K and KK: Topology and Operator Algebras

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Abstract. The theme of this paper is to discuss the relationship between algebraic K-theory, especially as it commonly occurs in problems in geometric and algebraic topology, and operator algebras. We begin with a brief survey of a few elements of algebraic K-theory. Then we review what is known about the comparison between algebraic and topological K-theory and discuss a few situations in which the algebraic K-theory of operator algebras seems to be interesting and useful for applications. We conclude by discussing the relationship between algebraic K-homology (as developed by Pedersen and Weibel) and Kasparov's KK-theory, and suggesting a kind of algebraic KK-theory modeled on Kasparov's theory.

§0. INTRODUCTION

This paper is about operator algebras, K-theory, and KK-theory. However, since *topological* K-theory and the index-theoretic applications of KK have been treated at great length in other lectures at this Summer Institute (notably the talks of N. Higson and P. Baum) and in my previous survey [**39**], as well as in the excellent book [**3**] by B. Blackadar, I shall discuss here only *algebraic* K-theory and the ways it relates to operator algebras. Except for Theorems 2.2, 2.4 and 2.5 and for the material of §3, this paper will mostly be a survey of known, and in some cases fairly old, facts. However, I believe that this area deserves to be better known, and that in particular, the topological applications of *algebraic* K-theory should be brought to the attention of operator algebraists.

The organization of this paper is briefly as follows: §1 consists of a brief review of the basic definitions of algebraic K-theory, as well as a discussion of a few important typical problems in geometric topology in which it arises. This section is intended for operator algebraists

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who would like to learn a little more about topology; any true topologists or algebraists who might for some reason be looking at this paper can skip this section. §2 then discusses similarities and differences between algebraic and topological K-theory, and reviews a few situations where the *algebraic* K-theory of operator algebras has proved interesting. The discussion incorporates some new results on algebraic K-theory of commutative C*-algebras. This section also includes some remarks on how operator algebras might contribute to the study of the geometric problems discussed in §1. §3, which is new or at least partially new, discusses the Pedersen-Weibel approach to negative K-theory and algebraic K-homology, and how it relates to Kasparov's version of K-homology. There is also a discussion of "algebraic KK-theory" and how it might be used in algebra.

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§1. Algebraic K-Theory

The simplest (and probably most naive) way of defining algebraic K-theory is as a machine for associating certain abelian groups, customarily denoted $K_i(R)$, to a ring R (with unit).

§1.1. K_0 and the Wall finiteness obstruction. The first of these groups, K_0, has already become relatively familiar to operator algebraists, since it occurs for instance in the classification of AF-algebras ([**9**], [**3**, §7]). The group $K_0(R)$ is by definition the *Grothendieck group*, i.e., group of formal differences, of equivalence classes of finitely generated right (say—one has to fix a convention, and this will be most convenient for us later on) projective R-modules. Recall here that a *projective* module is by definition a direct summand in a free module, so that the sorts of modules we are talking about are those of the form pR^n, where $p \in M_n(R)$ is an *idempotent*, i.e., $p^2 = p$. (As a general rule, we shall save the word *projection* for *self-adjoint* idempotents in a C*-algebra.) The group operation in $K_0(R)$ comes from the operation $\oplus$ on modules, or alternatively, from assembling two idempotents p and

q into a block matrix

$$\begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix}.$$

The equivalence relation is *stable* isomorphism, that is,

$$[P] - [Q] = [P'] - [Q']$$

if and only if there exists another finitely generated projective module M such that

$$P \oplus Q' \oplus M \cong P' \oplus Q \oplus M.$$

More occurrences of K_0 in the theory of operator algebras will be discussed in the next section, but here we shall mention an important way K_0 comes into algebraic and geometric topology, which is by way of the *Wall finiteness obstruction*. This obstruction really takes its values not in K_0 itself but rather in the so-called *reduced group* $\tilde{K}_0$, defined to be the quotient of K_0 by the canonical cyclic subgroup generated by the rank-one free module (or in terms of the idempotent picture, generated by the trivial idempotent 1). The reduced K-group is mostly only interesting for rings with a well-defined notion of rank, i.e., for rings R such that $R^n \cong R^m$ (as right R-modules) only when $n = m$.

In the topological applications, the ring R will usually be the integral group ring $\mathbb{Z}\pi$ of the fundamental group π of a space X, which let's say for sake of argument has the homotopy type of a connected CW-complex. Note that since R has a surjective homomorphism to $\mathbb{Z}$, coming from the trivial representation of π, the ring R has a well-defined notion of rank. We assume also that X is *finitely dominated*, which means that from the point of view of homotopy theory, X is a retract of a *finite* complex. More exactly, there exists a finite CW-complex Y and there exist maps $i : X \to Y$ and $r : Y \to X$ such that $r \circ i$ is homotopic to the identity map on X. This may sound like quite a specialized hypothesis, but actually it comes up fairly often, for instance when X is a compact connected ANR (absolute neighborhood retract) [**20**, Corollary 6.2]. The problem considered by C. T. C. Wall [**52**] was to determine when X then has the homotopy type of a finite complex. It turns out that this is the case precisely when a certain obstruction vanishes in $\tilde{K}_0(\mathbb{Z}\pi_1(X))$. Here we are using the word *obstruction* in a technical sense common among topologists; it denotes an element of a certain (usually abelian) group, defined by the geometric data, that if non-zero *obstructs*, or prevents, the solution of the problem.

We define the obstruction in the following way. Let $C_*(X)$ denote the cellular chain complex of X (or, if X is not itself a CW-complex, of a complex homotopy-equivalent to X). To be accurate, we mean the chains for computing homology with local coefficients, as determined by the fundamental group, so these are really the chains on the universal cover $\tilde{X}$, viewed as modules for the group ring $R = \mathbb{Z}\pi_1(X)$. Note that since $\pi_1(X)$ acts freely on $\tilde{X}$, this is really a chain complex of free R-modules. Now the corresponding chain complex for the finite CW-complex Y is actually a complex of finitely generated free R-modules, since Y has only finitely many cells. And the existence of the homotopy retraction $r : Y \to X$ guarantees that $C_*(X)$ is chain-homotopy-equivalent to a direct summand in $C_*(Y)$, that is, to a complex of finitely generated ***projective*** R-modules. The Wall obstruction is the image in $\tilde{K}_0(R)$ of the alternating sum of these modules. Surely it would vanish if X were given a finite cell decomposition to begin with, since then $C_*(X)$ would consist of finitely generated free modules. But it would still vanish if X were only homotopy-equivalent to a finite complex, since C_* remains unchanged up to chain-homotopy-equivalence under homotopy-equivalences of spaces, and since alternating sums are preserved under chain-homotopy-equivalence of chain complexes, by the "Euler–Poincaré principle." This shows that the vanishing of Wall's obstruction is necessary for X to be homotopically finite, and Wall **[52]** in fact showed that the vanishing is also sufficient, by a simple inductive construction.

Wall obstructions in various guises occur quite frequently in geometric topology, for instance in problems about putting a boundary on a non-compact manifold **[44]**, or about determining which groups can be fundamental groups of spherical space-forms (quotients of a sphere by a freely acting finite group of homeomorphisms) **[47]**, **[28]**. A slight variant of Wall's argument, involving a different ring R, gives ***equivariant*** finiteness obstructions for a finitely dominated G–CW-complex to be equivariantly homotopy-equivalent to a finite G–CW-complex, when G is a finite group **[2]**. Similar $\tilde{K}_0$-obstructions also occur in other problems in equivariant homotopy theory **[32]**.

§1.2. K_1 and Whitehead torsion. The next group of K-theory, $K_1(R)$, naturally arises from the familiar problem of trying to put matrices into a nice canonical form by performing elementary row and column operations, in the "stable limit" when the size of the matrices tends to infinity. Since, as is well-known, these operations come from multiplication on the right and left by the so-called *elementary matrices*, it is natural to introduce the groups

$$GL(R) = \varinjlim GL(n, R),$$

where as usual $GL(n, R)$ denotes the invertible $n \times n$ matrices with entries in R, and

$$E(R) = \varinjlim E(n, R),$$

where $E(n, R)$ is the subgroup of $GL(n, R)$ generated by the elementary matrices

$$E_{ij}(\alpha) = 1 + \alpha e_{ij}, \quad \alpha \in R, \quad i \neq j.$$

The first main result of algebraic K-theory is the following famous result of Whitehead [**31**, Lemma 3.1]:

THEOREM (WHITEHEAD). *$E(R)$ is exactly the commutator subgroup of $GL(R)$, and is itself a perfect group (i.e., is its own commutator subgroup).*

Thus it's natural to define

$$K_1(R) = GL(R)/E(R) = H_1(GL(R), \mathbf{Z}).$$

This classifies the canonical forms for matrices in $GL(R)$ modulo elementary row and column operations (involving adding a multiple of one row or column to another).

Once again, this group arises rather naturally in topology, with R the group ring of the fundamental group of a space. As in the case of K_0, it is not really the whole group $K_1(\mathbf{Z}\pi)$ that is of interest, but rather the quotient of this group by the "trivial" subgroup generated by $GL(1, \mathbf{Z}) = \{\pm 1\}$ and by π itself. This quotient

$$Wh(\pi) = K_1(\mathbf{Z}\pi)/(\{\pm 1\} \times \pi)$$

is called the *Whitehead group*.

The topological invariant that takes its values in $Wh(\pi)$ is called *Whitehead torsion*, and is explained in great detail in the books [**7**], [**41**]. Just to illustrate its importance, let us consider the problem of classification of (compact) manifolds up to homeomorphism or diffeomorphism, depending on whether one is in the topological or smooth category. When one is given two manifolds M and M' that one suspects are really the same, one might try to construct directly a homeomorphism or diffeomorphism between them, but this is usually a hopeless task. Therefore one usually tries instead to construct an *h-cobordism* between them, which is a compact manifold W with boundary, having the original two manifolds as its two boundary components, and such that the inclusions of both M and M' into W are homotopy equivalences. When everything is simply connected, the celebrated *h-cobordism theorem* of Smale (see, e.g. [**30**]) says that (in high dimensions) such an

h-cobordism is necessarily a product, i.e. that $W \cong M \times [0,1]$. Of course this implies the desired conclusion that $M \cong M'$, and even gives a bit more.

In the non-simply connected case, the connection with K-theory is given by:

S-COBORDISM THEOREM [41, §§6.19–6.21]. *The (homeomorphism or diffeomorphism) classes of h-cobordisms having M as one boundary component are in one-to-one correspondence with the elements of the group $Wh(\pi_1(M))$, with the trivial cobordism $M \times [0,1]$ corresponding to the 0-element of the group, provided that* $\dim M \geq 5$.

Thus the computation of the group $Wh(\pi)$ for various groups π is of significant geometric interest. The Whitehead group vanishes when π is a free or free abelian group, but can be non-zero for rather simple finite groups. The first non-trivial example, which will be rather instructive for our purposes later, is the case where π is cyclic of order 5, say with generator x. Then one finds that

$$(1 - x^2 - x^3) \times (1 - x - x^4) = 1 \in \mathbb{Z}[x]/(x^5 - 1),$$

so that $1 - x^2 - x^3$ defines an element of $GL(1, \mathbb{Z}\pi)$ and thus an element of $Wh(\pi)$. We claim this element is non-zero, in fact of infinite order. The simplest way to prove this is to observe that π has a complex representation σ in which x goes to $e^{2\pi i/5}$, and under this representation,

$$(1 - x^2 - x^3) \mapsto 1 - 2\cos(4\pi/5),$$

which has absolute value greater than 1. Now σ induces a map of rings $\sigma : \mathbb{Z}\pi \to \mathbb{C}$, which in turn maps $K_1(\mathbb{Z}\pi)$ to $K_1(\mathbb{C})$. However, since the determinant of any elementary matrix is one, we have a well-defined determinant

$$\det : K_1(\mathbb{C}) \to \mathbb{C}^{\times},$$

which in fact is an isomorphism. And under the induced map

$$\det \circ \sigma : K_1(\mathbb{Z}\pi) \to \mathbb{C}^{\times}, \tag{1.1}$$

the subgroup $\{\pm 1\} \times \pi$ must go to the unit circle, since σ is a *unitary* representation and unitary matrices have determinant one. Since no non-zero power of $1 - 2\cos(4\pi/5)$ has absolute value one, we must have an element in the Whitehead group of infinite order, as claimed.

