A Panorama of Hungarian Mathematics in the Twentieth Century: Non-Commutative Harmonic Analysis

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For present purposes, we shall define *non-commutative harmonic analysis* to mean the decomposition of functions on a locally compact G-space X,[1] where G is some (locally compact) group, into functions well-behaved with respect to the action of G. The classical cases are of course Fourier series, when $G = X = \mathbb{T}$, the circle group, and the Fourier transform, when $G = X = \mathbb{R}$, but we will mostly be concerned with the case when G is non-commutative. Since this subject is inextricably linked with the subject of representations of G (unitary representations, if we specialize to the case of L^2-functions), we will also consider the general theory of representations of locally compact groups and of various related structures, such as Lie algebras and Jordan algebras.

The subject of group representations was created by Georg Frobenius [9] in a remarkable series of papers in the 1890's, and continued in the first decade of the twentieth century in the work of his student Issai Schur [19]. However, Frobenius worked exclusively with finite groups, and his treatment was purely algebraic. It took a while before it was realized that Frobenius' theory had important implications for harmonic analysis. The generalization of the theory to compact groups was largely carried out by Hermann Weyl, and applications to harmonic analysis on compact groups did not come until the Peter-Weyl Theorem ([17]; reprinted in [31], pp. 387–404).

It is against this background that we shall consider the contributions of a few great Hungarian mathematicians: Alfred Haar, John von Neumann, and Eugene Wigner in the 1920's, 1930's, and 1940's; and in somewhat later generations, Béla Szőkefalvi-Nagy and Lajos Pukánszky. As there is room here to discuss only a few of their contributions, we refer the reader to the scientific obituaries [20], [25], [16], [10], [15], [33], [26], [6], and [5] for more details.

[1] This means that X is a locally compact space and we are given a continuous map $G \times X \to X$, $(g, x) \mapsto gx$, such that $(gh)x = g(hx)$ and $ex = x$ for all $g, h \in G$ and $x \in X$, where e is the identity element of G.

1 Haar, von Neumann, and Wigner

Alfred (Alfréd) Haar, Eugene Paul (Jenő Pál) Wigner, and John (János) von Neumann were all born in Budapest: Haar in 1885, Wigner in 1902, von Neumann in 1903. All three had the good fortune to have as their secondary school mathematics teacher László Rátz of the Evangelical Lutheran High School in Budapest. Rátz seems to have done a remarkable job in encouraging mathematical talent, and was also the founder of the *Mathematics Journal for Secondary Schools*, or *Középiskolai Matematikai Lapok*. (See [25] and "Eugene Paul Wigner: A Biographical Sketch" in [32], pp. 3–14.) These three students of Rátz were among the most important contributors to the development of non-commutative harmonic analysis.

1.1 Hilbert's Fifth Problem

Hilbert's Fifth Problem [12] asked "how far Lie's concept of continuous groups of transformations is approachable in our investigations without the assumption of the differentiability of the functions." Hilbert's Fifth Problem can be said to mark the beginning of the subject of non-commutative harmonic analysis.

Among the very first results in the direction of a solution was a paper of von Neumann, "Zur Theorie der Darstellungen kontinuierlichen Gruppen" (*Sitzungber. der Preuss. Akad.* (1927), 76–90; reprinted in [29], vol. 1, pp. 134–148). This paper basically proves that any continuous finite-dimensional representation of a Lie group is automatically differentiable, in fact analytic.

1.2 Invariant Measures and Analysis on Locally Compact Groups

It was soon realized that a reasonable attack on more substantial cases of Hilbert's Fifth Problem requires a means of doing analysis on a locally compact group, comparable to the sort of analysis one does with functions in Euclidean space. Since analysis on Euclidean space is based in large part on Lebesgue integration, the "search was on" for a means of invariant integration on a general locally compact group. In the case of an n-dimensional Lie group G, since G is an orientable manifold, G always admits left-invariant smooth measures, which can be identified with non-zero left-invariant differential n-forms, or in other words with non-zero elements of the one-dimensional vector space $\bigwedge^n \mathfrak{g}^*$, where $\mathfrak{g}$ is the Lie algebra of G, that is, the vector space of left-invariant vector fields. Note that one-dimensionality of $\bigwedge^n \mathfrak{g}^*$ implies that left-invariant measures on G are unique up to a scalar multiple.

