A Gentle Introduction to HyperKähler Manifolds

Geometry/Physics RIT

Henry Denson

Fall 2023
Kähler Manifolds

First, some basic definitions:

- An *almost complex structure* is a linear map $I \in \text{End}(TX)$ such that $I^2 = -1$ on the tangent spaces, varying smoothly over the manifold.

- An almost complex structure is *integrable* or a *complex structure* if there is a covering of $X$ by open sets $U_\alpha$ with holomorphic diffeomorphisms $\phi_\alpha : U_\alpha \rightarrow V_\alpha \subset \mathbb{C}^n$. If $I$ is integrable, then $(X, I)$ is a complex manifold.

- A *Hermitian metric* on $(X, I)$ is a Riemannian metric satisfying

  $$g(v, w) = g(Iv, Iw).$$

- For $g$ a Hermitian metric on $(X, I)$, the *fundamental form* is the $(1,1)$-form $\omega$ such that

  $$\omega(v, w) = g(Iv, w).$$
Kähler Manifolds

**Definition**

If $g$ is a Hermitian metric on $(X,I)$, we say $g$ is a Kähler metric if

$$\nabla I = 0$$

where $\nabla$ is the Levi-Civita connection on $TX$ induced by $g$. If $g$ is a Kähler metric, then $(X,I,g)$ is a Kähler manifold.
Kähler Form

For \((X,I,g)\) a Kähler manifold, denote the fundamental 2-form \(\omega\), called the Kähler form.

- \(\omega\) is a \((1,1)\)-form, written locally as
  \[
  \omega = i \sum_{i,j=1}^{n} g_{ij}(z) dz_i \wedge d\bar{z}_j
  \]

- \(\omega\) is closed
- \(\omega\) is positive definite: the matrix \((g_{ij}(z))\) is a positive definite hermitian matrix for every \(z\). In particular, \(\omega\) is non degenerate.
For \((X, I, g)\) a Kähler manifold, denote the fundamental 2-form \(\omega\), called the Kähler form.

- \(\omega\) is a \((1,1)\)-form, written locally as

\[
\omega = i \sum_{i,j=1}^{n} g_{ij}(z) dz_i \wedge d\bar{z}_j
\]

- \(\omega\) is closed
- \(\omega\) is positive definite: the matrix \((g_{ij}(z))\) is a positive definite hermitian matrix for every \(z\). In particular, \(\omega\) is non degenerate.

**Observation:**
For \((X, I, g)\) Kähler, \(\omega\) is a closed nondegenerate 2-form on \(X\), making \(X\) a symplectic manifold.
Kähler manifolds are abundant:

- $\mathbb{C}^n$ is Kähler, with
  \[ \omega = i \sum_{j=1}^{n} dz_j \wedge d\bar{z}_j \]

- $\mathbb{CP}^n$ is Kähler, with
  \[ \omega = i \frac{(1 + |z|^2) \sum_{i=1}^{n} dz_i \wedge d\bar{z}_i - \sum_{i,j=1}^{n} \bar{z}_i z_j dz_i \wedge d\bar{z}_j}{(1 + |z|^2)^2} \]

- Every Riemann surface is Kähler, with a volume form $f(x, y) dx \wedge dy$ which can be written as $g(z) dz \wedge d\bar{z}$, which is a Kähler form.
HyperKähler Manifolds

HyperKähler manifolds extend the notion of Kähler manifolds to manifolds based on the algebra of quaternions.

**Definition**

A *hyperKähler manifold* is a tuple \((X, g, I, J, K)\), where \((X, g)\) is a Riemannian manifold equipped with complex structures \(I, J, \text{ and } K\) with

\[ I^2 = J^2 = K^2 = IJK = -1 \]

such that \(X\) is Kähler with respect to each structure.

Then, denote the associated Kähler forms by \(\omega_I, \omega_J, \omega_K\).
This is certainly a stronger condition than Kähler, which only requires a single automorphism $I$ satisfying $I^2 = -1$.