§1.3. **K_2 and pseudo-isotopies.** The next of the K-groups, $K_2(R)$, measures some of the complexity of the group $E(R)$ of elementary matrices. Alternatively, just as $K_1(R)$ may be defined as $H_1(GL(R), \mathbf{Z})$, $K_2(R)$ is closely related to $H_2(GL(R), \mathbf{Z})$. It is not exactly this homology group, since we don't want to duplicate the information contained in K_1, and the group extension

$$1 \to E(R) \to GL(R) \to K_1(R) \to 1$$

gives us a map $H_2(GL(R)) \to H_2(K_1(R))$. The "correct" invariant, in the sense that it fits into a long exact sequence with K_0 and K_1, turns out to be

$$K_2(R) \stackrel{\text{def}}{=} H_2(E(R), \mathbf{Z}),$$

which of course maps to $H_2(GL(R))$. The group K_2 has a somewhat more concrete description, which is nicely described in [**31**]. Recall that E(R) was generated by the elementary matrices $E_{ij}(\alpha)$. These satisfy certain "obvious" relations such as

$$E_{ij}(\alpha) \cdot E_{ij}(\beta) = E_{ij}(\alpha + \beta)$$

and

$$[E_{ij}(\alpha), E_{jk}(\beta)] = E_{ik}(\alpha \cdot \beta), \quad i \neq k.$$

Thus one can form a group, usually denoted $St(R)$, the *Steinberg group*, with generators $X_{ij}(\alpha)$, $i \neq j$, $\alpha \in R$, subject to these "obvious relations" satisfied by the $E_{ij}(\alpha)$'s, and by construction, this group surjects onto $E(R)$. The kernel of this surjection, which measures the "non-obvious" relations in $E(R)$, turns out to be the same as the $K_2(R)$ defined as $H_2(E(R))$ [**31**, Theorems 5.1 and 5.10]. Even though the definition of K_2 is substantially more complicated than that of K_1, this group has also shown up in topology, notably in the work of Hatcher and Wagoner [**16**] (however, see the correction in the review, MR **51 #4278**) on *pseudo-isotopies*. A pseudo-isotopy of a manifold M is by defintion a diffeomorphism of $M \times [0, 1]$ which restricts to the identity on $M \times \{0\}$ and on $\partial M \times [0, 1]$. The work of Hatcher and Wagoner shows that K_2 of the group ring of the fundamental group of M comes up when one calculates the set of connected components of the space of pseudo-isotopies of a high-dimensional manifold. Without going into details, suffice it to say that this then affects the structure of the diffeomorphism group for any (high-dimensional) manifold.

§1.4. Higher K-theory. With the possible exception of $K_3(R)$, which is sometimes defined as $H_3(St(R), \mathbf{Z})$ [**14**], there seems to be no way of directly defining $K_i(R)$ for $i > 2$ in a way similar to and compatible with the definitions of K_0, K_1, and K_2. Of course, the words "compatible with" must be explained—we would like, when I is a two-sided ideal in R, to have *relative K-groups* $K_i(R, I)$ and a long exact sequence

$$\cdots \to K_{i+1}(R/I) \to K_i(R, I) \to K_i(R) \\ \to K_i(R/I) \to K_{i-1}(R, I) \to \cdots.$$

However, there are several ways of defining an infinite series of K-groups so that one gets such a long exact sequence and various other desirable properties. The most important of these are due to Quillen and are usually known as the "plus-construction" and "Q-construction." The plus-construction is the easiest to define—the idea is to start with $BGL(R)$, the classifying space of the (discrete) group $GL(R)$, and to change it to a new space $BGL(R)^+$ with the same homology groups but with *abelian* fundamental group. This is done by attaching 2-cells to kill the commutator subgroup $E(R)$ of

$$GL(R) = \pi_1(BGL(R)),$$

and then attaching additional 3-cells to correct the H_2. For a fairly readable exposition, see [**1**, pp. 84–88]. Then one defines

$$K_i(R) = \pi_i(BGL(R)^+) \qquad \text{for} \quad i \geq 1,$$

and of course this is compatible with our previous definition of $K_1(R)$ as $H_1(GL(R))$. In fact, the Hurewicz homomorphism gives a functorial map

$$K_i(R) = \pi_i(BGL(R)^+) \to H_i(BGL(R)^+, \mathbf{Z}) \\ = H_i(BGL(R), \mathbf{Z}) = H_i(GL(R), \mathbf{Z}),$$

which is the same as the map we discussed earlier for K_2, though this requires some proof. The long exact sequence for $I \triangleleft R$ now arises as the long exact sequence for the fibration obtained by taking the homotopy fiber of the natural map

$$BGL(R)^+ \to BGL(R/I)^+.$$

For many purposes, however, it is better to use the more categorical Q-construction of Quillen or various variants thereof, due to Gersten, Segal, May, Thomason, Waldhausen, and others. Since this material is very technical and an excellent bibliography is available in [**27**, §§D05, D10, D15, and D30], we shall only give here a crude idea of what this is all about, sufficient for our purposes in §3 below. The key idea is that instead of thinking of K-theory as providing an infinite series of groups $K_i(R)$, it is better to think of it as providing a single, more complicated object attached to the ring R. This object is a generalized homology theory, which we shall denote by $\mathbf{K}(R)$. (To be technical, we usually use the associated reduced homology theory on spaces with basepoint, which we denote by $\tilde{\mathbf{K}}(R)$.) The groups $K_i(R)$ are then obtained by evaluating this homology theory on spheres:

$$K_i(R) \stackrel{\text{def}}{=} \tilde{\mathbf{K}}(R)(S^i).$$

Thus from this more sophisticated point of view, K-theory should really be a functor from rings to homology theories.

The dual cohomology theory $\mathbf{K}(R)^*$ has been given a convenient concrete description by Karoubi in [**24**, Ch. III]—the group $\mathbf{K}(R)^0(X)$ is the set of formal differences of equivalence classes (under a relation we'll describe in a minute) of diagrams

$$\begin{array}{ccc} E & & \\ \big\downarrow{\scriptstyle \pi} & & \\ Y & \xrightarrow{\;f\;} & X, \end{array}$$

where $E \xrightarrow{\pi} Y$ is a flat R-bundle over Y, that is a bundle with locally constant transition functions, whose fibers are finitely generated projective R-modules, and f is an acyclic map, meaning roughly a homology equivalence, or more exactly, a map such that the homotopy fiber over any point of X has vanishing reduced homology. For instance, if $X = S^n$ for $n > 1$, Y will be a homology n-sphere, generally with large fundamental group. The equivalence relation is basically that two such diagrams

$$E \xrightarrow{\pi} Y \xrightarrow{f} X$$

and

$$E' \xrightarrow{\pi'} Y' \xrightarrow{f'} X$$

are considered equivalent if there is a third such diagram

$$E_1 \xrightarrow{\pi_1} Y_1 \xrightarrow{f_1} X$$

together with a commutative diagram

$$\begin{array}{ccc} Y' & \xrightarrow{f'} & X \\ {\scriptstyle \sigma'}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{id}} \\ Y_1 & \xrightarrow{f_1} & X \\ {\scriptstyle \sigma}\big\uparrow & & \big\uparrow{\scriptstyle \mathrm{id}} \\ Y & \xrightarrow{f} & X, \end{array}$$

such that

$$E \cong \sigma^* E_1, \quad \text{and} \quad E' \cong \sigma'^* E_1.$$

The group operation comes from the Whitney sum of R-bundles, and there is also a cup-product operation that comes from the tensor product operation on bundles.

Now by the machinery of modern algebraic topology [**1**, Ch. 1], homology theories are representable, via gadgets called ***spectra***, which in fact are often identified with homology theories. (A spectrum is really an infinite sequence $\{X_i\}_{i=0}^{\infty}$ of spaces, together with maps $X_i \to \Omega X_{i+1}$. These maps of spaces induce maps of homotopy groups

$$\pi_n(X_i) \to \pi_{n+1}(X_{i+1}).$$

The homotopy groups of the spectrum are by definition

$$\pi_n(\mathbf{X}) = \varinjlim \pi_{n+i}(X_i).$$

Note that if all the maps $X_i \to \Omega X_{i+1}$ are equivalences, then these are just the homotopy groups of X_0.) For those who prefer to think of single topological spaces (with extra structure), it is the same to deal with what are called ***infinite loop spaces***. For a generalized homology theory $\mathbf{E}$, the associated space $\mathbf{E}_0$ is characterized by the property that the dual *co*homology theory is computed by

$$\tilde{\mathbf{E}}^0(X) = [X, \mathbf{E}_0],$$

where [,] denotes the set of (based) homotopy classes of maps. Thus the coefficient groups of the homology theory are just the homotopy groups of the space $\mathbf{E}_0$, at least if we stick to positive degrees. In the case of algebraic K-theory, the infinite loop space $\mathbf{K}(R)_0$ is just Quillen's

$K_0(R) \times BGL(R)^+$. (We need the factor of K_0 since a connected space such as $BGL(R)^+$ won't have the right π_0.)

In fact, one can go one step further in generality. Instead of thinking of K-theory as a functor from rings to spectra, one can replace the ring R by a suitable category of its modules (since that's all that's needed anyway), and then generalize to a larger class of categories. Thomason's version of K-theory **[49]**, for instance, is based on a functor from *symmetric monoidal categories* to spectra. Symmetric monoidal categories are categories with a suitable $\oplus$ operation; the prototype is the category of finitely generated projective R-modules and isomorphisms, and it is this category that gives rise to Quillen's K-theory $\mathbf{K}(R)$. Later in this paper we shall discuss a more concrete realization of this homology theory, due to Pedersen and Weibel **[36]**.

§1.5. Lower K-theory. It is also possible, and in fact often desirable, to extend the infinite sequence of algebraic K-groups in the other direction, i.e., in the direction of negative index. To motivate this, note that the study of the Wall obstruction and of Whitehead torsion gives us a good geometrical reason for wanting to compute K_0 and K_1 for a group ring $\mathbf{Z}\pi$. If we replace the group π by the group $\pi \times \mathbf{Z}^n$, which corresponds geometrically to taking the product of our original space or manifold with an n-torus (the classifying space for $\mathbf{Z}^n$), then we are suddenly faced with computing

$$K_i(\mathbf{Z}(\pi \times \mathbf{Z}^n)) = K_i(\mathbf{Z}\pi[x_1, x_1^{-1}, \cdots, x_n, x_n^{-1}]), \quad i = 0, 1.$$

Hence we need to be able to compute K-theory of Laurent polynomial rings.

If F is any functor, e.g. K_i, from rings to abelian groups, Bass has defined a new functor LF by

$$LF(R) = \operatorname{coker}(F(R[x]) \oplus F(R[x^{-1}]) \to F(R[x, x^{-1}])).$$

Thus $L^nF(R)$ arises in computing $F(R[x_1, x_1^{-1}, \cdots, x_n, x_n^{-1}])$. It turns out that

$$L^nK_i \cong K_{i-n}, \qquad \text{provided} \quad i \geq n \geq 0,$$

and thus it makes sense to define $K_{-n} = L^nK_0$ for $n > 0$. As we have seen, these functors arise in computing Whitehead groups, so there is particular interest in $K_{-n}(\mathbf{Z}\pi)$. A fair amount about these groups is known; for instance, if π is finite, they vanish for $n > 1$ and are explicitly computed for $n = 1$ **[6]**. The K-theory long exact sequence for $I \triangleleft R$ extends to a two-sided long exact sequence involving the negative

K-groups. In fact, it is also possible to describe negative K-theory from the point of view of spectra, since a spectrum (unlike a single topological space) can have negative homotopy groups. It is known that there is a spectrum **[51]** whose positive and negative homotopy groups capture both higher *and* lower K-theory.

