Variants of K-theory and connections with noncommutative geometry and physics

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Index Theory and K-Theory

Outline

- Lecture 1: Variants of K-Theory
- Lecture 2: K-Theory Applied to String Theory

Classifying vector bundles

K-theory is one of the most natural cohomology theories: we study a space X (for us, always locally compact Hausdorff) by studying vector bundles over X. Fix a field $\mathbb{F} = \mathbb{C}$, $\mathbb{R}$, or $\mathbb{H}$. If X is a compact space, let $\mathrm{Vect}_{\mathbb{F}}(X)_n =$ isomorphism classes of rank-n $\mathbb{F}$-vector bundles over X. Then $\mathrm{Vect}_{\mathbb{F}}(X) = \bigcup_n \mathrm{Vect}_{\mathbb{F}}(X)$ is an abelian monoid (semigroup with unit element 0) under vector bundle direct sum, also known as Whitney sum. This semigroup is hard to deal with. It never has inverses since there are no vector bundles of negative rank, and it usually doesn't have cancellation either. However, by the basic classification theorem for vector bundles, it is a homotopy functor of X:

$$\mathrm{Vect}_{\mathbb{F}}(X)_n = \begin{cases} [X, BU(n)], & \mathbb{F} = \mathbb{C}, \\ [X, BO(n)], & \mathbb{F} = \mathbb{R}, \\ [X, BSp(n)], & \mathbb{F} = \mathbb{H}. \end{cases}$$

Basic notions of K-theory

We get something better by passing to the Grothendieck group

$$K_{\mathbb{F}}(X) = \begin{cases} [X, \mathbb{Z} \times BU], & \mathbb{F} = \mathbb{C}, \\ [X, \mathbb{Z} \times BO], & \mathbb{F} = \mathbb{R}, \\ [X, \mathbb{Z} \times BSp], & \mathbb{F} = \mathbb{H}. \end{cases}$$

We extend to non-compact X by taking K-theory with compact supports (basically, requiring everything to be trivialized off a compact set), and defining $K_{\mathbb{F}}^{-n}(X) = K_{\mathbb{F}}(X \times \mathbb{R}^n)$. This turns out to be a cohomology theory. Furthermore, it's periodic in n by Bott periodicity, with period 2 if $\mathbb{F} = \mathbb{C}$, period 8 if $\mathbb{F} = \mathbb{R}$ or $\mathbb{H}$. Traditionally we write K for $K_{\mathbb{C}}$, KO for $K_{\mathbb{R}}$, KSp for $K_{\mathbb{H}}$. There are also cup-product pairings $K \otimes K \to K$, $KO \otimes KO \to KO$, $KO \otimes KSp \to KSp$, and $KSp \otimes KSp \to KO$ coming from the tensor product of vector bundles. Note that we can't tensor two $\mathbb{H}$-bundles and get another one!

Bott periodicity and the coefficient groups

The Bott periodicity theorem computes all the coefficient groups for these theories, that is, the reduced cohomology groups of spheres, or equivalently, the cohomology groups of points. The table is as follows:

Theory	$j = 0$	1	2	3	4	5	6	7
K^{-j}	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0
KO^{-j}	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	0	$\mathbb{Z}$	0	0	0
KSp^{-j}	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	0

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KSp^{-j}	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	0

In fact the symmetry between KO and KSp is not an accident; there are natural isomorphisms $KSp^{-j} \cong KO^{-j-4}$ and $KO^{-j} \cong KSp^{-j-4}$ coming from the cup-product with the generator of KSp^{-4}.

The KSC-theory of Anderson and Green

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A self-conjugate vector bundle over X is a pair (E, χ), where $E \to X$ is a complex vector bundle over X and $\chi\colon E \to E$ is a conjugate-linear bundle automorphism. We identify two pairs (E_1, χ_1) and (E_2, χ_2) is there is a (complex linear) bundle isomorphism $\mu\colon E_1 \to E_2$ with χ_1 homotopic to $\mu^{-1}\chi_2\mu$ (among all self-conjugacies of E_1). For X compact, $KSC(X)$ is defined to the the Grothendieck group of isomorphism classes of pairs (E, χ), with group operation coming from the direct sum.

