First steps towards a noncommutative theory of nonlinear elliptic equations

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Special Session on E-Theory, Extensions, and Elliptic Operators
San Diego, January 9, 2008

Outline

Elliptic PDEs in Geometry

Many of the classical elliptic PDEs arise from variational problems in Riemannian geometry.

Examples:

- Harmonic map equation. Comes from looking for critical points of *energy of a map* $f\colon\ M^m \to N^n$,

  $$E(f) = \int_M \|\nabla f\|^2 \, d\text{vol}\,, \tag{1}$$

  M and N Riemannian manifolds.
  Special cases:
  - $M = \mathbb{R}$. Geodesics.
  - $N = \mathbb{R}$. Laplace(-Beltrami) equation.
  - $m = 2$, $N = \mathbb{R}^3$. Minimal surfaces.

Elliptic PDEs in Geometry (cont'd)

More Examples:

- Hilbert-Einstein equation. When $n = 4$ comes from looking for critical points of "total scalar curvature."
- Yamabe equation. The nonlinear elliptic equation that comes from trying to deform a given metric within a given conformal class to achieve constant scalar curvature. Variational formulation using $\left(\int_M \widetilde{R}\, d\widetilde{\text{vol}}\right) / \widetilde{\text{vol}}(M)^{2/p}$.
- Yang-Mills equation. Comes from looking for critical points of the energy of a connection ∇ on a vector bundle $E \to M$,

$$\int_M \|\Theta\|^2 \, d\text{vol}\,,$$

Θ the curvature 2-form.

Gelfand-Naimark Duality

Basic ideas of noncommutative geometry:
Recall: $X \rightsquigarrow C_0(X)$ is a contravariant equivalence of categories.
This sets up a dictionary:

Classical	Noncommutative
locally compact space	C^*-algebra
compact space	unital C^*-algebra
vector bundle	f. g. projective module
smooth manifold	C^*-algebra with "smooth subalgebra"
partial derivative	unbounded derivation

But it's pointless to go to the noncommutative world just "because it's there"—there should be a concrete motivation.

Motivation from Physics

- More concrete motivation comes from quantum physics.
- Many of the classical elliptic PDEs are also the field equations of physical theories.
- But the *uncertainty principle* forces quantum observables to be noncommutative.
- There is also increasing evidence [Connes, Connes-Douglas-Schwarz, Seiberg-Witten, Mathai-Rosenberg] that quantum field theories should allow for the possibility of noncommutative space-times.
- *Noncommutative sigma-models* will require the noncommutative harmonic map equation.

Connes' Noncommutative Differential Geometry

Set-up:

- A unital C^*-algebra, G a Lie group with action α on A, $\mathfrak{g}$ the Lie algebra of G, δ the differentiated action, $A^\infty = \{a \in A : t \mapsto \alpha_t(a)\ C^\infty\}$, Ξ^∞ f. g. projective (right) A^∞-module, $\Xi = \Xi^\infty \otimes_{A^\infty} A$, $\langle\ ,\ \rangle$ a Hilbert C^*-inner product on Ξ.
- ∇ a [unitary] connection on Ξ^∞:

$$\nabla_X(\xi \cdot a) = \nabla_X(\xi) \cdot a + \xi \cdot \delta_X(a),$$

$$\delta_X(\langle \xi, \eta \rangle) = \langle \nabla_X \xi, \eta \rangle + \langle \xi, \nabla_X \eta \rangle.$$

- Curvature:

$$\Theta(X, Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}$$

The Connes-Rieffel Theory of Noncommutative Yang-Mills

Suppose A has a G-invariant tracial state τ, extended to $\mathrm{End}_A(\Xi)$ as usual, and suppose $\mathfrak{g}$ has an invariant inner product (e.g., if G abelian or compact). Define

$$E = -\tau(\langle \Theta, \Theta \rangle).$$

This is the Yang-Mills action. Critical points satisfy the noncommmutative Yang-Mills equation.

Example

A_θ generated by two unitaries U, V satisfying $UV = e^{2\pi i \theta} VU$. A_θ is simple with unique trace τ if $\theta \in \mathbb{R} \smallsetminus \mathbb{Q}$. $G = \mathbb{T}^2$ acts by

$$(z_1, z_2) \cdot U = z_1 U, \quad (z_1, z_2) \cdot V = z_2 V, \quad |z_1| = |z_2| = 1.$$

$$A_\theta^\infty = \{ \sum_{m,n} c_{m,n} U^m V^n \mid c_{m,n} \text{ rapidly decreasing} \}.$$

The Connes-Rieffel Theory (cont'd)

Theorem (Pimsner-Voiculescu)

Assume $\theta \in \mathbb{R} \setminus \mathbb{Q}$. Then τ sets up an order isomorphism of $K_0(A_\theta)$ with $\mathbb{Z} + \theta\mathbb{Z} \subset \mathbb{R}$.

Theorem (Rieffel)

Finitely generated projective A_θ modules are classified by $K_0(A_\theta)_+$.