It is perhaps worth mentioning one more geometric application of negative K-theory, which will implicitly play a role in §3 below. This involves the concept, which has been increasingly important in geometric topology, of ***topology with control.*** For simplicity, we consider one of the simplest illustrations of this idea, as developed in **[34]**. Namely, we consider h-cobordisms W between two manifolds M and M' as before, but this time with a ***control map*** $p : W \to \mathbf{R}^k$. The control map p is required to be proper, and its restriction to either M or M' is required to be surjective. Of course, none of the manifolds W, M, or M' will be compact. We use the map p to measure "distances," that is, we define

$$\text{"dist"}\ (x,\, y) = |p(x) - p(y)|.$$

Then we require that W have ***bounded fundamental group***, i.e. that there be a fixed constant C such that for every $x,\ y \in W$, and for every homotopy class of paths from x to y, there be a representative for the class of length $< |p(x) - p(y)| + C$, and similarly that null-homotopic loops be contractible within a set of diameter $< C +$ the diameter of the loop. The result of **[34]** then gives a necessary and sufficient condition for a "bounded" h-cobordism W to have a bounded product structure, in terms of an invariant in $\tilde{K}_{-k+1}(\mathbf{Z}\pi_1(W))$, provided that $\dim W > 5$. If $k = 0$, this reduces to the usual s-cobordism theorem.

Of course, one way a bounded h-cobordism can arise is from a compact h-cobordism W' with fundamental group $\pi \times \mathbf{Z}^k$. The projection of the fundamental group onto $\mathbf{Z}^k$ induces a map $p' : W' \to T^k$, and taking coverings, we get a map

$$p = \widetilde{p'} : \widetilde{W'} = W \to \tilde{T}^k = \mathbf{R}^k.$$

Theorem 1.7 of **[34]** identifies the associated invariant as the image of the original Whitehead torsion in $Wh(\pi \times \mathbf{Z}^k)$. But there are controlled non-compact problems that do not arise so simply from compact situations.

§2. Algebraic K-Theory and Operator Algebras

§2.1. Algebraic versus topological K-theory.

Now we discuss the special situation where the ring R is a unital Banach algebra, say a C^*-algebra. In this case, there's another kind of K-theory attached to

R, namely *topological* K-theory, as developed, say, in **[3]**. One way of viewing this is by way of the fact that the Banach algebra structure makes $GL(R)$ into a *topological* group, so that it has a classifying space as such, $BGL(R)^{\text{top}}$. This is to be distinguished from the classifying space we considered earlier, which we now write for emphasis as $BGL(R)^{\text{disc}}$. By the Bott periodicity theorem, it turns out that $BGL(R)^{\text{top}}$ is already an infinite loop space, so that the plus-construction does nothing to it. Its homotopy groups are by definition the topological K-groups; they turn out to be periodic of period 8 in the real case, 2 in the complex case. On the other hand, the (identity) map

$$GL(R)^{\text{disc}} \to GL(R)^{\text{top}}$$

will induce a map of classifying spaces. Applying the plus-construction gives a map of infinite loop spaces, or equivalently, a natural transformation of homology theories

$$\mathbf{K}^{\text{alg}}(R) \to \mathbf{K}^{\text{top}}(R)$$

from algebraic to topological K-theory. Taking homotopy groups, this can be viewed on the level of individual K-groups as a map $K_i^{\text{alg}}(R) \to K_i^{\text{top}}(R)$. Though we've only defined it for $i \geq 0$, it's not hard to extend it to the negative K-groups as well. (Note that for topological K-theory, these merely repeat the positive K-groups, by Bott periodicity.) For $i = 0$, the map is always an isomorphism. However, it's usually not an isomorphism for other values of i, and it's an important problem to learn as much about it as possible. Part of the reason is that topological K-theory is by its nature related to index theory and the analytic theory of operator algebras, and we would like as much as possible to relate this to some of the algebraic and topolgical problems discussed above in §1.

We shall quickly survey now the known results about the comparison problem between algebraic and topological K-theory. When $i = 1$,

$$K_1^{\text{alg}}(R) = GL(R)/E(R), \quad \text{while} \quad K_1^{\text{top}}(R) = GL(R)/GL(R)^0,$$

and since $E(R)$ lies in the connected component $GL(R)^0$ of $GL(R)$, the natural map is surjective. However, it's not even injective for $\mathbf{C}$, since $K_1^{\text{alg}}(\mathbf{C}) \cong \mathbf{C}^\times$ via the determinant, whereas $K_1^{\text{top}}(\mathbf{C}) = K_{\text{top}}^{-1}(pt) = 0$. As far is K_2 is concerned, it's known **[31**, pp. 59–62] that for commutative C*-algebras $C(X)$, the image of the natural map

$$\begin{aligned} K_2^{\text{alg}}(C(X)) \to K_2^{\text{top}}(C(X)) &\cong K_0(C(X)) \text{ (by Bott periodicity)} \\ &= K^0(X) \end{aligned}$$

is (for X connected) the reduced K-group $\tilde{K}^0(X)$. For the higher K-groups, there are results only for special C*-algebras. Just about the best theorem available is a result of Higson [**19**, Theorem 5.4.1], which says that the natural map from algebraic to topological K-theory is an isomorphism in all degrees for C*-algebras of the form $A \otimes \mathcal{Q}(B)$, where A is unital, B is σ-unital, and $\mathcal{Q}$ denotes the generalized Calkin algebra construction, i.e.

$$\mathcal{Q}(B) = \mathcal{M}(B \otimes \mathcal{K})/(B \otimes \mathcal{K}).$$

The tensor products here are all spatial C*-tensor products.

As far as negative K-theory is concerned, Karoubi [**22**, Théorème 3.6] showed that the natural map $K_{-1}^{\mathrm{alg}}(A) \to K_{-1}^{\mathrm{top}}(A)$ is surjective for any unital C*-algebra. There is no comparable result for K_{-2}, as one can see by looking at the simplest example $A = \mathbb{C}$.

To better understand the nature of the comparison problem, it is important to keep in mind that topological K-theory has two key properties which are not shared by algebraic K-theory in general. These are *homotopy invariance* and *excision*. The first of these says that if A is a Banach algebra, the map $C([0,\ 1],\ A) \to A$ given by evaluation at any point $t \in [0,\ 1]$ gives an isomorphism in K-theory, independent of t. Excision says that the relative groups $K_i^{\mathrm{top}}(A, I)$ associated to $I \triangleleft A$ are independent of A, and can be defined purely in terms of the ideal I. The analogue of a continuous function in algebra is a polynomial, so the analogue of homotopy invariance in algebraic K-theory should concern the evaluation map

$$K_i(R[x]) \to K_i(R)$$

coming from setting $x = 0$. The kernel of this map is by definition Bass' $NK_i(R)$, sometimes called $\mathrm{Nil}_{i-1}(R)$, and though this is 0 for regular rings, it is often non-zero. In fact, when it's non-zero, it's not even finitely generated [**37**]. And as pointed out in [**31**, Remark 4, pp. 34-35], excision and the Mayer-Vietoris property also fail for algebraic K-theory.

These drawbacks led Karoubi and Villamayor[**25**] to introduce a theory, often denoted $KV_*(R)$, with properties intermediate between algebraic and topological K-theory. The Karoubi-Villamayor groups are defined for any ring R (without a topology), and satisfy algebraic homotopy-invariance and excision. When R is regular, though not in general, $KV_*(R) \cong K_*(R)$. And for any ring R, $KV_i(R) \cong K_i(R)$ for $i \leq 0$. When R is a unital Banach algebra, the map $K_*^{\mathrm{alg}}(R) \to K_*^{\mathrm{top}}(R)$ factors through $KV_*(R)$. Also, for general rings, there's a "Gersten-Anderson spectral sequence":

$$E^1_{pq} = N^p K_q(R) \Longrightarrow KV_{p+q}(R), \quad p \geq 0,\ q \geq 1.$$

For details on all of this, see section D25 of [**27**]. It is possible that this machinery will yet prove of value in studying the map from algebraic to topological K-theory. For instance, Higson [**19**, Theorem 6.3.14] has shown that the map $KV_* \to K_*^{\text{top}}$ is an isomorphism for stable C*-algebras. This provides further evidence for the conjecture [**22**] of Karoubi that the map $K_*^{\text{alg}} \to K_*^{\text{top}}$ is an isomorphism for such algebras. Of course, if $A \cong A \otimes \mathcal{K}$ is a stable C*-algebra, then A certainly is not unital, so the algebraic K-theory must be suitably interpreted. One possibility is to use the relative groups $K_*(\mathcal{M}(A), A)$, for which the conjecture is true by a corollary of the theorem of Higson cited previously. Another possibility, for which the conjecture has *also* been proved by Higson [**19**, Ch. IV] in the case of K_2, is to use a variant of the usual definition of K-theory of unital rings, taking into the fact that A is almost unital in the sense that any element is a linear combination of products, in fact in a semi-canonical way.

One final point about comparison between algebraic and topological K-theory is that there is substantial evidence that the difference between the two tends to disappear when one uses K-theory with finite coefficients. To make this more precise, recall that algebraic K-theory assigns a homology theory (or a spectrum) $\mathbf{K}(R)$ to a ring R, and we can then introduce coefficients into the homology theory (by smashing with a Moore space). Thus for any positive integer k one gets a cofiber sequence of spectra

$$\mathbf{K}(R) \xrightarrow{k} \mathbf{K}(R) \to \mathbf{K}(R; \mathbb{Z}/k).$$

K-theory with $\mathbb{Z}/k$-coefficients is by definition given by the homotopy groups of the spectrum on the right, and is related to usual K-theory by the Bockstein long exact sequence coming from the homotopy exact sequence of the (co)fibration. This exact sequence can be rephrased as a universal coefficient exact sequence:

$$(2.1) \quad 0 \to K_i(R) \otimes \mathbb{Z}/k \to K_i(R; \mathbb{Z}/k) \to \operatorname{Tor}(K_{i-1}(R), \mathbb{Z}/k) \to 0,$$

which splits under favorable circumstances (k odd or divisible by 4).

Intuitively, the advantage of introducing finite coefficients is that algebraic K-groups of algebras over $\mathbb{C}$ tend to contain a lot of "divisible junk" which is killed with finite coefficients. This procedure is so efficient that K-theory with finite coefficients has excision [**53**], [**23**, Appendice 1], provided one restricts attention to $\mathbf{Q}$-algebras, just as for topological K-theory. Karoubi [**23**, Théorème 2.5] has shown that for any Banach algebra A, the natural map

$$K_i^{\mathrm{alg}}(A;\, \mathbf{Z}/k) \to K_i^{\mathrm{top}}(A;\, \mathbf{Z}/k)$$

is an isomorphism for $i = 1$ (this follows from the fact that elements of $GL(A)^0$ close to the identity have k-th roots), and is surjective for $i > 1$. It has also been shown (see [**12**], [**38**]—the two authors were working independently at the same time, both using methods developed by Suslin [**46**]) that the map

$$K_i^{\mathrm{alg}}(A;\, \mathbf{Z}/k) \to K_i^{\mathrm{top}}(A;\, \mathbf{Z}/k)$$

is an isomorphism for all $i \geq 1$ when A is a commutative Banach algebra with unit. This work is closely related to an analogous theorem for algebraic varieties or schemes [**50**], the difference being that Thomason formally inverts the "Bott periodicity element" in K_2. Karoubi [**23**, Théorème 2.5] and Higson [**19**, Theorem 6.4.1] have proved the corresponding result for stable C*-algebras. The stability plays the role of inverting the Bott element, since $K_{-2}(\mathcal{K};\, \mathbf{Z}/k)$ (one can even replace the compact operators by a Schatten ideal) contains an "inverse Bott element" [**23**, Proposition 3.5]. Thus there seems to be good evidence for the conjecture that for any C*-algebra, algebraic and topological K-theory with finite coefficients coincide in positive degrees.