KSC-theory (cont'd)

As with the theories K, KO, and KSp, we extend KSC to a theory with compact supports on locally compact spaces, and then take $KSC^{-n}(X) = KSC(X \times \mathbb{R}^n)$. As before, this is a cohomology theory.

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The coefficient groups for KSC turn out to be:

j	0	1	2	3	4	5	6	7
KSC^{-j}	$\mathbb{Z}$	$\mathbb{Z}/2$	0	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}/2$	0	$\mathbb{Z}$

with a periodicity of period 4. We will see an explanation for this shortly.

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with a periodicity of period 4. We will see an explanation for this shortly. The torsion generator of KSC^{-1} comes from the trivial bundle $S^1 \times \mathbb{C} \to S^1$ with $\chi(t, z) = (t, t\bar{z})$. It is of order 2 since conjugating by $\mu(t, z) = (t, t^n z)$ we can change $t\bar{z}$ to $t^{1+2n}\bar{z}$ for any $n \in \mathbb{Z}$.

Atiyah's KR-theory

Still another version of K-theory, called KR or Real K-theory (with a capital R!) was introduced by Atiyah in the famous paper "K-theory and reality" in 1968. This is a theory defined on the category of Real spaces, locally compact spaces X with an involution ι (a self-homeomorphism of X with $\iota^2 = 1$). Think of the complex points of an algebraic variety defined over $\mathbb{R}$, with ι the action of $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$.

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For (X, ι) a compact Real space, we define $KR(X)$ (usually the ι will be implicit) to be the Grothendieck group of the Real vector bundles over X, pairs (E, χ), with E a complex vector bundle over X and $\chi\colon\ E \to E$ a involutive conjugate-linear isomorphism compatible with ι. Note that when ι is trivial, this is equivalent to giving a real vector bundle over X, and E is just its complexification.

KR periodicity

We extend KR to a theory with compact supports on locally compact spaces. As usual it comes with a cup-product coming from the tensor product of vector bundles. Let $\mathbb{R}^{p,q}$ be $\mathbb{R}^p \oplus \mathbb{R}^q$ with the involution ι that is the identity on the first summand and -1 on the second summand. (Caution: Atiyah calls this $\mathbb{R}^{q,p}$ with p and q reversed. People seem to be divided 50/50 on the notation.) Let $S^{p,q}$ denote the unit sphere in $\mathbb{R}^{q,p}$; topologically this is S^{p+q-1}, but the involution depends on p and q. For instance it is the antipodal map in the case of $S^{0,q}$. Let $KR^{p,q}(X) = KR(X \times \mathbb{R}^{p,q})$. The Bott element β lives in $KR^{1,1}(\mathrm{pt})$.

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Theorem (Atiyah)

Cup-product with β is an isomorphism $KR^{p,q}(X) \to KR^{p+1,q+1}(X)$ for any X and any p, q. Thus $KR^{p,q}(X)$ only depends on $p-q$, and it's periodic with period 8 in this index.

Special cases of KR-theory

Recall that if the involution on X is trivial, then $KR^{p,q}(X) \cong KO^{q-p}(X)$.

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There are natural isomorphisms $KR(X \times S^{0,1}) \cong K(X)$ *and* $KR(X \times S^{0,2}) \cong KSC(X)$. $KR(X \times S^{0,4})$ *is 8-periodic. For* $p \geq 3$ *there are short exact sequences*

$$0 \to KR^{-q}(X) \to KR^{-q}(X \times S^{0,p}) \to KR^{p+1-q}(X) \to 0.$$

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This "explains" the 4-periodicity of KSC. But the theorem is false ☹ in general; it contradicts the 8-periodicity of $KR^{-q}(S^{0,4})$.

KR for free involutions

In the case X compact and ι free, what happens more precisely is this. Locally, $X \cong Y \times S^{0,1}$ (where $S^{0,1}$ is two points, interchanged by the involution), and $KR^*(X) \cong K^*(Y)$. However, this is not true globally. However, there is a spectral sequence, the analogue of the Atiyah-Hirzebruch spectral sequence, $H^p(X/\iota, \utilde{KR^q(S^{0,1})}) \Rightarrow KR^{p+q}(X)$. Here $\utilde{KR^q(S^{0,1})}$ is a sheaf locally isomorphic to $\mathbb{Z}$ for q even, and is 0 for q odd.

