Exercises for Oberwolfach Seminar on Topological K-Theory of Noncommutative Algebras

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Note: My slides for the Seminar are available at:
`http://www.math.umd.edu/users/jmr/OberwolfachS05.pdf`
The file is a work in progress, but I am updating it as I go along.

1 Lecture I: A Survey of Bivariant K-Theories

1. Verify that $K_*(\_\_;\mathbb{Q})$ and $K_*(\_\_;\mathbb{Q}/\mathbb{Z})$, as we defined them for local Banach algebras as $K_*(\_\_\widehat{\otimes}D)$ for a suitable tensor product and suitable auxiliary algebras D, are indeed homology theories (homotopy invariant, half-exact, with long exact sequences).

2. Show that the definition of $K_*(\_\_;\mathbb{Z}/n)$ defined using the mapping cone of the unital map $\mathbb{C} \to M_n(\mathbb{C})$ agrees with the "classical choice," where we take for D the commutative algebra $C_0(X \smallsetminus \{x_0\})$, where X is the mod-n "Moore space," a CW complex with 3 cells defined by attaching a 2-cell to S^1 by a map of degree n, and x_0 is a basepoint (the 0-cell in X).

3. Check the details of the theorem that DK^*, as defined in the slides, is a cohomology theory on algebras.

2 Lecture II: Twisted K-Theory

1. Check the calculation of the twisted K-theory

$$K_{\delta_n}^{-*}(S^3) = K_*(CT(S^3, \delta_n)),$$

   where δ_n has as Dixmier-Douady invariant n times the fundamental class on S^3.

2. (harder) Use the last exercise and the Atiyah-Hirzebruch spectral sequence (the spectral sequence induced by the skeletal filtration) to show that if X is a finite CW complex and $\delta \in H^3(X, \mathbb{Z})$, there is a spectral sequence

$$H^p(X, K_q(\mathbb{C})) \Rightarrow K_\delta^{p+q \bmod 2}(X),$$

   in which the first non-trivial differential is $d_3 = \_\_ \cup \delta + \mathrm{Sq}^3$.

3 Lecture III: Connes' Thom Isomorphism

1. Deduce from Connes' Thom isomorphism theorem that for a connected, simply connected solvable Lie group G of dimension n, $K_*(C^*(G))$ depends only on n mod 2 and not on anything else. (Hint: G has a closed connected normal subgroup of codimension 1.)

2. Let α and α' be exterior equivalent actions of a locally compact group G on a C^*-algebra A. Prove that $A \rtimes_\alpha G$ and $A \rtimes_{\alpha'} G$ are $*$-isomorphic. Show in fact that one can choose the isomorphism to be the identity on the natural copies of A in the multiplier algebras.

3. Let $\mathbb{R}$ act on $T^2 = \mathbb{R}^2/\mathbb{Z}^2$ by flow along lines of slope θ, i.e., by

$$\alpha_t(x, y) = (x + t, y + \theta t) \mod \mathbb{Z} \times \mathbb{Z}.$$

   Compute the K-theory of the crossed product $T^2 \rtimes_\alpha \mathbb{R}$ (as a group). **Harder:** Find specific generators for $K_*(T^2 \rtimes_\alpha \mathbb{R})$. **Note:** This is an example of an *induced action*. Thus the K-theory can also be computed by the Pimsner-Voiculescu sequence for the action of $\mathbb{Z}$ on $\mathbb{T}$ by rotation by $2\pi\theta$.

4 Lecture IV: Applications to Physics

1. Suppose a compact group T acts freely on a (reasonably nice) space X, with the quotient map $X \to Z$ a principal T-bundle, and suppose $E \xrightarrow{p} X$ is a principal G-bundle over X, for G some other group (in our applications PU). Show that the T-action on X lifts to an action on E by bundle automorphisms if and only if p is pulled back from a G-bundle over Z.

2. Let $p : T \to Z$ be a principal $\mathbb{T}$-bundle, with T and Z locally compact. Let $\mathbb{T}$ act on $C_0(T)$ in the obvious way. Show that $C_0(T) \rtimes \mathbb{T} \cong C_0(X, \mathcal{K})$, and that the dual action θ of $\mathbb{Z}$ on $C_0(Z, \mathcal{K})$ has the property that $C_0(Z, \mathcal{K}) \rtimes_\theta \mathbb{Z} \cong C_0(T, \mathcal{K})$.

3. With notation as in the last exercise, verify that

$$\operatorname{Ind}_{\mathbb{Z}}^{\mathbb{R}} C_0(Z, \mathcal{K}) \cong CT(S^1 \times Z, \delta),$$

   where $\delta = a \times [p]$, $a \in H^1(S^1)$ the usual generator and $[p] \in H^2(Z)$ the characteristic class of the $\mathbb{T}$-bundle $p : T \to Z$.