However, we are still left with the rather puzzling issue of the lower K-groups. This contributes to some of the confusion in the literature, since many authors, to avoid a definition of $K_0(R;\, \mathbf{Z}/k)$ which will involve $\mathrm{Tor}(K_{-1}(R),\, \mathbf{Z}/k)$, give a somewhat *ad hoc* definition of this group that doesn't satisfy the universal coefficient theorem (2.1) as we've stated it and thus can't match up well with $K_0^{\mathrm{top}}(R;\, \mathbf{Z}/k)$.

In fact, the lower K-groups with finite coefficients are quite interesting for operator algebras, since they provide a measure of how "stable" the algebra is. The following improvement in the Fischer-Prasolov Theorem completely answers the question of what these groups are for *commutative* C*-algebras.

THEOREM 2.2. *The functor*

$$F^{-*} : X \mapsto K_*^{\mathrm{alg}}(C_0(X);\, \mathbf{Z}/k)$$

is a generalized cohomology theory on the category of (second-countable) locally compact spaces and proper maps, and in fact coincides with *connective K-theory with $\mathbf{Z}/k$-coefficients:*

$$K_i^{\text{alg}}(C_0(X);\, \mathbf{Z}/k) \cong \widetilde{bu}^{-i}(X^+;\, \mathbf{Z}/k), \quad i \in \mathbf{Z}.$$

(Here X^+ denotes the one-point compactification of X, not the Quillen plus-construction.)

The same holds for real-valued functions with bu replaced by bo.

PROOF: Since K-theory with finite coefficients satisfies (algebraic) excision, it will necessarily give a cohomology theory (on the category of second-countable locally compact spaces and ***proper*** maps) once we prove that it is homotopy-invariant. The reason is that if Y is a closed subspace of X, the short exact sequence of algebras

$$0 \to C_0(X \smallsetminus Y) \to C_0(X) \to C_0(Y) \to 0$$

will by algebraic excision give a long exact sequence of K-groups in which the relative groups only depend on the complement of Y in X, so that (topological) excision is satisfied. The proof of homotopy invariance will show at the same time that the theory is (countably) additive, i.e., that

$$F^*(\coprod X_i) \cong \bigoplus F^*(X_i).$$

This is enough to determine the behavior of the theory under inverse limits of compact spaces—it must have the property that

$$F^*(\varprojlim X_i) \cong \varinjlim F^*(X_i),$$

and since arbitrary compact metric spaces are inverse limits of finite complexes, we can determine the theory completely by knowing what spectrum gives it on finite complexes. (For the mechanics of this sort of reasoning, see **[43]**, especially §5.) Now the homotopy groups of this spectrum are just those of $\mathbf{K}^{\text{alg}}(\mathbf{C};\, \mathbf{Z}/k)$, since by definition this is the spectrum that computes the theory when X is a point. And since $\mathbf{C}$ is a field, its lower K-groups all vanish, so the spectrum is connective, i.e., has vanishing negative homotopy groups. Finally, by **[46]**, the map from algebraic to topological K-theory induces an isomorphism of non-negative K-groups of $\mathbf{C}$ with finite coefficients, which says just that $\mathbf{K}^{\text{alg}}(\mathbf{C};\, \mathbf{Z}/k)$ is the connective K-theory spectrum $\mathbf{bu}(\mathbf{Z}/k)$. Now we can see that this is also the spectrum that computes F for general finite complexes. Indeed, suppose the one-point compactification of X is a finite complex.

The natural transformation from algebraic to topological K-theory will induce (by the result of Fischer and Prasolov) an isomorphism

$$K_i^{\mathrm{alg}}(C_0(X);\ \mathbb{Z}/k) \to K_i^{\mathrm{top}}(C_0(X);\ \mathbb{Z}/k)$$
$$\cong \tilde{K}^{-i}(X^+;\ \mathbb{Z}/k) \cong \widetilde{bu}^{-i}(X^+;\ \mathbb{Z}/k), \quad i \geq 0,$$

since connective and ordinary K-theory coincide in *negative* degrees. But connective K-theory is characterized (see [**1**, p. 145]) as the *unique* connective cohomology theory equipped with a natural transformation to topological K-theory which gives an isomorphism in non-positive degrees.

Thus we've reduced the theorem to proving homotopy-invariance. This and more is provided by the Fischer-Prasolov Theorem for the higher K-groups, so it's enough to prove homotopy-invariance of the lower K-groups with finite coefficients. Since these are in turn determined by the integral lower K-groups (via (2.1)), which in turn are direct summands in the K_0 of Laurent polynomial rings, the theorem will follow from the following.

LEMMA 2.3. *Let A be a separable commutative (unital) C^*-algebra and let r be a positive integer and $\alpha : C([0,1], A) \to A$ be "evaluation at zero." Then the map*

$$\alpha_* : K_0(C([0,1], A)[x_1^{\pm}, \cdots, x_r^{\pm}]) \to K_0(A[x_1^{\pm}, \cdots, x_r^{\pm}])$$

is injective.

PROOF: The proof will be based on the homotopy lifting property for a suitable fibration, so it is necessary first to set up what might seem like peculiar notation. However, a little thought shows that the essence of the problem is to understand a little more about the Laurent polynomial ring

$$R_r = \mathbb{C}[x_1^{\pm}, \cdots, x_r^{\pm}]$$

and the group $GL(R_r)$. Note that since the negative K-groups and Nil-groups vanish for $\mathbb{C}$,

$$K_0(R_r) \cong \mathbb{Z} \quad \text{and}$$

$$K_1(R_r) \cong \mathbb{C}^{\times} \times (\text{free abelian group on} \quad x_1, \cdots, x_r),$$

and thus $GL(R_r)$ is generated by scalar matrices, elementary matrices, and the 1×1 matrices $(x_1), \cdots, (x_r)$. In fact, by stability theorems proved independently by Suslin [**45**, §7] and by Swan [**48**], all finitely generated projective modules over R_r are free, and for $n \geq 3$, $GL(n, R_r)$

coincides with the subgroup $G_{n,r}$ generated by diagonal scalar matrices, by $E(n, R_r)$, and by

$$\begin{pmatrix} x_1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix}, \dots, \begin{pmatrix} x_r & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix}.$$

We also define $P_{n,k,r}$ to be the set of idempotents of rank k in $M_n(R_r)$, i.e., those conjugate under $G_{n,r}$ to

$$\begin{pmatrix} 1_k & 0 \\ 0 & 0_{n-k} \end{pmatrix}.$$

By Suslin's and Swan's result, all idempotents are of this form for suitable n and k. Finally, we let

$$V_{n,k,r} = P_{n,k,r} \times P_{n,k,r},$$

$$W_{n,k,r} = \{ (p, q, g) : (p, q) \in V_{n,k,r}, \quad g \in G_{n,r}, \quad gpg^{-1} = q \}.$$

We topologize these sets by noting that $M_n(R_r)$ is naturally filtered. We say an element has *length* $\leq s$ if all its matrix entries involve only monomials

$$x_1^{i_1} x_2^{i_2} \cdots x_r^{i_r}, \quad |i_1| + \cdots + |i_r| \leq s.$$

The elements of $M_n(R_r)$, $P_{n,k,r}$, $G_{n,r}$, $V_{n,k,r}$, and $W_{n,k,r}$ of length $\leq s$ constitute algebraic varieties over $\mathbb{C}$, since they involve only finitely many complex coordinates subject to certain polynomial constraints. Therefore they have natural topologies and are homeomorphic to finite-dimensional cell complexes. We give all the sets $M_n(R_r)$, $P_{n,k,r}$, $G_{n,r}$, $V_{n,k,r}$, and $W_{n,k,r}$ the inductive limit topologies for the filtration by length. Thus for a sequence to be convergent, the lengths of the elements in the sequence must eventually stabilize, and the coefficients of the relevant Laurent polynomials must converge. The advantage of this is that $G_{n,r}$ becomes a topological group operating continuously on $P_{n,k,r}$ by conjugation. (The continuity of all the operations except for matrix inversion is easy to check. So the only real difficulty is to show that matrix inversion is continuous on $G_{n,r}$. It is here that our knowledge of the generators of $GL(n, R_r)$ comes in handy, since we know that each of the generators, and thus every element of the group, has determinant of the form $cx_1^{i_1} x_2^{i_2} \dots x_r^{i_r}$, $c \in \mathbb{C}^\times$ and $i_j \in \mathbb{Z}$. Then the classical formula for the inverse of a matrix in terms of determinants of cofactors shows that inversion is continuous.)

We mention as motivation for our definitions that an element of

$$M_n(C(X)[x_1^{\pm}, \ldots, x_r^{\pm}])$$

can be always be viewed as a map

$$X \to M_n(\mathbb{C}[x_1^{\pm}, \ldots, x_r^{\pm}])$$

. With our definitions, this map is continuous, and all such continuous maps arise this way. Similarly with P, V, and so on.

We claim now that the natural projection from $W_{n,k,r}$ to $V_{n,k,r}$ is a (Hurewicz) fibration (that is, that it has the homotopy lifting property with respect to maps from compact metric spaces), with fiber the centralizer $H_{n,k,r}$ in $G_{n,r}$ of the matrix

$$\begin{pmatrix} 1_k & 0 \\ 0 & 0_{n-k} \end{pmatrix}.$$

To prove this, note that all the spaces involved are paracompact, so that the fibration property is local. By homogeneity, it is enough to know that the map $G_{n,r} \to G_{n,r}/H_{n,k,r}$ has a local cross-section and that the map

$$g \mapsto g \begin{pmatrix} 1_k & 0 \\ 0 & 0_{n-k} \end{pmatrix} g^{-1}$$

is open near the identity matrix, from $G_{n,r}$ to $P_{n,k,r}$ (so that $P_{n,k,r}$ is homeomorphic to $G_{n,r}/H_{n,k,r}$). However, matrices of the form

$$\begin{pmatrix} 1_k & b \\ 0 & 1_{n-k} \end{pmatrix} \begin{pmatrix} 1_k & 0 \\ a & 1_{n-k} \end{pmatrix},$$

with a and b suitable rectangular blocks, give the required local cross-section in $G_{n,r}$ for a neighborhood of the identity coset in $G_{n,r}/H_{n,k,r}$, and map in $P_{n,k,r}$ to

$$\begin{pmatrix} 1_k + ba & -b - bab \\ a & -ab \end{pmatrix}.$$

From this one can read off the bicontinuity of the map $G_{n,r}/H_{n,k,r} \to P_{n,k,r}$, since given a convergent sequence in the latter converging to

$$\begin{pmatrix} 1_k & 0 \\ 0 & 0_{n-k} \end{pmatrix},$$

it eventually consists of matrices of fixed length of the above form, from which one can continuously recover the a and the b (say by the implicit function theorem).

We're finally ready to use all this machinery. Suppose $A = C(X)$ with X a compact metric space, and consider an element

$$[p] - [q] \in \ker(\alpha_*) \subseteq K_0(C([0,1],\, A)[x_1^{\pm}, \cdots, x_r^{\pm}]).$$

Stabilize p and q if necessary, by taking direct sums with a large enough matrix of the form

$$\begin{pmatrix} 1_s & 0 \\ 0 & 0_t \end{pmatrix},$$

so that they give conjugate idempotents in some

$$M_n(A[x_1^{\pm}, \cdots, x_r^{\pm}]).$$

We may represent p and q by continuous functions from $X \times [0,1]$ into the idempotents in $M_n(R_r)$. (The continuity here involves the fact that the lengths are bounded!) Since we know that such idempotents are classified only by rank, p and q can be assumed to each define a continuous function from $X \times [0,1]$ to some $V_{n,k,r}$. (Rank is a locally constant function, so if it's not globally constant, we can divide X into finitely many closed and open sets and argue separately on each one. The fact that we end up in $V_{n,k,r}$ is due to the fact that the assumption

$$[p] - [q] \in \ker(\alpha_*)$$

means that p and q are equivalent over $X \times \{0\}$, and thus certainly have the same rank there.)