One of Haar's greatest mathematical contributions was his proof, in the paper "Der Massbegriff in der Theorie der kontinuierlichen Gruppen," (*Ann. of Math.* (2) **34** (1933), 147–169; reprinted in [11], 600–622), that *every* locally compact group G admits a left-invariant measure. Haar's paper also appeared slightly earlier in Hungarian (*Mat. Term. Ért.* **49** (1932), 287–307; reprinted in [11], 579–599). While various improved reformulations of Haar's method have been given, notably the elegant ones due to Weil [30] and Cartan [3], to this author's knowledge, no one has ever improved on his main idea, which is to compare the relative size of two compact sets A and B with dense interiors, by letting $h(A;B)$ be the minimal number of translates of B required to cover A. From this data Haar constructs his measure m by letting

$$m(A) = \lim_n \frac{h(A;B_n)}{h(C;B_n)},$$

where C is fixed once and for all and where the sets B_n run over a compact neighborhood base of the identity. Haar was well aware that his theorem made possible harmonic analysis on non-Lie topological groups, and he remarks at the end of his paper that it immediately follows that one can prove the Peter-Weyl Theorem ([17]; reprinted in [31], pp. 387–404), giving a decomposition of $L^2(G)$ into an orthogonal direct sum of matrix coefficients of irreducible representations, for any compact group G, not just a Lie group.

Haar and von Neumann were in close contact at the time of this work, and a paper of von Neumann on Hilbert's Fifth Problem, "Die Einführung analytischer Parameter in topologischen Gruppen" ([29], vol. 2, pp. 366–386) was submitted to the *Annals* the same day as Haar's paper and published right next to it. In it, von Neumann proves that every compact group which is *topologically* locally Euclidean is a Lie group, i.e., admits an analytic structure. From this and the Peter-Weyl Theorem, it follows that every compact group is an inverse limit of Lie groups.

Two other important papers of von Neumann follow up on the theme of Haar's work. In "Zum Haarschen Maß in topologischen Gruppen" (*Compositio Math.* **1** (1934), 106–114; also [29], vol. 2, pp. 445–453), von Neumann gives an easier proof of the existence of Haar measures on a compact group G, by proving that if f is a continuous function on G, then the closed convex hull of the translates of f contains a unique constant function (the value of the constant being of course $\int f(g)\,dg$). This remains the easiest proof of existence of Haar measure for compact groups. Then in "The uniqueness of Haar's measure" (*Mat. Sb.* **1** (1936), 721–734; also [29], vol. 4, pp. 91–104), von Neumann gives a proof that a left (or right) Haar measure is unique up to scalar multiples, just as in the case of invariant smooth measure on a Lie group. This important result was proved independently, using different methods, by André Weil [30] and Henri Cartan [3].

1.3 Representation Theory and Quantum Physics

Among the most important motivations for the development of non-commutative harmonic analysis in the years between the two World Wars was the development of quantum mechanics. Indeed, Weyl, von Neumann, and Wigner all approached the subject of non-commutative harmonic analysis with quantum mechanics in mind, and Wigner always considered himself more of a physicist than a mathematician.

As early as his paper "Über nicht kombinierende Terme in der neueren Quantentheorie" of 1926 (*Z. für Physik* **40** (1926–27), 492–500 and 883–892; reprinted in [32], pp. 34–52), Wigner realized that Frobenius' theory of representations of the symmetric group was relevant to the study of wave functions of multiparticle systems. This can be regarded as an example of non-commutative harmonic analysis in the sense of this article, with $G = S_n$. In his paper, Wigner thanks von Neumann for telling him about the work of Frobenius and Schur.

The following year, 1927, Wigner spent at Göttingen as Hilbert's assistant. There he met several mathematicians and physicists and began to collaborate with Pascual Jordan. In their paper "Über das Paulische Äquivalenzverbot of 1928 (*Z. für Physik* **47** (1928), 631–651; reprinted in [32], pp. 109–129), Jordan and Wigner first reformulated the Pauli exclusion principle in terms of representations of the "canonical anticommutation relations" (CAR). The Clifford algebras defined by the CAR of course play a pivotal role in the Dirac equation of the electron, and in fact the connection between Dirac and Wigner was more than scientific: Dirac later married Wigner's sister.

