since these are sheaf homology groups they involve contributions from many dimensions. For nontrivial extensions $1\to\pi\to\mathrm{E}\to\Gamma\to 1$ one can prove analogous results.

11. Another generalization of paragraph 6 makes use of Waldhausen's algebraic K-theory of spaces [26]. The result in this context is the following:

Theorem [12]. — *Let* Γ *be a a torsion-free group as in the previous theorem. Then the* A-*theory assembly map splits. That is,*

$$\mathrm{B}\,\Gamma_+ \wedge \mathrm{A}(*) \to \mathrm{A}(\mathrm{B}\,\Gamma)$$

splits.

This has the following implication for ordinary K-theory.

Corollary. — *If* Γ *(perhaps with torsion) has a subgroup of finite index satisfying the hypothesis of the previous theorem, then*

$$\sum_{i\leqq n} \mathrm{H}_i(\mathrm{B}\,\Gamma;\ \mathrm{K}_{n-i}(\mathbf{Z})\otimes\mathbf{Q}) \to \mathrm{K}_n(\mathbf{Z}\,\Gamma)\otimes\mathbf{Q}$$

injects.

For Γ cocompact and torsion-free in an almost connected Lie group, injection has been improved to isomorphism [9]. Borel has computed the coefficients $\mathrm{K}_*(\mathbf{Z})\otimes\mathbf{Q}$ [2].

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References

[1] M. Atiyah and I. Singer, The index of elliptic operators, III, *Ann. of Math.*, 87, 1968, pp. 546-604.

[2] A. Borel, Stable real cohomology of arithmetic groups, *Ann. scient. Éc. Norm. Sup.*, (4), 7, 1974, pp. 235-272.

[3] S. Cappell, On homotopy invariance of higher signatures, *Invent. Math.*, 33, 1976, pp. 171-179.

[4] F. Connoly and T. Kosniewski, in preparation.

[5] T. Fack, Sur la conjecture de Novikov, *Contemp. Math.* (to appear).

[6] F. Farrell and W.-C. Hsiang, On Novikov's conjecture for nonpositively curved manifolds, I, *Ann. of Math.*, 113, 1981, pp. 197-209.

[7] F. Farrell and W.-C. Hsiang, Topological characterization of flat and almost flat Riemannian manifolds M^n, *Amer. J. of Math.*, 105, 1983, pp. 641-672.

[8] F. Farrell and L. Jones, K-theory and dynamics, I, *Ann. of Math.*, 124, 1986, pp. 531-569.

[9] F. Farrell and L. Jones, Algebraic K-theory of discrete subgroups of Lie groups, *Proc. Nat. Acad. Sci. U.S.A.*, 84, 1987, pp. 3095-3096.

[10] F. Farrell and L. Jones, *Topological rigidity for hyperbolic manifolds*, preprint.

[11] S. Ferry, Homotoping ε-maps to homeomorphisms, *Amer. J. of Math.*, 101, 1979, pp. 562-582.

[12] S. Ferry and S. Weinberger, in preparation.

[13] G. Kasparov, Equivariant KK-theory and the Novikov conjecture, *Invent. Math.*, 91, 1988, pp. 147-202.

[14] R. Lashof and H. Rothenberg, Equivariant smoothing theory, *Proc. Symp. Pure Math.*, 32, 1978, pp. 211-266.

[15] G. Lusztig, Novikov's higher signature and families of elliptic operators, *J. Diff. Geom.*, 7, 1972, pp. 229-252.

[16] F. QUINN, Applications of topology with control, *Proc. I.C.M.*, Berkeley, 1986 (to appear).

[17] F. QUINN, *Weakly stratified spaces*, preprint.

[18] J. ROSENBERG and S. WEINBERGER, *Higher G-indices (on smooth and Lipschitz manifolds) and applications*, submitted.

[19] J. ROSENBERG and S. WEINBERGER, *An equivariant Novikov conjecture*, in preparation, with an appendix by J. P. May.

[20] M. ROTHENBERG and S. WEINBERGER, Group actions and equivariant Lipschitz analysis, *Bull. Amer. Math. Soc.*, 17, 1987, pp. 109-112.

[21] R. SCHOEN and S.-T. YAU, Compact group actions and the topology of manifolds with non-positive curvature, *Topology*, 18, 1979, pp. 361-380.

[22] M. STEINBERGER, The equivariant topological s-cobordism theorem, *Invent. Math.*, 91, 1988, pp. 61-104.

[23] M. STEINBERGER and J. WEST, Approximation by equivariant homeomorphisms, *Trans. Amer. Math. Soc.*, 302, 1987, pp. 297-318.

[24] D. SULLIVAN and N. TELEMAN, An analytic proof of Novikov's theorem on rational Pontrjagin classes, *Publ. Math. Inst. Hautes Études Sci.*, 58, 1983, pp. 79-81.

[25] F. WALDHAUSEN, Algebraic K-theory of amalgamated free products, *Ann. of Math.*, 108, 1978, pp. 135-256.

[26] F. WALDHAUSEN, The Algebraic K-theory of topological spaces, I, *Proc. Symp. Pure Math.*, 32, 1978, pp. 35-60.

[27] C. T. C. WALL, *Surgery on Compact Manifolds*, Academic Press, 1970.

[28] S. WEINBERGER, Letter to Connolly.

[29] M. YAMASAKI, *Lattices in Lie groups and Seifert fibrations*, preprint.

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