Now the fact that p and q are conjugate over $X \times \{0\}$ translates into the assumption that our map into the base of the fibration

$$W_{n,k,r} \to V_{n,k,r}$$

can be lifted over $X \times \{0\}$. So by the homotopy lifting property, there is a global lift. By the peculiar definition of the topology on $G_{n,r}$, this means there is an element of $GL(n,\, C(X \times [0,1])[x_1^{\pm}, \cdots, x_r^{\pm}])$ conjugating p to q, showing that $[p] = [q]$ in K_0. This completes the proof of the lemma. ∎

To complete the proof of the theorem, we should check the additivity of F^{-*}. For $* > 0$, this follows from a standard property of K-theory with compact supports by the Fischer-Prasolov Theorem. So once again, we only need prove the result for lower K-theory, which again comes down to consideration of $K_0(A[x_1^{\pm},\cdots,x_r^{\pm}])$. Suppose

$$A = C(X), \quad X = (\coprod X_i)^+ = \bigvee X_i^+.$$

(The wedge is topologized as a compact metric space.) We have seen that elements of $K_0(A[x_1^{\pm},\cdots,x_r^{\pm}])$ arise from formal differences of maps

$$X \to P_{n,k,r}$$

(in the notation of the proof of (2.3)), and any such map is determined by its restriction to the X_i's, so that

$$F^r(\coprod X_i) \hookrightarrow \prod F^r(X_i).$$

Furthermore, the image lies in the direct *sum*, since if p and q are such maps which agree at the basepoint, then they have the same rank nearby, and locally define a map into $V_{n,k,r}$. In a small enough neighborhood, the fibration

$$W_{n,k,r} \to V_{n,k,r}$$

is trivial, so there's a local lifting, and thus $[p]-[q]$ is trivial in $F^r(X_i)$ for i sufficiently large.

As for the statement about the real case, note that everything goes through exactly the same way, using the *real* case of the Fischer-Prasolov Theorem and taking all matrices above to be real. ∎

We should point out that we've actually proved more than just the statement of Theorem 2.2; we've actually shown the following statement about *integral* negative algebraic K-theory.

THEOREM 2.4. *If X is a second-countable locally compact space, there is a natural isomorphism*

$$K_i^{\mathrm{alg}}(C_0(X)) \cong \widetilde{bu}^{-i}(X^+), \quad i \le 0,$$

and similarly in the real case with bu replaced by bo.

PROOF: Define a $\mathbb{Z}$-graded functor of X by

$$F^i(X) = \begin{cases} K_{-i}^{\mathrm{top}}(C_0(X)) = K^i(X) = \tilde{K}^i(X^+), & i \le 0, \\ K_{-i}^{\mathrm{alg}}(C_0(X)), & i > 0. \end{cases}$$

Recall that we already know by [25] that negative K-theory agrees with KV-theory and thus satisfies algebraic excision, hence gives long exact sequences with relative groups that only depend on the ideal involved. Since one can splice together the exact sequences in toplogical K-theory and negative algebraic K-theory, the functor F comes with a long exact sequence for each pair $Y \subseteq X$ (with Y closed). Hence the homotopy-invariance result above shows it is a cohomology theory. Now one can argue as before that this theory is connective K-theory. ∎

In fact (and I thank Chuck Weibel for this observation), the method of proof of Theorem 2.2 also yields the following variant of the Quillen-Suslin results on Serre's problem.

THEOREM 2.5. *Let X be a compact metric space, and let $C(X)$ denote the ring of either complex-valued or real-valued continuous functions on X. If X is contractible, then for any s and t, every finitely generated projective module over $C(X)[y_1, y_2, \ldots, y_s, x_1^{\pm}, \ldots, x_r^{\pm}]$ is free. Even if X isn't contractible, every finitely generated projective module over $C(X)[y_1, y_2, \ldots, y_s]$ is extended from a projective module over $C(X)$.*

PROOF: Consider the first statement, and adopt the notation of the proof of Lemma 2.3, except for allowing some polynomial generators as well as Laurent polynomial generators. A finitely generated projective module over

$$C(X)[y_1, y_2, \ldots, y_s, x_1^{\pm}, \ldots, x_r^{\pm}]$$

will be given by a map p from X into some $P_{n,k,s,r}$, and we can take q to be the constant map from X to the matrix

$$\begin{pmatrix} 1_k & 0 \\ 0 & 1_{n-k} \end{pmatrix}.$$

Putting p and q together, we get a continuous map from X to $V_{n,k,s,r}$. Since X is contractible, we can lift to a map from X to $W_{n,k,s,r}$, proving that our module is free.

For the second case, where we only allow polynomial generators but don't require X to be contractible, take p to be as before and let q be the map from X to *scalar* idempotent matrices corresponding to setting all the y's to 0. Consider the map

$$X \times [0, 1] \to V_{n,k,s,0}$$

defined by

$$(x, t) \mapsto (p(x)(ty_1, ty_2, \ldots, ty_s), q(x)).$$

Since we can lift the map to $W_{n,k,s,0}$ over $X \times \{0\}$, the map given by p and q has a lift by the homotopy lifting property, and so the module is extended. ∎

Probably the above argument could be pushed to show that lower algebraic K-theory with finite coefficients is a homotopy functor also on non-commutative C*-algebras. The proof would be a bit nastier because of various negative K-groups and Nil-groups that come in, even though these should wash out after passage to finite coefficients.

§2.2. Algebraic K-theory for operator algebras. In this next section, we recall a few contexts in which the *algebraic*, as opposed to topological, K-theory of operator algebras has arisen in problems of direct interest to analysts. Since algebraic and topological K-theory coincide in the case of K_0, and since not much is known about the direct interpretation of higher and lower K-groups, we shall mention mostly occurrences of K_1, together with one occurrence of K_2 and of the higher K-groups. It is an interesting challenge to look for other potential uses of algebraic K-theory, especially higher K-theory, for operator algebras.

Thus we begin with algebraic K_1, which we recall is the abelianization of the stable general linear group. Accordingly, any suitable notion of a *determinant* for a unital operator algebra A, in other words, a homomorphism from $GL(A)$ to an abelian group, must factor through $K_1(A)$. (In this section we drop the superscript "alg," since topological K_1 will not be of interest unless we say so specifically.) As we mentioned earlier, the classical determinant gives an isomorphism from $K_1(\mathbf{C})$ to $\mathbf{C}^\times$. But there are other determinants for operator algebras, notably, the determinant for operators of the form $1+$ (trace class), and the Fuglede-Kadison determinant [**13**] for finite factors. When A is a II_1-factor, the latter gives a well-defined notion of $|\det(x)|$ for $x \in A$, x invertible, defined by writing

$$x = u|x|, \quad u \quad \text{unitary}, \quad |x| = e^t, \quad t = t^*,$$

and setting

$$|\det(x)| \stackrel{\text{def}}{=} e^{\operatorname{Tr}(t)}.$$

Here Tr is the normalized trace on A, so $\operatorname{Tr}(1) = 1$. Note that though the Fuglede-Kadison determinant is initially defined only on $GL(1, A)$, it extends to $GL(n, A)$ with the same properties, since this is just $GL(1, M_n \otimes A)$, and $M_n \otimes A$ is a new II_1-factor. A rather beautiful extension of the work of Fuglede and Kadison is then:

THEOREM [**11**, PROPOSITION 2.5]. *The Fuglede-Kadison determinant induces an isomorphism*

$$K_1(A) \to \mathbf{R}_+^\times.$$

The idea of Fuglede and Kadison has also been used to produce notions of determinant for other kinds of C*-algebras equipped with a trace (see [15], [10]). In K-theoretic language, this amounts to studying the structure of K_1 through homomorphisms to $\mathbf{C}^\times$ or other abelian groups. Exel's determinants are defined on all of $GL(A)$, hence on all of $K_1(A)$, whereas some writers (cf. [15]) consider only determinants defined on $GL(A)^0$, the connected component of the identity. This is not so much of a problem since

$$GL(A)/GL(A)^0 \cong K_1^{\text{top}}(A),$$

a "known" invariant. Exel's work in particular shows that analysis of the action of $\text{Aut}(A)$ on $K_1(A)$ can be a fruitful tool for studying automorphisms. The work of de la Harpe and Skandalis shows that knowledge of the traces on a C*-algebra often, but not always, completely determines K_1.

Next we mention an occurrence of K_2 and an apparent use of the higher K-groups. This stems from the work of Helton and Howe ([17], [18]) on algebras of almost commuting operators. In the simplest case of this theory, they consider a family of bounded self-adjoint operators that commute modulo the trace-class operators. Such families occur both in the theory of single operators (from the study of hyponormal operators, Toeplitz operators, etc.) and in more geometric settings, e.g., the study of pseudo-differential operators on the circle. These almost-commuting operators generate a C*-algebra of the Brown-Douglas-Fillmore type, that is, an extension of some $C(X)$ by $\mathcal{K}$, but the fact that the commutators are trace-class gives more, namely (cf. [17, Theorem 3.1]) that the non-closed (unital) algebra generated by the operators can be completed to a Fréchet algebra A fitting into a short exact sequence of topological algebras:

$$0 \to \mathcal{L}^1 \to A \to C^\infty(X) \to 0, \tag{2.6}$$

where the notion of C^∞-function on X comes from the embedding of X into a suitable $\mathbf{C}^n$ (recall X is just the joint essential spectrum of the generating operators). Here we are using the notation $\mathcal{L}^p$ for the Schatten p-class. If one prefers to deal with Banach algebras (as is more convenient for introducing topological K-theory), it is possible also to replace C^∞ by C^r for a suitable value of r.

In this situation, the (multiplicative) commutator of two invertible operators in A will lie in the "determinant class," and so the Fredholm determinant gives a pairing

$$(a,\, b) \mapsto \det(aba^{-1}b^{-1}). \tag{2.7}$$

The theory of this pairing is developed in [**17**, §10] and in [4]. It turns out that if the joint essential spectrum X lies in the plane and if a and b have "symbols" f and g in $C^\infty(X)$, then the pairing (2.7) is given by an integral formula

$$\exp\left(\frac{1}{2\pi i}\int \frac{\{f,g\}}{fg}\,dm\right), \tag{2.8}$$

where the numerator involves the Poisson bracket on $\mathbf{R}^2$ and dm is a certain measure introduced by Pincus. In some cases, the formula has to be suitably interpreted.

Now it was discovered by L. Brown [**5**] that the pairing (2.7) has a nice interpretation in terms of algebraic K-theory. Namely, consider the extension (2.6) and the associated long exact sequence of algebraic K-groups. The Fredholm determinant defines a map

$$K_1(A, \mathcal{L}^1) \to K_1(\mathcal{B}(\mathcal{H}), \mathcal{L}^1) \xrightarrow{\det} \mathbf{C}^\times,$$

and composing with the connecting map of the long exact sequence of (2.6), we get a map

$$K_2(C^\infty(X)) \to \mathbf{C}^\times.$$

On the other hand, there is a cup-product in algebraic K-theory, described in very concrete terms in [**31**, §8], giving a skew-symmetric bilinear pairing

$$K_1(C^\infty(X)) \times K_1(C^\infty(X)) \to K_2(C^\infty(X)),$$

and the determinant pairing (2.7) is just the composite of these two maps.