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An example of the spectral sequence for KR

Recall that $KSC^* = KR^*(S^{0,2})$. So take $X = S^{0,2}$, $X \to X/\iota$ a 2-to-1 covering map. We have $E_2^{p,q} = 0$ unless $p = 0$ or 1 and q is even. For $q \equiv 0 \pmod 4$, we have $E_2^{p,q} = H^p(S^1, \mathbb{Z}) = \mathbb{Z}$ for $p = 0, 1$. For $q \equiv 2 \pmod 4$, we have $E_2^{p,q} = H^p(S^1, \underset{\sim}{\mathbb{Z}})$. This cohomology with local coefficients is the same as $H^p_{\text{group}}(\mathbb{Z}, \underline{\mathbb{Z}})$, where $\underline{\mathbb{Z}}$ is the $\mathbb{Z}$-module isomorphic to $\mathbb{Z}$ as an abelian group, but on which 1 (the generator of the group) acts by -1. The spectral sequence looks like:

$$
\begin{array}{rc|cc}
 & q & p = 0 & p = 1 \\
 & 4 & \mathbb{Z} & \mathbb{Z} \\
 & 3 & 0 & 0 \\
\text{periodicity } \cong & 2 & 0 & \mathbb{Z}/2 \\
 & 1 & 0 & 0 \\
 & 0 & \mathbb{Z} & \mathbb{Z} \quad \cdots\cdots \to p.
\end{array}
$$

We see that KSC^* is 4-periodic with groups as shown before.

Connection with noncommutative geometry

All the variants of K-theory that we have discussed: K, KO, KSp, KSC, and KR can be unified by thinking of them as topological K-theory for various Banach algebras (in fact, C^*-algebras) over $\mathbb{R}$. For X locally compact, we have

$$\begin{cases} K^{-q}(X) = K_q(C_0(X)), \\ KO^{-q}(X) = KO_q(C_0^{\mathbb{R}}(X)), \\ KSp^{-q}(X) = KO_q(C_0^{\mathbb{H}}(X)), \\ KSC^{-q}(A) = KO_q(C_0^{\mathbb{R}}(X) \otimes T), \end{cases}$$

where $T = \{f \in C([0,1]) \mid f(0) = \overline{f(1)}\}$. In addition, if $\underline{(X, \iota)}$ is a Real space, then $KR^{-q}(X) = KO_q(\{f \in C_0(X) \mid f(x) = \overline{f(\iota x)}\})$.

Twistings from noncommutative geometry

All of the K-groups K, KO, KSp, KSC, and KR have twisted versions that are special cases of the K-theory of real continuous-trace (CT) algebras. I originally studied these back in the 1980's for purely operator-algebraic reasons, but in the next lecture we will see how they arise in modern physics.

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$$\{a \in A_+ \mid \operatorname{Tr} \pi(a) < \infty \, \forall \pi \in \widehat{A}, \text{ and } \pi \mapsto \operatorname{Tr} \pi(a) \text{ continuous on } \widehat{A}\}$$

are dense in A_+. A real C^*-algebra A is said to have continuous trace if its complexification does. Note that commutative real C^*-algebras automatically have continuous trace.

Structure theory of CT algebras

A structure theory for (complex) continuous-trace algebras was developed by Dixmier and Douady in the 1960's. They showed that if X is locally compact and second countable, and if A is a separable (complex) CT algebra with spectrum X, then A is determined up to stable isomorphism (or Morita equivalence) by a Dixmier-Douady class $\delta \in H^3(X, \mathbb{Z})$. This class classifies a principal PU-bundle over X, and since $PU(\mathcal{H}) = \operatorname{Aut} \mathcal{K}(\mathcal{H})$, there is an associated bundle of algebras $\mathcal{A}$ over X with fibers $\mathcal{K}$, and $A \otimes \mathcal{K} \cong \Gamma_0(X, \mathcal{A})$.

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The real case is more complicated. A real CT algebra is built out of three pieces of real, quaternionic, and complex type, respectively. These are locally isomorphic to $C_0^{\mathbb{R}}(X) \otimes \mathcal{K}(\mathcal{H}_{\mathbb{R}})$, $C_0^{\mathbb{R}}(X) \otimes \mathcal{K}(\mathcal{H}_{\mathbb{H}})$, and $C_0(X) \otimes \mathcal{K}(\mathcal{H}_{\mathbb{C}})$, respectively.