Theorem (Connes-Rieffel)

Let $A = A_\theta$ as above. Given a projective module Ξ^∞, the minima of E are precisely the connections of constant curvature, and if Ξ is not a multiple of another projective module, then the moduli space of Yang-Mills connections on Ξ^d may be identified with $(T^2)^d/\Sigma_d$.

The Work of Dąbrowski, Krajewski, and Landi

We move now to the noncommutative harmonic map equation. A map $f\colon M^m \to N^n$, say with M and N compact, dualizes to a unital $*$-homomorphism $\varphi\colon A \to B$, $A = C(N)$ and $B = C(M)$. Case of Dąbrowski, Krajewski, and Landi: $N = S^0$. A unital $*$-homomorphism $C(S^0) = \mathbb{C} \oplus \mathbb{C} \to B$ is the same as a nonunital $*$-homomorphism $\mathbb{C} \to B$, i.e., a choice of

$$e = e^* = e^2 \in B.$$

When $A = A_\theta$, $G = \mathbb{T}^2$ as above, the natural "energy" analogous to (1) is

$$E(e) = \tau\big((\delta_1(e))^*\delta_1(e) + (\delta_2(e))^*\delta_2(e)\big). \qquad (2)$$

Result of Dąbrowski, Krajewski, and Landi

The “Euler-Lagrange equation” for critical points of (2) is nonlinear second order. But absolute minima occur when e satisfies the nonlinear first order equation for being *self-dual* or *anti-self-dual*. Dąbrowski, Krajewski, and Landi write down explicit solutions.

The Noncommutative Laplace Equation

A (periodic) harmonic function on a compact Riemannian manifold M^n is a harmonic map $f\colon\ M^n \to S^1$. This is dual to a unital map $C(S^1) \to C(M)$. Noncommutative analogue: $\varphi\colon\ C(S^1) \to A$, or equivalently, a unitary $u \in A$. Harmonicity amounts to looking for critical points of $\tau\big((\nabla(u))^*\nabla(u)\big)$.

Example

$A = M_2(C(S^3))$, $u \in C(S^3, U(2))$, want to minimize energy in homotopy class of the generator of $K^1(S^3)$. Solution is

$$u(z_1, z_2) = \begin{pmatrix} z_1 & z_2 \\ -\overline{z_2} & \overline{z_1} \end{pmatrix}, \quad |z_1|^2 + |z_2|^2 = 1.$$

The Noncommutative Laplace Equation on A_θ

Let $A = A_\theta$ with action of $G = \mathbb{T}^2$ as before. $K_1(A_\theta)$ is free abelian on the classes of U and V.

Theorem

The scalar multiples of $U^m V^n$ are critical points of the energy

$$E(u) = \tau\left((\delta_1(u))^* \delta_1(u) + (\delta_2(u))^* \delta_2(u)\right),$$

and are local minima. Any critical point u depending on U alone is a power of U.

The Noncommutative Laplace Equation (cont'd)

Sketch of Proof.

Since δ_1 and δ_2 generate one-parameter groups of automorphisms, $\tau \circ \delta_j \equiv 0$. We start by deriving the "Euler-Lagrange equations" from the formula for E. If u is unitary, then any nearby unitary is of the form ue^{ith}, $h = h^*$, and

$$\left.\frac{d}{dt}\right|_{t=0} E(ue^{ith}) = \tau\Big(-i\delta_1(h)u^*\delta_1(u) + i\delta_1(u)^*u\delta_1(h) \\ + \text{similar expression with } \delta_2\Big).$$

So u is a critical point iff $\forall h = h^*$,

$$\tau\Big(\delta_1(h)\,\mathrm{Im}\big(\delta_1(u)^*u\big) + \delta_2(h)\,\mathrm{Im}\big(\delta_2(u)^*u\big)\Big) = 0. \qquad (3)$$

The Noncommutative Laplace Equation (cont'd)

Sketch of Proof (cont'd).

In (3), the Im's can be omitted since u unitary $\Rightarrow \delta_j(u)^* u$ skew-adjoint. If $u = e^{i\lambda} U^m V^n$, then $\delta_1(u)^* u = -2\pi i m$ and $\delta_2(u)^* u = -2\pi i n$, so (3) becomes

$$\tau\big(m\delta_1(h) + n\delta_2(h)\big) = 0,$$

which is satisfied since $\tau \circ \delta_j \equiv 0$.
Furthermore, if u depends on U alone, then $\delta_2(u) = 0$. So if u is a critical point, then $\tau\Big(\delta_1(h) \cdot \delta_1(u)^* u\Big) = 0\ \forall h = h^*$. Since the range of δ_1 contains U^m unless $m = 0$ and τ induces a nonsingular pairing, $\delta_1(u)^* u$ is a scalar, and so $u = e^{i\lambda} U^m$ for some m.

The Noncommutative Laplace Equation (cont'd)

Sketch of Proof (cont'd).