In [4], Brown raises the question of trying to find an appropriate generalization of this observation to the "higher-dimensional case," since as shown in [**18**], the phenomenon of operators that commute modulo $\mathcal{L}^1$ is essentially one-dimensional in nature. The correct model for the higher-dimensional case seems to be given by the crypto-integral algebras of Helton and Howe, such as (to give the most important case) the algebra $\Psi^0(M)$ of pseudo-differential operators of order ≤ 0 on a compact smooth manifold without boundary, M^n. The commutator ideal of $\Psi^0(M)$ is $\Psi^{-1}(M)$, the pseudo-differential operators of order ≤ -1, and if these operators are thought of as acting on $\mathcal{H} = L^2(M)$ (with respect to some smooth measure), then $\Psi^{-1}(M) \subseteq \mathcal{L}^{n+\varepsilon}(\mathcal{H})$. Thus completing $\Psi^0(M)$ will give an algebra A and an extension

$$0 \to \mathcal{L}^{n+1} \to A \to C^\infty(X) \to 0, \tag{2.9}$$

where X is the cosphere bundle of M, a manifold of dimension $2n-1$. Once again, we can make A into a Banach algebra (by abuse of notation, denoted by the same letter) by replacing C^∞ by C^r, r sufficiently large. The long exact sequence of (2.9) in *topological* K-theory gives the usual index map of the Atiyah-Singer Theorem:

$$\partial : K_1^{\text{top}}(C^r(X)) \to K_0(\mathcal{L}^{n+1}) \cong \mathbf{Z}.$$

However, the analogue of Helton-Howe theory involves the *odd* algebraic K-theory of $\mathcal{L}^{n+1}$, and thus is in some sense orthogonal to ordinary index theory. The correct map has been identified by Connes and Karoubi [8], and can be viewed in two different ways. Perhaps the simplest is to note that the algebra introduced in [8], $\mathcal{M}^{2n-2}$, fits into an exact sequence

$$0 \to \mathcal{L}^{2n-1} \to \mathcal{M}^{2n-2} \to \mathcal{B}(\mathcal{H}) \to 0,$$

and since $\mathcal{B}(\mathcal{H})$ (being *flasque*) has vanishing K-theory, we obtain a natural isomorphism

$$(2.10) \qquad K_*(\mathcal{M}^{2n-2}) \xrightarrow{\cong} K_*(\mathcal{M}^{2n-2}, \mathcal{L}^{2n-1}) \cong K_*(\mathcal{B}(\mathcal{H}), \mathcal{L}^{2n-1}).$$

The generalized Helton-Howe "determinant invariant," for $n \geq 2$, can then be thought of as the following composite

$$(2.11) \quad K_{2n}(C^\infty(X)) \xrightarrow{\partial} K_{2n-1}(A, \mathcal{L}^{n+1}) \to K_{2n-1}(\mathcal{B}(\mathcal{H}), \mathcal{L}^{2n-1}) \xrightarrow{\text{Connes-Karoubi}} \mathbf{C}^\times.$$

Here ∂ is the boundary map of the long exact sequence coming from (2.9), and the Connes-Karoubi map comes from (2.10) and the construction of [8, §V]. The alternative construction of (2.11) is by another point of view from [8]—we can make a $2n$-summable odd Fredholm module for the commutative algebra $C^\infty(X)$, and this induces a map (again as in [8, §V])

$$K_{2n}(C^\infty(X)) \to \mathbf{C}^\times.$$

In principle it ought to be possible to give an explicit interpretation for this map as restricted to the image of the $2n$-th tensor power of $C^\infty(X)$ under the cup-product, similar to (2.7), and an integral formula as in (2.8).

§2.3. Algebraic K-theory of operator algebras applied to topology. In this section we shall mention ways in which operator algebras might be of use in studying two of the topological applications of K-theory from §1, the Wall obstruction and Whitehead torsion. It seems operator algebras might also be of use in studying negative K-theory, but we will save this topic for §3.

We begin with the Wall obstruction. The idea here is work in progress by John Miller [**29**], which is so pretty it deserves some extra publicity. If A is a unital C*-algebra, Miller develops a theory of "elliptic Fredholm complexes" $\{\mathcal{H}_i, d^i\}$ over A, where the $\mathcal{H}_i$'s are countable inductive limits of Hilbert A-modules. Then he shows how to associate to such a complex an *index* in $K_0(A)$. The idea of the construction is that if M is a manifold, connected but generally non-compact, and $A = C^*(\pi_1(M))$, then one can get such a complex out of the DeRham complex of the universal cover of M, provided that M is "finitely dominated over A." This latter condition is strictly weaker than just being finitely dominated in the usual sense, and when this is the case, the image of the index in $\tilde{K}_0(A)$ coincides with the image of the Wall obstruction under the natural map

$$\tilde{K}_0(\mathbf{Z}\pi_1(M)) \to \tilde{K}_0(A) \qquad (2.12)$$

induced by the inclusion of rings. In principle this can be used to study classical Wall obstructions for open manifolds, such as occur in [**44**], the advantage of this approach being that the index can be controlled by *analysis* on M and its universal cover. One might in turn be able to relate this to, say, curvature properties. The only problem, as noted in [**29**], is that there are no known cases in which the map (2.12) is non-zero. In fact, one would not even expect any, since there is a long-standing conjecture that $\tilde{K}_0(\mathbf{Z}\pi)$ vanishes for any torsion-free group π, and even that for general π, it is all "induced" from finite subgroups. (This latter statement needs to be suitably interpreted, since there can be a degree shift; for instance $K_{-1}(\pi)$ "shows up" in $\tilde{K}_0(\mathbf{Z}(\mathbf{Z}\times\pi))$.) On the other hand, for a finite group π, $\tilde{K}_0(\mathbf{Z}\pi)$ is finite, whereas $\tilde{K}_0(C^*(\pi))$ is torsion-free and $K_{-1}(C^*(\pi))$ vanishes, even if one uses the real C*-algebra in place of the complex one. Nevertheless, Miller's construction is still interesting for two reasons:

(1) if the conjecture about $\tilde{K}_0$ is false, this might detect a counterexample, and
(2) Miller's obstruction is often defined and non-trivial for manifolds that aren't finitely dominated in the classical sense.

The homotopy-theoretic meaning of Miller's invariant in this latter case is not yet totally clear (at least to me).

Next we consider the Whitehead torsion obstruction in $Wh(\pi)$, a quotient of $K_1(\mathbf{Z}\pi)$. This can potentially be studied by using the Fuglede-Kadison determinant. As we indicated back in §1, non-zero elements of Whitehead groups can often be detected by using a finite-dimensional unitary representation of the group and applying the determinant. However, this method has the drawback that most infinite groups have very few finite-dimensional representations, though every group has enough (infinite-dimensional) unitary representations to separate points.

Thus suppose we are given a (possibly horrible) group π and and element $a \in GL(\mathbf{Z}\pi)$, which we suspect represents a non-zero element of $Wh(\pi)$. How can we show that a *cannot* be transformed into the image of $\{\pm 1\} \times \pi$ via elementary row and column operations?

If σ is a II_1-factor representation of π, we can consider (just as in (1.1))

$$|\det(\sigma(a))|, \tag{2.13}$$

where $|\det|$ denotes the Fuglede-Kadison determinant of the finite factor, which recall is well-defined on K_1. Since σ sends ± 1 and π to unitary matrices, the quantity (2.13) only depends on the image of a in the Whitehead group, and gives us a well-defined group homomorphism $Wh(\pi) \to \mathbf{R}_+^\times$.

§3. Algebraic and Topological K-Homology and KK-Theory

§3.1. Pedersen-Weibel K-homology. We begin this section with a review of some of the work of E. Pedersen and C. Weibel on explicit realization of the homology theory corresponding to the spectrum whose homotopy groups give the algebraic (negative) K-groups. Then we shall discuss how their construction is related to Kasparov's KK-groups for operator algebras.

We begin with a notion that already made its appearance in the theory of controlled h-cobordisms, which we referred to briefly in the "Lower K-theory" subsection of §1 above. Thus let $\mathcal{S}$ denote the category of *metric* spaces and *locally bounded* maps, in which the morphisms are maps that are *not necessarily continuous*, but that only change distances by a controlled amount. In other words, if X and Y are metric spaces, a map $f : X \to Y$ is a morphism in the category if there are positive functions ρ_1 and ρ_2 (depending on the map f) such that

$$\operatorname{dist}(x, y) < \rho_1(f(x), \operatorname{dist}(f(x), f(y))),$$
$$\operatorname{dist}(f(x), f(y)) < \rho_2(\operatorname{dist}(x, y)), \qquad x, y \in X.$$

Thus the map collapsing X to a point will be a morphism in the category if and only if X has finite diameter. Note that this is definitely a *metric* notion; it doesn't just depend on the topology of X, which is relatively unimportant from this point of view. In fact, in this category, the inclusion of $\mathbb{Z}$ into $\mathbb{R}$ (with both spaces given their usual metrics) has a left inverse $x \mapsto [x]$, where $[x]$ denotes the greatest integer $\leq x$. The objects of the category $\mathcal{S}$ will be the parameter spaces for controlled topology.

Now let R be a ring (with unit) and let X be a metric space, that is, an object of $\mathcal{S}$. Pedersen and Weibel define a category $\mathcal{C}_X(R)$ [**35**, Remark 1.2.3], which one can call the category of *configurations of finitely generated projective R-modules over X*, as follows. The objects of the category are collections $A = \{A_x\}_{x\in X}$, of finitely generated projective R-modules, such that for any ball B in X, $A_x \neq 0$ for only finitely many $x \in B$. A morphism $\varphi : A \to C$ is defined to be given by R-module homomorphisms $\varphi^x_y : A_x \to C_y$, $x, y \in X$, such that there exists $k = k(\varphi)$ with $\varphi^x_y = 0$ if $\operatorname{dist}(x, y) > k$. The interesting point is that if $f : X \to Y$ is a morphism in $\mathcal{S}$, then it induces a functor $\mathbf{f}_* : \mathcal{C}_X(R) \to \mathcal{C}_Y(R)$, sending an object $A = \{A_x\}_{x\in X}$ to $f_*(A) = \{\bigoplus_{x\in f^{-1}(y)} A_x\}_{y\in Y}$, as well as a functor $\mathbf{f}^* : \mathcal{C}_Y(R) \to \mathcal{C}_X(R)$, sending $C = \{C_y\}_{y\in Y}$ to $f^*(C) = \{C_{f(x)}\}_{x\in X}$. Note that the boundedness of f is essential here in showing that $f_*(A)$ and $f^*(C)$ satisfy the local finiteness condition for admissible configurations.

LEMMA 3.1. *Let X be a subspace of the metric space Y, and suppose there is a retraction $r : Y \to X$ in $\mathcal{S}$, satisfying the extra condition that the function ρ_1 is* **uniform**, *i.e., a function of the distance but not of the point $x \in X$. (This is the analogue in $\mathcal{S}$ of a* **deformation** *retraction.) Then the inclusion $i : X \to Y$ induces an equivalence of categories $\mathbf{i}_* : \mathcal{C}_X(R) \to \mathcal{C}_Y(R)$, for any ring R.*

PROOF: Clearly $\mathbf{r}_* \circ \mathbf{i}_* = \mathbf{id}_*$ on $\mathcal{C}_X(R)$. On the other hand, $\mathbf{i}_* \circ \mathbf{r}_*$ on $\mathcal{C}_Y(R)$ merely "slides configurations over to X," and so is naturally equivalent to the identity functor. ∎

COROLLARY 3.2. *For any ring R and any positive integer n, $\mathcal{C}_{\mathbb{R}^n}(R)$ is equivalent to $\mathcal{C}_{\mathbb{Z}^n}(R)$, and for any* **compact** *(or bounded) non-empty metric space X, $\mathcal{C}_X(R)$ is equivalent to $\mathcal{C}_{\mathrm{pt}}(R)$, the category of finitely generated projective R-modules.*

PROOF: For the first assertion, use the product of n copies of the retraction $\mathbb{R} \to \mathbb{Z}$ given by the greatest-integer function $[\_\_]$. For the second assertion, note that the inclusion of a point into X has as a left inverse the map collapsing X to a point. ∎

Now, for any finite-dimensional compact metrizable space X, Pedersen and Weibel define a metric space $O(X)$, the *open cone* on X, by embedding X into some sphere S^{n-1} and letting

$$O(X) = \{t \cdot x \mid t \in [0, \infty), \qquad x \in X\}$$

with the metric induced from $\mathbf{R}^n$. Of course, the actual metric obtained depends on the choice of embedding of X into a sphere, but we only need to consider $O(X)$ up to isomorphism in the category $\mathcal{S}$, which will be independent of this choice. Notice that if $X = S^{n-1}$, $n \geq 1$, then $O(X) = \mathbf{R}^n$ (metrically), and that if X is a point, then $O(X) = [0, \infty)$. We will generally assume that X has a basepoint, which we can take to be the vector $(1, 0, \ldots, 0)$. The main result of Pedersen and Weibel on K-homology is the following.