The work of Wigner and Jordan was the precursor of the famous paper "On an algebraic generalization of the quantum mechanical formalism" by Jordan, von Neumann, and Wigner (*Ann. of Math.* (2) **35** (1934), 29–64; reprinted in [32], pp. 298–333 and in [29], vol. 2, pp. 408–444) that founded the study of what are now called Jordan algebras. In their paper, Jordan, von Neumann, and Wigner obtain the classification of the finite-dimensional simple Jordan algebras over $\mathbb{R}$, including the exceptional ones coming from the Cayley octonians. We now know that one can trace the existence of the exceptional compact Lie groups G_2, F_4, E_6, E_7, and E_8 to these exceptional Jordan algebras.

We have mentioned the CAR, the anticommutation relations that govern the behavior of fermions. Of equal importance are the "canonical commutation relations" (CCR) for bosons, that the position operators Q_k and momentum operators P_j should satisfy

$$Q_k P_j - P_j Q_k = i\hbar\delta_{kj}, \qquad Q_k^* = Q_k, \qquad P_j^* = P_j, \tag{1}$$

where $\hbar$ is Planck's constant. These simple relations, familiar to every physics student, hide a serious mathematical difficulty: the equations (1) have no finite-dimensional solutions,[2] in fact no solutions in bounded operators! And for unbounded operators which are not everywhere defined, what is the meaning of the commutator? The problem of making rigorous sense of (1), and of showing that there is essentially only one irreducible solution (satisfying certain nice regularity properties), was solved by Hermann Weyl, Marshall Stone, and von Neumann. The final result, in von Neumann's important paper "Die Eindeutigkeit der Schrödingerschen Operatoren" (*Math. Ann.* **104** (1931), 570–578; also [29], vol. 2, pp. 221–229), turned out to be important not only for theoretical physics but also for the future development of unitary representation theory.

[2]The reason is that for any finite-size matrices Q and P, $\mathrm{Tr}\,(QP - PQ) = 0$, so $QP - PQ$ cannot be a non-zero multiple of the identity.

The idea is to note that (1) amounts to looking for a representation through skew-adjoint operators of what is now called the *Heisenberg Lie algebra* $\mathfrak{g}$ of dimension $2n+1$, with a basis $X_1, \ldots, X_n, Y_1, \ldots, Y_n, Z$ satisfying

$$[X_j, Y_k] = \delta_{kj} Z, \qquad [Z, X_j] = 0, \qquad [Z, Y_k] = 0. \tag{2}$$

Now to the Lie algebra $\mathfrak{g}$ we can attach a simply connected nilpotent Lie group G. Topologically it looks like $\mathbb{R}^{2n+1}$, but the multiplication is slightly twisted. We can impose a regularity condition on our Lie algebra representations by requiring that they come from unitary representations π of G, that is, strongly continuous homomorphisms from G to the unitary group of some Hilbert space, for which the corresponding "infinitesimal representation" is $d\pi(Z) = i\hbar$, $d\pi(X_j) = iP_j$, $d\pi(Y_k) = iQ_k$. Von Neumann's Theorem says that up to unitary equivalence, there is one and only one irreducible such representation, and every unitary representation π of G with $d\pi(Z) = i\hbar$ is a multiple of this irreducible representation.

1.4 Invariant Means and Almost Periodic Functions and Groups

In von Neumann's paper "Zum Haarschen Maß in topologischen Gruppen," cited above, there appears an interesting "Zusatz während der Korrektur," in which von Neumann mentions that he had noticed that his argument for constructing the Haar measure on a compact group can be extended to some non-compact groups as well, but that in this case it gives rise not to Haar measure but to an invariant "mean" $f \mapsto \int f$ for which the constant function 1 has "mean value" 1. (On the other hand, Haar measure on a non-compact group is never a finite measure, so on a non-compact group, constant functions are never integrable with respect to Haar measure.) The study of such means was to lead to a whole other direction in non-commutative harmonic analysis. Von Neumann's Zusatz asserts that "die Ausführung erscheint demnächst in den *Annals of Mathematics*." Evidently he misspoke; the paper von Neumann refers to, "Almost periodic functions in a group, I" was to come out in the *Transactions* of the *Amer. Math. Soc.* (**36** (1934), 445–492; also [29], vol. 2, 454–501), not the *Annals*.