Consequentially, hyperKähler manifolds and metrics are far less abundant:

- A Kähler metric on $M$ may be modified to another by addition of a Hermitian form $\partial \bar{\partial} f$ for a small $C^\infty f$, giving an infinite dimensional space of Kähler metrics. In contrast, if a hyperKähler metric exists, then there is only a finite dimensional space.

- Kähler manifolds themselves are plentiful (as noted above), in particular, complex submanifolds of Kähler manifolds are Kähler. HyperKähler manifolds are relatively scarce, and constructing them can be challenging.
Definition

If \((X,I)\) is a complex manifold, a \((2,0)\)-form \(\Omega\) on \(X\) is a **holomorphic symplectic form** if \(d\Omega = 0\) and \(\Omega\) is nondegenerate (induces an isomorphism \(T^{(1,0)}X \rightarrow (T^{(1,0)}X)^*\)). For such a form, \((X, I, \Omega)\) is a holomorphic symplectic form.

Such manifolds have real dimension \(4n\) for some \(n\). Further, holomorphic symplectic manifolds locally admit Darboux coordinates \((p_1, \ldots, p_n, q_1, \ldots, q_n)\) such that

\[
\Omega = \sum_{i=1}^{n} dp_i \wedge dq_i.
\]
Theorem

Suppose $X$ is a manifold with real dimension $4n$, and holomorphic 2-form $\Omega$ such that $d\Omega = 0$ and

$$T_{\mathbb{C}}X = \ker\Omega \oplus \ker\bar{\Omega}.$$ 

Then there is a unique complex structure $I$ on $X$ for which $\Omega$ is a holomorphic symplectic form.
## The Berger Classification

<table>
<thead>
<tr>
<th>$\text{Hol}(g)$</th>
<th>$\text{dim}(M)$</th>
<th>Type of manifold</th>
<th>Comments</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\text{SO}(n)$</td>
<td>$n$</td>
<td>Orientable manifold</td>
<td>—</td>
</tr>
<tr>
<td>$\text{U}(n)$</td>
<td>$2n$</td>
<td>Kähler manifold</td>
<td>Kähler</td>
</tr>
<tr>
<td>$\text{SU}(n)$</td>
<td>$2n$</td>
<td>Calabi–Yau manifold</td>
<td>Ricci-flat, Kähler</td>
</tr>
<tr>
<td>$\text{Sp}(n) \cdot \text{Sp}(1)$</td>
<td>$4n$</td>
<td>Quaternion-Kähler manifold</td>
<td>Einstein</td>
</tr>
<tr>
<td>$\text{Sp}(n)$</td>
<td>$4n$</td>
<td>Hyperkähler manifold</td>
<td>Ricci-flat, Kähler</td>
</tr>
<tr>
<td>$\text{G}_2$</td>
<td>7</td>
<td>$\text{G}_2$ manifold</td>
<td>Ricci-flat</td>
</tr>
<tr>
<td>$\text{Spin}(7)$</td>
<td>8</td>
<td>$\text{Spin}(7)$ manifold</td>
<td>Ricci-flat</td>
</tr>
</tbody>
</table>

**Figure:** Berger Classification (via Wikipedia)
On a hyperKähler manifold, parallel translation preserves $I$, $J$, and $K$ (since they are covariantly constant) so $\text{Hol}(g) \subseteq O(4n)$ and $\text{Hol}(g) \subseteq GL(n, \mathbb{H})$ (quaternionic invertible matrices). The maximal intersection is $\text{Sp}(n)$, the group of $n \times n$ quaternionic unitary matrices.
HyperKähler Manifolds are Holomorphic Symplectic

Sp(n) is also an intersection of U(2n) and \( sp(2n, \mathbb{C}) \) (the linear transformations of \( \mathbb{C}^{2n} \) which preserves a nondegenerate skew form). then, a hyperKähler manifold is a complex manifold with holomorphic symplectic forms given by the Kähler 2-forms

\[ \Omega_I = \omega_J + i\omega_K \]

(and cyclic permutations).
HyperKähler Forms Determine Complex Structure

**Theorem**

If $X$ is hyperKähler, then

$$I = \omega_K^{-1} \omega_J$$

and analogous for $J$ and $K$.