THEOREM 3.3 (PEDERSEN-WEIBEL). *For X a finite CW-complex,*

$$K_*(\mathcal{C}_{O(X)}(R)) \cong \tilde{\mathbf{K}}(R)_{*-1}(X).$$

Here the groups on the right are as in §2 above, using the *nonconnective* K-theory spectrum, and the groups on the left are the K-groups of a category. These are obtained by taking the symmetric monoidal subcategory of isomorphisms in $\mathcal{C}$ and plugging into the "infinite loop machine" to produce a spectrum. The groups in question are just the homotopy groups of this spectrum. However, all this "abstract nonsense" can be made much more concrete by taking $* = 1$. Then the group on the left can be defined by the usual procedure for defining K_1; it is the stable abelianization of automorphism groups in the category. More explicitly, K_1 of a category with short exact sequences is the abelian group generated by symbols $[A, \alpha]$, where A is an object of the category and α an automorphism of A, subject to the relations

$$[A, \alpha\beta] = [A, \alpha] + [A, \beta]$$

and

$$[A, \alpha] + [C, \gamma] = [B, \beta]$$

whenever there is a commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} & & \\
0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0.
\end{array}$$

The first goal of this section will be to prove the following theorem, which has to do with the comparison between algebraic and topological K-theory. Recall that by [**42**] or [**40**], Kasparov's KK-functor is closely related to ***topological*** K-homology, in the sense that for a σ-unital C*-algebra R, one can prove that

$$KK(C_0(X),\, R) \cong \mathbf{K}(R)_0^{\mathrm{top}}(X^+),$$

for X^+ (X with a point adjoined at infinity) a finite CW-complex.

THEOREM 3.4. *Let R be a unital C^*-algebra. There is a natural transformation Ψ of homology theories (corresponding to the natural transformation from algebraic to topological K-homology), which we shall make explicit below, which for any non-empty compact metrizable space X, maps*

$$K_1(\mathcal{C}_{O(X)}(R)) \to \widetilde{KK}(C(X),\, R).$$

Here the reduced Kasparov group is defined as the elements in

$$KK(C(X),\, R)$$

which are zero when restricted to the constant functions $\mathbf{C} \subseteq C(X)$. When $X = S^0$, the map Ψ is an isomorphism, which may be identified with the identity map on $K_0(R)$.

PROOF: Let us begin with an automorphism α of some configuration A of finitely generated (right) R-modules over X. Out of this data we will construct a Kasparov $(C(X),\, R)$-bimodule which restricts on $\mathbf{C}$ to a trivial bimodule. To begin with, we may assume $O(X) \subseteq \mathbf{R}^n$ and that $A_0 = 0$, where 0 is the origin in $\mathbf{R}^n$, since we may "slide" R-modules a short distance using Lemma 3.1 if necessary. We let

$$\mathcal{E} = \mathcal{E}_0 \oplus \mathcal{E}_1,$$

where $\mathcal{E}_0$ and $\mathcal{E}_1$ are each suitable Hilbert R-module completions of the direct sum of all the $A_{t\cdot x}$, $t \cdot x \in O(X)$. The R-valued inner product $\langle\ \ ,\ \ \rangle_1$ on $\mathcal{E}_1$ is the usual one, inherited from the canonical R-valued inner products on R^m, $m \in \mathbf{N}$. However, as α may not be norm-bounded with respect to the norm on $\mathcal{E}_1$, we let $\mathcal{E}_0$ have the inner product

$$\langle \xi,\, \eta\rangle_0 = \langle \alpha(\xi),\, \alpha(\eta)\rangle_1.$$

The local finiteness conditions on A and on α insure that this is well-defined on the algebraic direct sum of the $A_{t\cdot x}$. Furthermore, the invertibility of α and the fact that α commutes with the right action of

R guarantee that $\langle \ , \ \rangle_0$ satisfies the axioms for an R-valued inner product. It's now easy to check that α extends to an isometry $\mathcal{E}_0 \to \mathcal{E}_1$ of Hilbert R-modules, with α^{-1} as its Hilbert module adjoint. Thus if

$$F = \begin{pmatrix} 0 & \alpha^{-1} \\ \alpha & 0 \end{pmatrix} \quad \text{on} \quad \mathcal{E} = \mathcal{E}_0 \oplus \mathcal{E}_1,$$

then $F \in \mathcal{L}(\mathcal{E})$ and $F = F^* = F^{-1}$.

Now define a $*$-homomorphism

$$\psi : C(X) \to \mathcal{L}(\mathcal{E})$$

by defining $\psi(f)$ on $A_{t \cdot x}$ (in either $\mathcal{E}_0$ or $\mathcal{E}_1$) to be multiplication by $f(x)$. The action is well-defined by our assumption that $A_0 = 0$. We claim that $(\mathcal{E}, \psi, F)$ defines a Kasparov $(C(X), R)$-bimodule.

To check that claim, we need to show that

$$[\psi(f), F] \in \mathcal{K}(\mathcal{E}) \qquad \text{for} \quad f \in C(X).$$

For this purpose, let $\xi, \eta \in A_{t_1 \cdot x_1}$, where ξ is viewed as sitting in $\mathcal{E}_0$ and η is viewed as sitting in $\mathcal{E}_1$. Then

$$[\psi(f), F] \begin{pmatrix} \xi \\ \eta \end{pmatrix} = \sum_{t_2, x_2} (f(x_2) - f(x_1)) \begin{pmatrix} (\alpha^{-1})^{t_1 \cdot x_1}_{t_2 \cdot x_2}(\eta) \\ \alpha^{t_1 \cdot x_1}_{t_2 \cdot x_2}(\xi) \end{pmatrix}.$$

Now there exists $k > 0$ such that

$$(\alpha^{-1})^{t_1 \cdot x_1}_{t_2 \cdot x_2} = 0, \quad \alpha^{t_1 \cdot x_1}_{t_2 \cdot x_2} = 0 \quad \text{for} \quad \|t_2 \cdot x_2 - t_1 \cdot x_1\| > k.$$

By continuity of f, given $\varepsilon > 0$, we can choose M sufficiently large such that

$$|t_1| > M, \quad \|t_2 \cdot x_2 - t_1 \cdot x_1\| \le k \Longrightarrow |f(x_2) - f(x_1)| < \varepsilon.$$

This shows $[\psi(f), F]$ can be approximated to within any $\varepsilon > 0$ by an operator of "finite rank," as required.

Thus we can define Ψ by

$$\Psi([A, \alpha]) = [(\mathcal{E}, \psi, F)] \in KK(C(X), R).$$

The image lies in $\widetilde{KK}(C(X), R)$, since by construction ψ is unital and F unitary, so that $(\mathcal{E}, \psi|_{\mathbb{C}}, F)$ is a trivial Kasparov $(\mathbb{C}, R)$-module. To check that Ψ is a homomorphism, well-defined on the Grothendieck group K_1, note that it is obvious from the construction that

$$\Psi([A \oplus B, \alpha \oplus \beta]) = \Psi([A, \alpha]) + \Psi([B, \beta]),$$

so that to finish the proof, it is necessary to show that

$$\Psi([A, \alpha \cdot \beta]) = \Psi([A, \alpha]) + \Psi([A, \beta])$$

and that

$$\Psi\left(\left[A \oplus B, \begin{pmatrix} 1 & \eta \\ 0 & 1 \end{pmatrix}\right]\right) = 0 \quad \text{in} \quad KK.$$

This is easily done by using the standard homotopies

$$\begin{pmatrix} 1 & \eta \\ 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and

$$\begin{pmatrix} \alpha\beta & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \beta & 0 \\ 0 & 1 \end{pmatrix} \sim \begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \beta \end{pmatrix},$$

the latter by "rotation."

For the last part of the proof, suppose $X = S^0$. Then $O(X) = \mathbf{R}$, which by Corollary 3.2 we may replace by $\mathbf{Z}$, and by the main theorems of [**33**] or [**35**], $K_1(\mathcal{C}_{\mathbf{Z}}(R)) \cong K_0(R)$. We similarly have

$$\widetilde{KK}(C(S^0), R) \cong KK(\mathbf{C}, R) \cong K_0(R),$$

and we want to show that in this case, Ψ reduces to the identity map. Thus let P be a finitely generated projective R-module. The element $[A, \alpha]$ of $K_1(\mathcal{C}_{\mathbf{Z}}(R))$ corresponding to $[P] \in K_0(R)$ is by the proof of [**33**, Lemma 1.15] given by letting $A_n = P$ for all $n \in \mathbf{Z}$, $\phi_n^m = 0$ unless $n = m+1$, $\phi_{m+1}^m : A_m \to A_{m+1}$ the identity on P. Applying the construction above, we see $\Psi([A, \alpha])$ corresponds to the shift on $\bigoplus_{-\infty}^{\infty} P$, with

$$\psi(f) = \begin{cases} \text{multiplication by } f(1) \text{ on summands with } n \geq 0, \\ \text{multiplication by } f(-1) \text{ on summands with } n < 0, \end{cases}$$

for $f \in C(S^0)$. When $R = \mathbf{C}$, this is one of the standard realizations of the usual generator $\mathbf{1}$ of $\widetilde{KK}(C(S^0), \mathbf{C}) \cong KK(\mathbf{C}, \mathbf{C})$. For general R, we clearly get the external Kasparov product of $[P] \in K_0(R)$ with $\mathbf{1}$, *i.e.*, $[P]$ again. So Ψ is indeed the identity map. ∎

Now let us further specialize Theorem 3.4, by taking $X = S^0$, $R = \mathbf{C}$, and identifying $\widetilde{KK}(C(S^0), \mathbf{C})$ as usual with $\mathbf{Z}$. As before, we also replace $O(X)$ by $\mathbf{Z}$. Then an object of $\mathcal{C}_{\mathbf{Z}}(R)$ is just given by a collection of finite-dimensional $\mathbf{C}$-vector spaces, indexed by $\mathbf{Z}$. An endomorphism (in $\mathcal{C}_{\mathbf{Z}}(R)$) of such an object is then given by a (two-sided) infinite matrix, with a block decomposition into *finite blocks,* having *finite* bandwidth with respect to this decomposition, *e.g.,*

$$\begin{pmatrix} \ddots & \ddots & & & & \\ \ddots & \begin{bmatrix} * & * \\ * & * \end{bmatrix} & \begin{bmatrix} * \\ * \end{bmatrix} & & & \\ & \begin{bmatrix} * & * \end{bmatrix} & * & * & & \\ & & * & * & \ddots & \\ & & & \ddots & \ddots \end{pmatrix}.$$

The theorem then reduces to the following (non-trivial) result of linear algebra (or, if you prefer, operator theory).

COROLLARY 3.5. *Suppose α is a (two-sided) infinite block matrix with finite blocks, such that α also has an inverse of the same form. Let α_+ denote the* **one-sided infinite** *truncation of α to its lower right-hand corner. Then α_+ is Fredholm (in the sense of having finite-dimensional kernel and cokernel for its action on (two-sided infinite) vectors with only* **finitely** *many entries), with index independent of the point of truncation, and α is a product of a block diagonal matrix and elementary matrices (matrices such as α of the form $1 +$ (nilpotent)) if and only if* $\operatorname{Ind} \alpha_+ = 0$.

