More on the Webster-Yoo Survey

Geometry/Physics RIT

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Dualities

2D Mirror Symmetry: suggests that symplectic and complex manifolds come in pairs

$$\begin{array}{c}\text{Algebraic}\\ \text{Geometry}\end{array} \overset{\begin{array}{c}\text{2D Mirror}\\ \text{Symmetry}\end{array}}{\longleftrightarrow} \begin{array}{c}\text{Symplectic}\\ \text{Geometry}\end{array}$$

Dualities

2D Mirror Symmetry: suggests that symplectic and complex manifolds come in pairs

$$\begin{tikzcd} \begin{matrix}\text{Algebraic}\\ \text{Geometry}\end{matrix} \arrow[r, leftrightsquigarrow, "\begin{matrix}\text{2D Mirror}\\ \text{Symmetry}\end{matrix}"] & \begin{matrix}\text{Symplectic}\\ \text{Geometry}\end{matrix} \end{tikzcd}$$

3D Mirror Symmetry: explains the "symplectic duality" between seemingly unrelated pairs of symplectic varieties by observing that the Coulomb branch of a theory is the Higgs branch of its dual:

$$\mathfrak{M}_A(\mathcal{T}) = \mathfrak{M}_B(\mathcal{T}^\vee)$$

and that the dual pairs of symplectic varieties may be realized as the Higgs branches of dual theories, $(\mathfrak{M}_B(\mathcal{T}), \mathfrak{M}_B(\mathcal{T}^\vee))$.

Symplectic Singularities and Resolutions

Definitions

First, some background on symplectic singularities and resolutions, roughly corresponding to Webster-Yoo section 1.2.

Symplectic Resolution

A *symplectic resolution* consists of

1. A singular affine variety X_0
2. A smooth variety X with an algebraic symplectic form which resolves the singularity

X_0 here is an example of a *symplectic singularity*, which is a singular affine variety having a symplectic form that is well behaved at singularities.

Springer Resolution

A particularly important example of a symplectic resolution is the Springer resolution:

Springer Resolution

The *Springer resolution* is a symplectic resolution, with X_0 the variety of nilpotent elements in a semisimple Lie algebra $\mathfrak{g}$, and X the cotangent bundle of the flag variety of $\mathfrak{g}$.

Property: The Springer resolution corresponding to $\mathfrak{g}$ is fundamental, in the sense that all other aspects of Lie theory can be obtained from the Springer resolution.

Examples

Example 1: Cotangent bundle of $\mathbb{CP}^{n-1}$:

$$T^*\mathbb{CP}^{n-1} = \{(\ell, \phi) \in \mathbb{CP}^{n-1} x M_{nxn}(\mathbb{C}) | \phi(\mathbb{C}^n) \subset \ell,\ \phi(\ell) = \{0\}\}$$

Projection to the second component is a resolution of $M_{nxn}^{rk=1}(\mathbb{C})$. Since this cotangent bundle has a canonical symplectic form, this is a symplectic resolution. Let $X_n^{(A)}$ denote the space $T^*\mathbb{CP}^{n-1}$.

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Example 2: $\mathbb{Z}/n\mathbb{Z} \curvearrowright \mathbb{C}^2$ by

$$k \mapsto \begin{pmatrix} \exp(2\pi ik/n) & 0 \\ 0 & \exp(-2\pi ik/n) \end{pmatrix}$$

Then $\mathbb{C}^2/(\mathbb{Z}/n\mathbb{Z})$ has a unique symplectic resolution $X_n^{(B)}$.

Examples

Looking closer at $X_n^{(A)}$ and $X_n^{(B)}$ indicates some underlying relation. Consider:

—The action of a maximal torus $T^{(A)}$ on $X_n^{(A)}$ (resp. B) preserving the symplectic structure.

—The affine variety X_0 has unique maximal decomposition into finitely many smooth pieces with induced symplectic structure, generalizing the decomposition of nilpotent matrices.

Relation of $X_n^{(A)}$ and $X_n^{(B)}$

1.

$$\mathfrak{t}^{(A)} \simeq H^2(X_n^{(B)}) \quad \mathfrak{t}^{(B)} \simeq H^2(X_n^{(A)})$$

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2. Both torus actions have the same number of fixed points, which is n, implying that the sum of $X_n^{(\sim)}$ is n.