Twisted K-theory

Twisted (complex) K-theory of X with twisting $\delta \in H^3(X, \mathbb{Z})$ can be defined simply to be $K_*(A)$, where A is a CT algebra with spectrum X and Dixmier-Douady class δ. When $\delta = 0$, A is Morita equivalent to $C_0(X)$, and we get back $K^{-*}(X)$.

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Basic Ideas of String Theory

The basic idea of string theory is to replace point particles (in conventional physics) by one-dimensional "strings." At ordinary (low) energies these strings are extremely short, on the order of the Planck length,

$$l_P = \sqrt{\frac{\hbar G}{c^3}} \approx 1.616 \times 10^{-35}\,\mathrm{m}\,.$$

A string moving in time traces out a two-dimensional surface called a worldsheet. The most basic fields in string theory are thus maps $\varphi\colon\ \Sigma \to X$, where Σ is a 2-manifold (the worldsheet) and X is spacetime.

String theory offers [some] hope for combining gravity with the other forces of physics and quantum mechanics.

Strings and Sigma-Models

Let Σ be a string worldsheet and X the spacetime manifold. String theory is based on the nonlinear sigma-model, where $\varphi\colon \Sigma \to X$ and the leading terms in the action are

$$S(\varphi) = \frac{1}{4\pi\alpha'} \int_\Sigma \|\nabla\varphi\|^2 \, d\mathrm{vol} + \int_\Sigma \varphi^*(B), \tag{1}$$

the energy of the map φ (in Euclidean signature) plus the Wess-Zumino term based on the B-field B. $1/(2\pi\alpha')$ is the string tension. B is a locally defined 2-form on X (really associated to a gerbe) but $H = dB$ is a globally defined integral form, whose cohomology class h we will call the H-flux.
We have to add to this various gauge fields (giving rise to the fundamental particles) and a "gravity term" involving the scalar curvature of the metric on X. Usually we also require supersymmetry; this means the theory involves both bosons and fermions and there are symmetries interchanging the two.

D-Brane Charges and K-Theory

Physicists talk about both closed and open strings. Both kinds of strings are given by compact manifolds, but in the "open" case there is a boundary. So to get a reasonable theory one has to impose Dirichlet or Neumann boundary conditions on some submanifold Y of X where the boundary of Σ must map. These submanifolds are traditionally called D-branes, "D" for *Dirichlet* and *brane* from *membrane*.

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Orientifolds

One can construct many more string theories out of the basic Type II theories by considering orientifold theories. In these theories, the spacetime manifold X is equipped with an involution ι. The inclusion $\varphi\colon \Sigma \to X$ of a string worldsheet into X is required to be equivariant for the involution Ω on Σ given by the worldsheet parity operator. The Chan-Paton bundle on a D-brane then has to have a conjugate-linear involution compatible with ι, and so D-brane charges live in (a variant of) $KR^*(X, \iota)$.

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The involution ι does not have to be free. In general, its fixed set will have several components, called O-planes ("O" for orientifold). On a given O-plane, the restriction of the Chan-Paton bundle must have a real or symplectic structure, giving a class in KO^* or KSp^* of the O-plane. We refer to O^+ and O^- planes in these two cases.

KR with a sign choice

Ordinary *KR*-theory restricts to *KO*-theory on the fixed set of the involution ι. So if we have both O^+ and O^- planes, a variant is needed that keeps track of the signs of the O-plane. We call this *KR-theory with a sign choice*.

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Orientifold and KR theories on a circle

First consider the case of $X = S^1$ with one of its "standard" orientifold structures, $S^{2,0}$ (the case of trivial involution), $S^{1,1}$ (the unit circle in $\mathbb{C}$ with complex conjugation), and $S^{0,2}$ (the circle with the antipodal map). Since superstring theory is always 10-dimensional, one should really cross with $\mathbb{R}^9$ to get the associated string theory spacetime.