PROOF: The fact that α_+ is Fredholm is a consequence of the finite bandwidth condition, since any vector in the kernel of α_+ must be supported within a fixed distance of the point of truncation, and similarly all vectors (of finite support) supported sufficiently far away from the point of truncation are in the image of α_+. Furthermore, changing them point of truncation only changes α_+ by something of finite rank, and thus leaves the index unchanged. The rest of the statement now follows immediately from Theorem 3.4 and the formula for Ψ. ∎

§3.2. Algebraic KK-theory. As one can see from §3.1 above, there appears to be a close connection between Kasparov's KK-theory and algebraic K-theory. We intend now to make this relationship a little more precise. To begin with, we would like to explain how Kasparov theory can be obtained in a way very similar to algebraic K-theory.

First assume that A and B are C*-algebras with B σ-unital. Kasparov theory may be defined by first defining a category $\mathcal{C}(A, B)$ whose objects are the Kasparov (A, B)-bimodules $(\mathcal{E}, \phi, F)$ with $\mathcal{E}$ countably generated over B, where we only view the F as defined up to perturbation by elements of $\mathcal{K}(\mathcal{E})$. In other words, we identify $(\mathcal{E}, \phi, F)$ and $(\mathcal{E}, \phi, F')$ if $F - F' \in \mathcal{K}(\mathcal{E})$. The morphisms in the category are taken to be unitary **isomorphisms**, i.e., $T \in \mathcal{L}(\mathcal{E}, \mathcal{E}')$ defines a morphism

$$(\mathcal{E}, \phi, F) \to (\mathcal{E}', \phi', F')$$

if and only if T is unitary ($T^*T = 1_{\mathcal{E}}, TT^* = 1_{\mathcal{E}'}$) of degree 0, T intertwines ϕ and ϕ', and T intertwines F and F' up to a compact, *i.e.*, $TFT^* - F' \in \mathcal{K}(\mathcal{E}')$.

An object in $\mathcal{C}(A, B)$ is called **degenerate** if there's a representative $(\mathcal{E}, \phi, F)$ in its class (recall the F is only defined up to compacts) for which

$$[F, \phi(a)] = 0, \quad (F^2 - 1)\phi(a) = 0, \quad (F - F^*)\phi(a) = 0$$

for *all* $a \in A$. In such a case, note that

$$(\mathcal{E}^\infty, \phi^\infty, F^\infty) = (\mathcal{E} \oplus \mathcal{E} \oplus \cdots, \phi \oplus \phi \oplus \cdots, F \oplus F \oplus \cdots)$$

is well-defined.

$\mathcal{C}(A, B)$ is a symmetric monoidal category under direct sum, and if $(\mathcal{E}, \phi, F)$ is degenerate,

$$(\mathcal{E}, \phi, F) \oplus (\mathcal{E}^\infty, \phi^\infty, F^\infty) \cong (\mathcal{E}^\infty, \phi^\infty, F^\infty),$$

so that $(\mathcal{E}, \phi, F)$ is stably trivial.

We let $KK(A, B)$ = group completion of the isomorphism classes of elements of $\mathcal{C}(A, B)$. This is an abelian group and agrees with Kasparov's definition (see [**3**]). We could also apply the infinite loop machine of [**49**] and form the spectrum

$$\mathbf{KK}(A, B) \stackrel{\text{def}}{=} \operatorname{Spt} \mathcal{C}(A, B),$$

and

$$\begin{aligned}\pi_0(\operatorname{Spt} \mathcal{C}(A, B)) =& \text{group completion of } \pi_0(\mathcal{B}\mathcal{C}(A, B)) \\ =& KK(A, B),\end{aligned}$$

where $\mathcal{B}$ denotes the classifying space functor of a category. The spectrum we have constructed gives the homology theory associated to KK.

All this now motivates the following construction. We intend to define similarly an *algebraic* version of KK, which in the one-variable case reduces to standard algebraic K-theory. Note that such a construction (of something called $K(A, B)$) was given by C. Kassel in **[26]**, though it is our opinion that Kassel's construction is not that close to the definition of KK: it does not (as does Kasparov theory) come from the study of problems about *almost commuting* operators, nor does it reduce in the one-variable case to Quillen's higher algebraic K-theory (it reduces to Karoubi-Villamayor theory instead).

We suppose we are given a commutative ground-ring k and two unital k-algebras, A and B. In standard applications, k will be either $\mathbf{Z}$ or a field, and we will sometimes suppress mention of k. An **algebraic Kasparov** (A, B)**-bimodule** (over k) will be a triple (E, ϕ, F) with E a $\mathbf{Z}_2$-graded *countably generated* projective right B-module, $\phi : A \to \mathrm{End}_B(E)$ a unital k-linear ring homomorphism of degree 0, and $F \in \mathrm{End}_B(E)$ of degree 1 with $F^2 - 1$ and each $[F, \phi(a)]$, $a \in A$, contained in the ideal $\mathcal{F}_B(E) \subseteq \mathrm{End}_B(E)$ of "finite-rank operators," *i.e.*, B-linear maps contained in some $\mathrm{End}_B(P)$ (as naturally embedded in $\mathrm{End}_B(E) = \mathrm{End}_B(P \oplus P^{\perp})$) for some finitely generated projective B-submodule P of E (depending on a in the second case). Note of course that if E is itself finitely generated, then there are no restrictions on F.

We define a category $\mathcal{C}(A, B)$ as before, whose objects are equivalence classes of such bimodules, in which we identify (E, ϕ, F) with (E, ϕ, F') if $F - F' \in \mathcal{F}_B(E)$. The morphisms in $\mathcal{C}$ are defined by conjugation by B-module *iso*morphisms $E \to E'$. As before, this is a symmetric monoidal category under $\oplus$, and we get a spectrum

$$\mathbf{KK}(A, B) \stackrel{\mathrm{def}}{=} \mathrm{Spt}\, \mathcal{C}(A, B)$$

with

$$\begin{aligned} KK(^{\mathrm{alg}}_{k})(A, B) =& \pi_0(\mathbf{KK}(A, B)) \\ =& \text{Grothendieck group of iso. classes} \\ & \text{of Kasparov } (A, B)\text{-bimodules.} \end{aligned}$$

One fairly trivial example of a Kasparov bimodule, though nevertheless useful, and mentioned in **[26]**, occurs when $A = M_n(B)$. Then $E_0 = B^n$ is a Morita equivalence bimodule between A and B, and along with $E_1 \equiv 0$ and $F \equiv 0$ it defines an element of $KK(A, B)$. This also works the other way around, in fact, between any two Morita-equivalent rings.

Suppose now that $k = A = \mathbb{Z}$. Then for any Kasparov bimodule (E, ϕ, F), the ϕ is trivial and the only restriction on F is that $F^2 - 1 \in \mathcal{F}_B(E)$. One can easily see then that we can write our object in the form

$$(\text{degenerate}) \oplus (P, 0, 0)$$

with P finitely generated projective. Alternatively, one can stabilize so that

$$E_0 = E_1 = B^\infty,$$

and use [**26**, Proposition 6.1] to deduce that giving F amounts to giving an invertible element of what is called in [**25**] ΣB, whereas we know that $K_1(\Sigma B) \cong K_0(B)$. In either event, using the fact that degenerate elements die in the Grothendieck group, we deduce that

$$KK(\mathbb{Z}, B) = \pi_0(\mathbf{KK}(\mathbb{Z}, B)) = K_0(B).$$

In fact, the whole category $\mathcal{C}(\mathbb{Z}, B)$ is equivalent to the category of formal differences of finitely generated projective B-modules, so that

$$\mathbf{KK}(\mathbb{Z}, B) \simeq \mathbf{K}(B),$$

and we recover ordinary (higher) algebraic K-theory.

Of course, one of the most useful features of Kasparov's KK is the product, the construction of which relies on the so-called "Kasparov technical theorem." Unfortunately, we have not been able to find an appropriate analogue of this result in our context, so we are forced to content ourselves with a fairly simple case. However, we should point out at least that **KK** is a covariant functor in the second variable and a contravariant functor in the first variable, which means in effect that there is a well-defined product (on either side) between a KK-class and a homomorphism.

We will show now that an algebraic Kasparov (A, B)-bimodule sets up maps in algebraic K-theory $K_*(A) \to K_*(B)$. For this we need to know that we get a "lax" functor of symmetric monoidal categories

$$\mathcal{C}(\mathbb{Z}, A) \to \mathcal{C}(\mathbb{Z}, B).$$

Given a finitely generated projective A-module P, it's defined by some idempotent $e = e^2$ in a matrix ring over A. Since we've already explained how to get a Kasparov bimodule from A to $M_n(A)$ which induces the usual transfer on algebraic K-theory, we may as well replace $M_n(A)$ by A and assume $e \in A$. Consider an object of $\mathcal{C}(\mathbb{Z}, A)$, represented by a Kasparov bimodule (E, ϕ, F). Then $\phi(e)$ is an idempotent in $\mathrm{End}_B(E)$ and defines a submodule eE. Since F almost commutes with $\phi(e)$, it's cutdown to eE is well-defined up to elements of $\mathcal{F}_A(eE)$, as required. The map we get on objects this way is clearly compatible with direct sums.

As for morphisms, note that if $a \in A$ conjugates e to f, then $\phi(a)$ is a B-map from eE to fE, *etc.* So we get the desired functor and thus a pairing

$$KK(A, B) \otimes K_i(A) \to K_i(B).$$

Unfortunately, algebraic KK as we have defined it seems at the moment to be almost impossible to compute, except in fairly trivial cases. We shall therefore just say a few words about it and mention some examples that might indicate how it might come up in some situations. By the way, note that if A has characteristic p (*i.e.*, $p \cdot 1 = 0$ in A), then there are no unital maps from A into an algebra of characteristic 0 or some other prime, so that for instance $KK(A, \mathbb{Z}) = 0$. When dealing with algebras over a field, it seems best to use the version of the theory with $k =$ the ground field.

Here now is an example of an element of $KK(k[t, t^{-1}], k)$ which one might think of as an algebraic analogue of a pseudodifferential operator on S^1 (think of the algebra of trigonometric polynomials inside $C^\infty(S^1)$).

Define a Kasparov bimodule by letting $E_0 = E_1 = k[t, t^{-1}]$ with the identity action of $k[t, t^{-1}]$ and by letting

$$F = \begin{cases} \text{multiplication by } +1 \text{ on span } (1, t, t^2, \ldots), \\ \text{multiplication by } -1 \text{ on span } (t^{-1}, t^{-2}, \ldots). \end{cases}$$

Of course we have $F^2 = 1$, but the more interesting thing is that F, the analogue of the Hilbert transform, approximately commutes with the multiplication operators. There is also no way to perturb F by an operator of finite rank to make it commute exactly. This should be a prototype of a kind of algebraic Kasparov module corresponding to a differential or pseudodifferential operator on an affine variety.

Finally, here's an example modeled on a construction in **[21]**, that might be a prototype of a construction useful for studying the algebraic K-theory of group rings. Suppose a group Γ acts on a tree X. Define a class in $KK(k[\Gamma], k)$ as follows.

$$\text{Let } \begin{cases} E_0 = \text{free abelian group on the vertices } \Delta^0, \\ E_1 = \text{free abelian group on the edges } \Delta^1. \end{cases}$$

Fix an origin x_0 in X and as in [**21**] let

$$\beta : \Delta^0 \setminus \{x_0\} \to \Delta^1$$

be defined by $\beta(x) =$ edge pointing "in" from x to x_0. Define $F : E_0 \to E_1$ by

$$F(\delta_x) = \begin{cases} 0 & \text{if } x = x_0, \\ \delta_{\beta(x)} & \text{if } x \neq x_0. \end{cases}$$

Then F is "almost" an isomorphism. The action of $k[\Gamma]$ on E of course comes from the action of Γ on X, and as in [**21**], this action almost commutes with F. Presumably, the Kasparov bimodule one gets this way could be used in algebraic K-theory the same way Julg and Valette use its completion for studying topological K-theory of the group C*-algebra.

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