Finally let's show that $u = e^{i\lambda} U^m V^n$ is a local *minimum* for E. For simplicity take $m = 1$, $n = 0$. (The general case is similar.) Expanding shows that

$$E(Ue^{iht}) = 4\pi^2 + t^2 \tau\Big(\delta_1(h)^2 + \delta_2(h)^2\Big) + O(t^3).$$

The term in t^2 vanishes exactly when $\delta_1(h) = \delta_2(h) = 0$, i.e., h is a constant, and in that case $E(Ue^{iht}) = 4\pi^2$ (exactly). Otherwise, the coefficient of t^2 is strictly positive and $E(Ue^{iht})$ has a strict local minimum at $t = 0$. □

Maps Between Noncommutative Tori

This section is joint work with Mathai Varghese, Adelaide.

Theorem

Fix Θ and θ in $(0, 1)$, both irrational, and $n \in \mathbb{N}$, $n \geq 1$. There is a unital $*$-homomorphism $\varphi\colon A_\Theta \to M_n(A_\theta)$ if and only if $n\Theta = c\theta + d$ for some $c, d \in \mathbb{Z}$, $c \neq 0$. Such a $*$-homomorphism φ can be chosen to be an isomorphism onto its image if and only if $n = 1$ and $c = \pm 1$.

For simplicity let's take $n = 1$. Denote the canonical generators of A_Θ and A_θ by U and V, u and v, respectively. The natural analogue of $E(f)$ in our situation is

$$E(\varphi) = \tau\Big(\delta_1(\varphi(U))^*\delta_1(\varphi(U)) + \delta_2(\varphi(U))^*\delta_2(\varphi(U)) + \delta_1(\varphi(V))^*\delta_1(\varphi(V)) + \delta_2(\varphi(V))^*\delta_2(\varphi(V))\Big). \tag{4}$$

Harmonic Maps Between Noncommutative Tori

For the automorphism $\varphi_A\colon\ u \mapsto u^p v^q,\ v \mapsto u^r v^s$, with $A = \begin{pmatrix} p & q \\ r & s \end{pmatrix} \in SL(2,\mathbb{Z})$, we obtain

$$\begin{aligned} E(\varphi_A) &= \operatorname{Tr}\Big(\delta_1(u^p v^q)^*\delta_1(u^p v^q) + \delta_2(u^p v^q)^*\delta_2(u^p v^q) \\ &\qquad + \delta_1(u^r v^s)^*\delta_1(u^r v^s) + \delta_2(u^r v^s)^*\delta_2(u^r v^s)\Big) \qquad (5) \\ &= 4\pi^2\Big(p^2+q^2+r^2+s^2\Big). \end{aligned}$$

Conjecture

The value (5) of $E(\varphi_A)$ is minimal among all $E(\varphi)$, $\varphi\colon\ A_\theta^\infty \circlearrowleft$ a $*$-endomorphism inducing the matrix $A \in SL(2,\mathbb{Z})$ on $K_1(A_\theta) \cong \mathbb{Z}^2$.

Results on the Minimum Energy Conjecture

Theorem

The Conjecture is true if $\varphi\colon\ A_\theta^\infty \circlearrowleft$ maps u to a scalar multiple of itself. (In this case, $p = s = 1$ and $q = 0$.) The minimum is achieved precisely when $\varphi(v) = \lambda u^r v$, $\lambda \in \mathbb{T}$.

Theorem

Each φ_A is a critical point for E, and the Conjecture is "locally true" at the critical point φ_A. In other words, there is no continuous family of deformations of $\varphi_A\colon\ A_\theta^\infty \circlearrowleft$ which decreases the energy functional E, and E remains constant in a continuous family of deformations of φ_A only in the case of gauge transformations (multiplication by the images of u and v each by a scalar of modulus 1).

Results on the Minimum Energy Conjecture (cont'd)

Theorem

The Conjecture is true for automorphisms, at least assuming θ satisfies a Diophantine condition (known to hold for almost all θ). In other words, for generic θ, if φ is an automorphism of A_θ^∞ inducing the map given by $A \in SL(2, \mathbb{Z})$ on $K_1(A_\theta)$, then

$$E(\varphi) \geq E(\varphi_A),$$

with equality if and only if $\varphi(U) = \lambda\varphi_A(U)$, $\varphi(V) = \mu\varphi_A(V)$, for some $\lambda, \mu \in \mathbb{T}$.

Summary

- The important geometric elliptic PDE's, like the harmonic map equation, have noncommutative analogues.
- The noncommutative Euler-Lagrange equations are usually very messy. Usually easier to work directly with variational problems.
- Even irrational rotation algebras provide lots of interesting examples.
- Unsolved problems for A_θ:
  - Show the only minimizers for $E(u)$ are $e^{i\lambda} U^m V^n$.
  - Complete study of energy of $*$-automorphisms.
  - What about $*$-endomorphisms, especially when θ a quadratic irrational?
  - What about variation of the metric?