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Orientifold and KR theories on a circle

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KR with a sign choice (cont'd)

Now the string background $S^{1,1} \times \mathbb{R}^9$ has two O-planes, each of the form $\{\mathrm{pt}\} \times \mathbb{R}^9$ for one of the two fixed points of the involution on S^1. What if we require the Chan-Paton bundles to be of symplectic type on one O-plane, and of orthogonal type on the other? D-brane charge for this theory would live in $KR^{-j-9}_{(+,-)}(S^{1,1})$, *KR*-theory with sign choice $(+,-)$. How do we even define this group?

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Computing KR with a sign choice

Note that $KR^{-j}_{(+,-)}(S^{1,1})$ maps to $KO^{-j} \oplus KSp^{-j}$ via restriction to the fixed set. And we have a map $K^{-j-1} \cong K^{-j}(\mathbb{R}) \to KR^{-j}_{(+,-)}(S^{1,1})$ coming from inclusion of the open set where ι is free.

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$$\cdots \to K^{-j-1} \to KR^{-j}_{(+,-)}(S^{1,1}) \to KO^{-j} \oplus KSp^{-j} \xrightarrow{\delta} K^{-j} \to \cdots .$$

The connecting map $\delta\colon\ KO^{-j} \oplus KSp^{-j} \to K^{-j}$ kills the torsion and amounts to complexification of real bundles and restriction of scalars from $\mathbb{H}$ to $\mathbb{C}$ for quaternionic bundles. Careful analysis then shows that $KR^{-j}_{(+,-)}(S^{1,1}) \cong KSC^{-j+1} \cong KR^{-j+1}(S^{0,2})$.

T-duality

One of the important dualities in string theory is called T-duality ("T" for "target space" or "torus"). This duality sets up an equivalence of string theories on two very different spacetime manifolds X and $X^\sharp$. The basic idea is that "winding" modes of the string theory on X are replaced by "momentum" modes on $X^\sharp$, and *vice versa*. Tori in X are replaced by their dual tori in $X^\sharp$. In the simplest case, that means that X has a circle factor of radius R and $X^\sharp$ has a circle factor of radius $\widetilde{R} = \frac{\alpha'}{R}$. The duality also involves changes in the metric and the B-field, known as the Buscher rules, after Buscher, who derived them in 1987–88.

Derivation of T-duality, following Buscher

Consider the simplest case. Take Σ a closed Riemannian 2-manifold and consider the action (1) for a map to a circle, gotten by integrating a 1-form ω on Σ:

$$S(\omega) = \frac{1}{4\pi\alpha'} \int_\Sigma \frac{R^2}{\alpha'} \omega \wedge *\omega.$$

Add a new parameter θ, and consider instead

$$S(\omega, \theta) = \frac{1}{4\pi\alpha'} \int_\Sigma \left(\frac{R^2}{\alpha'} \omega \wedge *\omega + 2\theta \, d\omega \right).$$

For an extremum of S with respect to variations in θ, we need $d\omega = 0$, so we get back the original theory. But instead we can take the variation in ω.

Derivation of T-duality (cont'd)

$$\begin{aligned}\delta S &= \frac{R^2}{4\pi\alpha'^2}\int_\Sigma \left(\delta\omega\wedge *\omega + \omega\wedge *\delta\omega + \frac{2\alpha'}{R^2}\theta\, d\delta\omega\right)\\ &= \frac{R^2}{4\pi\alpha'^2}\int_\Sigma \delta\omega\wedge\left(2*\omega + \frac{2\alpha'}{R^2}d\theta\right),\end{aligned}$$

so if $\delta S = 0$, $*\omega = \frac{-\alpha'}{R^2}d\theta$ and $\omega = \frac{\alpha'}{R^2} * d\theta$. If $\eta = d\theta$, substituting back into $S(\omega,\theta)$ gives

$$\begin{aligned}S'(\eta) &= \frac{1}{4\pi\alpha'}\int_\Sigma \left(\frac{R^2}{\alpha'}\left(\frac{\alpha'}{R^2}\right)^2 \eta\wedge *\eta + 2\frac{\alpha'}{R^2}\theta\, d*\eta\right)\\ &= -\frac{1}{4\pi\alpha'}\int_\Sigma \frac{\alpha'}{R^2}\eta\wedge *\eta\end{aligned}$$

which is just like the original action (with η replacing ω, $\widetilde{R} = \frac{\alpha'}{R}$ replacing R).