To explain von Neumann's discovery, we have to back up a bit and review Harald Bohr's theory of almost periodic functions [2]. In its simplest version, this refers to bounded uniformly continuous functions f on the line with a "Fourier series" expansion $f(x) \sim \sum_j c_j e^{i\lambda_j x}$ (convergent in a suitable sense), where the frequencies λ_j are not necessarily rationally related to one another, and thus f is not necessarily periodic. Of course we cannot expect expect the Fourier series of f to converge uniformly to f, since this is not always true even when f is literally periodic. Instead, Bohr found that a natural notion of convergence in this context is convergence in mean over bigger and bigger intervals, i.e., that

$$\lim_{N\to\infty} \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \Big| f(x) - \sum_{j=1}^{N} c_j e^{i\lambda_j x} \Big|^2 \, dx = 0,$$

and that there is a well-defined notion of *mean* for such functions, namely

$$\overline{f} = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f(x)\, dx.$$

Existence of the limit here is Bohr's "Mittelwertsatz" in [2]. What von Neumann noticed is that for almost periodic functions f on $\mathbb{R}$, the closed convex hull of the translates of f contains a unique constant function, and the value of this constant is the Bohr mean value of f.

Now in the earlier paper "Zur allgemeinen Theorie des Masses" (*Fund. Math.* **13** (1929), 76–116 and 333; reprinted in [29], vol. 1, pp. 599–643), von Neumann had studied the similar subject of means on *discrete* groups, and had defined a group G to be "messbar" (literally *measurable*, but we will instead use the now-standard terminology *amenable*) if it admits a left-invariant mean. Such a mean can be viewed as an "integration" process $f \mapsto \int f$ on bounded functions, for which the constant function 1 has "mean value" 1, and for which $\int f = \int \lambda(x) f$, if $\lambda(x)f$ denotes the left translate of f by $x \in G$, i.e., $(\lambda(x)f)(y) = f(x^{-1}y)$. Von Neumann proved that finite and abelian (discrete) groups are amenable, and that the class of amenable groups is closed under extensions and direct limits. It follows that there is a fairly large class of "obviously" amenable groups, what are now called *elementary amenable* groups: the smallest class containing the solvable and finite groups and closed under extensions and direct limits. On the other hand, von Neumann exhibited countable groups that are *not* amenable, for example, the free group on two generators. It is important to note that on infinite amenable groups, invariant means are not at all unique. For example, a bounded function f on the group $G = \mathbb{Z}$ is simply a two-sided bounded infinite sequence $f = \{f_n\}_{n\in\mathbb{Z}}$, and the possible means of f are all the various limit points of the sequence of approximating averages

$$\frac{1}{2N+1}\bigl(f_{-N} + \cdots + f_0 + \cdots + f_N\bigr)$$

as $N \to \infty$.

Let us come back to von Neumann's work on almost periodic functions, continued in two later papers (the first with Salomon Bochner, "Almost periodic functions in a group, II," *Trans. Amer. Math. Soc.* (**37** (1935), no. 1, 21–50; also [29], vol. 2, 528–557; and the second with E. Wigner, "Minimally almost periodic groups," *Ann. of Math.* (2) **41** (1940), 746–750; in [29], vol. 4, 220–224 and in [32], pp. 390–394). To make a long story short, von Neumann defines two classes of groups, which are in some sense opposite extremes. *Minimally almost periodic groups* admit no non-constant almost periodic functions (in a natural sense extending Bohr's). *Maximally almost periodic groups* admit enough almost periodic functions to separate points. André Weil [30] eventually cleaned up the theory and showed that, for every maximally almost periodic group G, there exists a compact group G^* which contains G as a dense subgroup and such that every continuous real-valued almost periodic function on G can be uniquely extended to a continuous (and hence almost periodic) function on G^*.

In fact, a locally compact group is maximally almost periodic if and only if it has a continuous embedding into a compact group. Thus as von Neumann pointed out, the Bohr mean on almost periodic functions really comes from the construction of Haar measure on compact groups, and in the case of an abelian locally compact group, it coincides with the restriction of an invariant mean on all bounded uniformly continuous functions. However, residually finite discrete groups[3] are always maximally almost periodic, but not always amenable, so even in the case of discrete groups, the Bohr mean on almost periodic functions does not always extend to a mean on all bounded functions.