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3. The stratifications on $X_0^{(A)}$ and $X_0^{(B)}$ have the same number of pieces.

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4. The above are related to the deeper fact that the "universal enveloping algebra" $X_n^{(\sim)}$ has a category of representations called "category $\mathcal{O}$" and the categories $\mathcal{O}$ of $X_n^{(A)}$ and $X_n^{(B)}$ are *Koszul dual*: the homeomorphisms between projective modules in one category describe the extensions between simple modules in the other.

Symplectic Duality

Far from being unique to $X_n^{(A)}$ and $X_n^{(B)}$ as defined above, these properties apply to many pairs of symplectic singularities.

More examples include finite and affine type-A quiver varieties, and smooth hypertoric varieties

Field Theory and Mirror Symmetry

Physics

Now, for some relevant definitions from physics (corresponding to part 2 of the Webster-Yoo survey). In particular, we ask:

—What are 3D, $\mathcal{N} = 4$ SUSY QFTs and their topological twists?

—How do they relate to symplectic duality?

We will see that a 3D $\mathcal{N} = 4$ QFT $\mathcal{T}$ gives a choice of two topological twists, which give two TQFTs corresponding to the Coulomb and Higgs branches.

Quantum Field Theory

Generally, a quantum field theory consists of

1. **(spacetime)** a d-dimensional Riemannian manifold (M,g)
2. **(field)** a fibre bundle B over M and a space $\mathcal{F}(M) = \Gamma(M, B)$ of sections of B over M
3. **(action functional)** a functional $S : \mathcal{F} \to \mathbb{R}$

We can think of $\mathcal{F}$ as the space of all possible states of a physical system, while S controls which states are likely to be realized.

Examples

Free Scalar Field Theory

1. M compact
2. B trivial bundle $M \times \mathbb{R}$ so $\mathcal{F}$ is the smooth functions $C^{\infty}(M)$.
3. $S : C^{\infty}(M) \to \mathbb{R}$ given by $S(\phi) = \int_M \phi \Delta_g \phi \mathrm{Vol}_g$ (Δ_g is the Laplacian of the metric).

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Gauge Theory

For a compact Lie group G_c, with $\mathcal{F}$ consisting of connections on a principal G_c-bundle over M, we have a gauge theory with gauge group G_c.

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σ-Model

For X a manifold, a σ-model is a theory with $\mathcal{F}$ consisting of all maps from M to X (the "target"). $B = M \times X$.

Correlation Functions

Definition

Given observables O_i, which depend only on the values of the fields on open sets that do not overlap, define

$$\langle O_1, ..., O_n \rangle = \int_{\phi \in \mathcal{F}(M)} e^{-S(\phi)/\hbar} O_1(\phi) ... O_n(\phi) D\phi$$

These correlation functions for a theory $\mathcal{T}$ are the main objects of study.

Topological Quantum Field Theory

Definition (Atiyah-Segal)

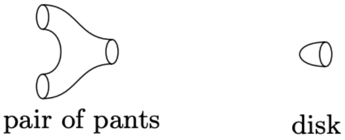
A d-dimensional TQFT is a symmetric monoidal functor Z from $(\text{Bord}_d, \coprod, \phi)$ to $(\text{Vec}_{\mathbb{C}}, \otimes, \mathbb{C})$.

For closed d-manifold M, we have that M is a bordism from ϕ to ϕ, in which case $Z(M)$ is a complex number, which may be physically thought of as

$$Z(M) = \int_{\phi \in \mathcal{F}(M)} e^{-S(\phi)/\hbar} D\phi.$$

The complex vector space $Z(M)$ attached to a closed (d-1)-manifold N is the Hilbert state of spaces on N. In particular, for $N = S^{d-1}$, $Z(S^{n-1})$ is the vector space of local operators in the TQFT (state-operator correspondance).

Commutative Algebra Structure



d=2: In this case, it may be shown that the linear maps $m : Z(S^1) \otimes Z(S^1) \to Z(S^1)$ and $u : Z(S^1) \to Z(S^1)$ associated with the pair of pants and disc (plus the reversed versions), respectively, induce a commutative Frobenius algebra structure on $Z(S^1)$.
For higher d, it may also be shown that $Z(S^{d-1})$ has a commutative $\mathbb{C}$-algebra structure, and therefore is the coordinate ring for an algebraic variety.

$QFT \Rightarrow TQFT$

Topological quantum field theories can be obtained from supersymmetric QFTs by applying a *topological twist*.