Thinking of $\omega_J$ as a map $TX \to T^*X$ defined by $v \mapsto \omega_J(v, \cdot)$ and $\omega_K^{-1}$ as a map $T^*X \to TX$ with $\omega_K(v, \cdot) \mapsto v$, then $\omega_K^{-1} \omega_J : TX \to TX$ is the complex structure $I$.

**Proof:** That is, show $\omega_K(IV, \cdot) = \omega_J(v, \cdot)$. But this is immediate, as

$$\omega_K(IV, \cdot) = g(KIV, \cdot) = g(Jv, \cdot) = \omega_J(v, \cdot)$$
Forms on a Manifold Determine the Metric

HyperKähler Forms Determine the Metric

If X is hyperKähler, then

\[ g = -\omega_1 \omega^{-1}_K \omega_j \]

(and cyclic permutations)

Further, given a holomorphic symplectic forms can be used to find a hyperKähler metric on a manifold not assumed to be hyperKähler:

Symplectic Forms Give HyperKähler Metric

Suppose X is a manifold with symplectic forms \( \omega_1, \omega_2, \omega_3 \) such that

\[ -\omega_1 \omega_3^{-1} \omega_2 = -\omega_2 \omega_1^{-1} \omega_3 = -\omega_3 \omega_2^{-1} \omega_1 = g \]

and that \( g \) is a positive definite symmetric bilinear form. Then \( g \) is a hyperKähler metric on \( X \) with Kähler forms \( \omega_i \).
A HyperKähler Form is Kähler for $S^2$-Many Complex Structures

If a metric is Kähler for three complex structures, then it Kähler for $S^2$-many complex structures.

**Theorem**

Suppose $(X, g, I_1, I_2, I_3)$ is a hyperKähler manifold. Then for any $\vec{s} = (s_1, s_2, s_3) \in S^2 \subset \mathbb{R}^3$, define

$$l_{\vec{s}} = \sum_{i=1}^{3} s_i l_i \quad \omega_{\vec{s}} = \sum_{i=1}^{3} s_i \omega_i.$$ 

Then $(X, g, l_{\vec{s}})$ is a Kähler manifold with form $\omega_{\vec{s}}$.

Thus, specifying three complex structures is equivalent to specifying a whole family $l_{\vec{s}}, \vec{s} \in S^2$. 
Existence of HyperKähler Metrics

For a compact Kähler manifold with holomorphically trivial canonical bundle, by Calabi-Yau there is a Kähler metric with vanishing Ricci tensor. Then, by another theorem (Bochner), any holomorphic form on a compact Kähler manifold with zero Ricci tensor is covariantly constant. It follows from these results that any compact Kähler manifold with a holomorphic symplectic form, there is a hyperKähler metric on this manifold.

This is a very useful criterion for showing the existence of hyperKähler metrics on certain complex manifolds, notably the K3 surface. However, it only shows existence, and does not explicitly give the structure.
Constructing HyperKähler Manifolds

This is done, by obtaining some differential equation, and solving for the metric.

There are two main methods for constructing hyperKähler metrics:

- **Twistor Theory:** This essentially encodes the data for the metric in terms of holomorphic geometry. In a sense, this reduces the differential equations to the Cauchy-Riemann equation, but explicitly writing the metric is still challenging in general. Deriving global properties of metrics obtained by this method is very challenging, but the following method makes these properties more accessible.

- **HyperKähler Quotients:**
**Example—Flat Space**

$\mathbb{H}$ is a four dimensional real vector space under the natural identification $x_0 + x_1 i + x_2 j + x_3 k \mapsto (x_0, x_1, x_2, x_3)$, and so we may identify $T_p \mathbb{H} \cong \mathbb{H}$ to get almost complex structures $(I,J,K)$, and the norm $\|q\|^2 = q \bar{q}$ induces a metric $g$ on $\mathbb{H}$.

Then, left multiplication by $i, j, k$ give complex structures $I, J, K$ on $\mathbb{H}$. These are covariantly constant, so $g$ is Kähler for each, and thus $\mathbb{H}$ is hyperKähler.
References:

This presentation was based on

- *Moduli of Higgs Bundles* by Neitzke
- *HyperKähler Manifolds* by Hitchin