T-Duality on circle orientifolds

Now suppose the target space is S^1, but with an involution. Let's suppose (this is the simplest case) that Σ, the string worldsheet, is $S^{1,1} \times \mathbb{R}$, where $\mathbb{R}$ represents time and the involution on $S^{1,1}$ is worldsheet parity reversal Ω.

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T-duality is supposed to interchange winding and momentum modes in the sigma-model. The winding number for $z \mapsto z^n$ (from S^1 to S^1) is n; this mode is always equivariant when the involution is complex conjugation on both circles (the case of $S^{1,1}$), but when the target space is $S^{2,0}$, only the case $n = 0$ is equivariant, and when the target space is $S^{0,2}$, equivariance means $\bar{z}^n = z^{-n} = -z^n$, so there are no equivariant maps.

T-Duality on circle orientifolds (cont'd)

Let's look at this in more detail. When the target space is $S^{1,1}$, if $z \in \mathbb{T}$ and $t \in \mathbb{R}$ are the coordinates on $\Sigma = S^{1,1} \times \mathbb{R}^{1,0}$ and $x\colon \Sigma \to S^{1,1}$, then quantization forces x to be periodic in t also, so x descends to the quotient space $S^{1,1} \times S^{2,0}$. But equivariance in the $S^{2,0}$ means the map is trivial in time, i.e., the momentum is 0. So maps $\Sigma \to S^{1,1}$ have arbitrary winding but vanishing momentum.

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When we perform T-duality, winding and momentum are interchanged, so we have vanishing winding and arbitrary momentum. This is precisely the situation for $S^{2,0}$, so the orientifold targets $S^{1,1}$ and $S^{2,0}$ are T-dual to one another. This is reflected in the KR-theory: $KR^*(S^{1,1}) \cong KO^* \oplus KO^{*+1}$, while $KR^*(S^{2,0}) \cong KO^* \oplus KO^{*-1}$. These are the same up to a shift in degree by 1!

T-duality for circle orientifolds (cont'd)

Other arguments from physics, which we don't have time to go into here, have convinced physicists that the orientifold string theory with string background $S^{0,2} \times \mathbb{R}^9$ is T-dual to a theory with two O-planes with opposite sign. We see this reflected in our calculation that $KR^{-j}_{(+,-)}(S^{1,1}) \cong KSC^{-j+1} \cong KR^{-j+1}(S^{0,2})$.

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The types of elliptic curve orientifolds

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IIB	$\emptyset$	$S^{2,0} \times S^{0,2}$	$KSC^*(S^1) \cong KSC^* \oplus KSC^{*-1}$

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IIA	$S^1 \amalg S^1$	$S^{1,1} \times S^{2,0}$	$KO^{*+1}(T^2)$
IIA	S^1	not a product	complicated
IIA	$\emptyset$	$S^{1,1} \times S^{0,2}$	$KSC^*(S^{1,1}) \cong KSC^* \oplus KSC^{*+1}$

T-duality of elliptic curve orientifolds

Now using the T-duality between $S^{1,1}$ and $S^{2,0}$, we can see how the various orientifolds are related through T-duality. We get the following diagram of T-dualities:

$$
\begin{tikzcd}
\text{IIB} & \text{IIA} \\
S^{2,0} \times S^{2,0} \arrow[r, leftrightarrow] & S^{1,1} \times S^{2,0} \\
S^{1,1} \times S^{1,1} \arrow[ur, leftrightarrow] & \boxed{\text{antiholomorphic, species 1}} \\
S^{0,2} \times S^{0,2} \arrow[r, leftrightarrow] & S^{1,1} \times S^{0,2}
\end{tikzcd}
$$

Note the compatibility with the table of KR^* groups.

Twistings and sign choices

A few string theories where the underlying spacetime manifold is $T^2 \times \mathbb{R}^8$ are missing above. The theories have D-brane charges involving twisted groups, or sign choices forced by the presence of of O-planes of opposite sign. An example is Witten's "Toroidal compactification without vector structure" (*JHEP* 1998). This theory is closely related to $KO^*(T^2, w)$ with $w \neq 0$ in $H^2(T^2, \mathbb{Z}/2)$, and is also T-dual to an orientifold theory with 3 O^+ planes and one O^- plane. So the full power of what we've been discussing is needed to study these cases.