SUSY Field Theory

Roughly, these a field theories that admit nontrivial "odd" symmetries called supercharges. That is, the $\mathcal{F}$ ($\mathbb{Z}/2\mathbb{Z}$ graded) carries the action of a Lie superalgebra $\mathfrak{a}$ called a super-Poincaré algebra with even part $\mathfrak{a}_0 = \mathfrak{so}(d) \ltimes \mathbb{R}^d$ and odd part $\mathfrak{a}_1 = \Sigma$ the copies of spin representations of $\mathfrak{so}(d)$. A Lie bracket is given by the action of $\mathfrak{so}(d)$ on Σ, and a symmetric pairing $\Gamma : \Sigma \otimes \Sigma \to \mathbb{R}^d$ of $\mathfrak{so}(d)$ representations.

$QFT \Rightarrow TQFT$

To get a TQFT from a QFT, suppose Q is a supercharge s.t. [Q,Q]=0. Since Q is odd, $\frac{1}{2}[Q, Q] = Q^2$ acts as zero in representations of $\mathfrak{a}$, so one may consider $\mathcal{F}$ as a $\mathbb{Z}/2\mathbb{Z}$-graded complex and take its Q-cohomology. The result is a simpler theory, called a *topological twist* of the original theory.

Existence of topological twists of a theory depends on d and $\mathcal{N}$. In particular, for $d = 3$ and $\mathcal{N} = 4$, there are exactly 2 topological twists Q_A and Q_B.

(2D) Mirror Symmetry

For a Calabi-Yau manifold X, there is a $d = 2$, $\mathcal{N} = (2,2)$ SUSY $\sigma-$model $\mathcal{T}^{d=2}(X)$ with two topological twists corresponding to Q_A and Q_B, denoted $\mathcal{T}_A^{d=2}(X)$ and $\mathcal{T}_B^{d=2}(X)$. The A-twist depends on the symplectic topology of X, and the B-twist depends on the complex geometry of X.

Duality

Mirror symmetry identifies $\mathcal{T}^{d=2}(X)$ with $\mathcal{T}^{d=2}(X^\vee)$, where $X^\vee$ is the mirror dual manifold. Since this duality is preserved by twists we have $\mathcal{T}_A^{d=2}(X)$ and $\mathcal{T}_B^{d=2}(X^\vee)$ are also identified.

(3D) Mirror Symmetry

We have an analogous result for $d = 3$ $\mathcal{N} = 4$ theories: there is an equivalance of twisted theories $\mathcal{T}_A = Z_A^{\mathcal{T}}$ and $\mathcal{T}_B^\vee = Z_B^{\mathcal{T}}$.

Recall that $Z_A^{\mathcal{T}}(S^2)$ and $Z_A^{\mathcal{T}}(S^2)$ are commutative algebras, denoted $\mathcal{A}_A(\mathcal{T})$ (resp. B). Then we define

$$\mathfrak{M}_A(\mathcal{T}) = SpecZ_A^{\mathcal{T}} \quad \mathfrak{M}_B(\mathcal{T}) = SpecZ_B^{\mathcal{T}}$$

which are the Coulomb and Higgs branches, respectively. We have that

$$\mathfrak{M}_A(\mathcal{T}) \simeq \mathfrak{M}_B(\mathcal{T}^\vee) \quad \mathfrak{M}_B(\mathcal{T}) \simeq \mathfrak{M}_A(\mathcal{T}^\vee).$$

Advanced Directions

Stable Envelopes

Discussed in a recent paper by Aganagić and Okounkov, elliptic stable envelopes are classes on each symplectic resolution with a Hamiltonian $\mathbb{C}^*$ action. Many examples arise from the Coulomb/Higgs branches of 3D $\mathcal{N} = 4$ theories.

We also have equivariant stable envelopes, classes in equivariant elliptic cohomology which correspond to the thimbles flowing to the different $\mathbb{C}^*$-fixed points on the resolution. "These play an important role in the study of enumerative geometry and are expected to be one of the key mathematical manifestations of 3d mirror symmetry."

Koszul Duality of Category $\mathcal{O}$'s

See paper by Braden, Licata, Proudfoot, and Webster.

Conjecture

The categories $\mathcal{O}$ of mirror dual symplectic singularities (e.g. Higgs and Coulomb branches) are Koszul dual.