1.5 Von Neumann Algebras

Among von Neumann's greatest contributions was the development of the theory of what he called *rings of operators*, and what are now called *von Neumann algebras*. A von Neumann algebra is simply a subalgebra of $\mathcal{B}(\mathcal{H})$, the algebra of all bounded operators on a Hilbert space $\mathcal{H}$, which is stable under the involution $T \mapsto T^*$ and closed under the strong (or equivalently, weak) operator topology. The massive papers of von Neumann on this subject, almost all of them joint with Francis Joseph Murray, fill all of volume III of [29]. It would be impossible to do justice to them here, so we refer the reader to [10] for more information, but we just briefly point out what this huge body of work has to do with non-commutative harmonic analysis.

Suppose G is a locally compact group. A *unitary representation* π of G on a Hilbert space $\mathcal{H}$ means a homomorphism π from G to the group of unitary operators on $\mathcal{H}$, which is continuous with respect to the strong operator topology (or the weak operator topology—it gives exactly the same notion). Note that we do *not* require continuity in the norm topology for operators, since this fails for the standard example of a unitary representation, namely the left regular representation λ of G on $L^2(G)$ (L^2 being defined with respect to Haar measure). Then the possible decompositions of π into subrepresentations are governed by the structure of the *commutant* of the representation,

$$\pi(G)' = \{T \in \mathcal{B}(\mathcal{H}) : T\pi(g) = \pi(g)T \text{ for all } g \in G\}.$$

For example, Schur's Lemma says that π is irreducible if and only if $\pi(G)' = \mathbb{C}$. But $\pi(G)'$ is a von Neumann algebra, so the classification theory of von Neumann algebras comes into play at this point. For example, we call π *multiplicity-free* if $\pi(G)'$ is abelian and a *factor representation* if $\pi(G)'$ is a factor, that is, a von Neumann algebra with one-dimensional center. The Murray-von Neumann papers classify factors into three types. If $\pi(G)'$ is a type I factor, then π is simply a multiple of a single irreducible representation. But if $\pi(G)'$ is a type II or type III factor, then there is no canonical way to decompose π into irreducible representations, so that type II or type III factor representations should themselves be regarded as basic building blocks of representation theory. Some groups (for example, abelian or compact groups, or the Heisenberg

[3] This means groups like $SL(n, \mathbb{Z})$, the $n \times n$ matrices with integer entries and determinant 1, with enough homomorphisms to finite groups to separate points.

group defined by (2)) are *type I*, in the sense that the commutants of their unitary representations are *always* type I. For such groups, at least if G is second countable, von Neumann's theory of direct integral decompositions ("On rings of operators. Reduction theory," *Ann. of Math.* (2) **50** (1949), 401–485; [29], vol. 3, pp. 400–484) provides a canonical way of decomposing all unitary representations (on separable Hilbert spaces) into irreducible pieces, and there is a hope to copy many features of the Frobenius-Schur theory for finite groups. But for non-type I groups, one is forced to contend with non-type I factor representations. For example, Murray-von Neumann proved that the commutant of the regular representation of a discrete group G is a finite type II factor if and only if G is an ICC-group,[4] that is, if the identity of G is the only element whose conjugacy class is finite.

One of the other great contributions of Murray and von Neumann was the theory of the trace on a type II factor. If π is a finite-dimensional unitary representation, then one can define its character χ_π, a class function on G, by $\chi_\pi(g) = \operatorname{Tr} \pi(g)$. (This definition was introduced by Frobenius.) Furthermore, all one needs to know about π can be recovered from the character χ_π. It would be nice to do something similar for certain other factor representations. If π is a *finite* type II factor representation, then there is a continuous linear functional Tr on the von Neumann algebra generated[5] by $\pi(G)$, satisfying $\operatorname{Tr}(1) = 1$ and the usual trace property $\operatorname{Tr}(ab) = \operatorname{Tr}(ba)$, and so it is again possible to develop a theory of group characters similar to the one for finite groups. If π is an infinite-dimensional irreducible representation or a II_∞ factor representation, then $\pi(G)''$ admits a trace, but it is only partially defined, and in particular $\operatorname{Tr}(u)$ is undefined for u unitary. So in this case, while it is possible that $\chi_\pi(g) = \operatorname{Tr} \pi(g)$ might make sense as an equality of *distributions*, provided there are enough functions ϕ on G for which $\pi(\phi) = \int \phi(g)\pi(g)\, dg$ is trace-class, $\chi_\pi = \operatorname{Tr} \circ \pi$ does not make sense directly as a function on the group G.[6]

1.6 Development of Unitary Representation Theory of Non-Compact Lie Groups

The modern development of the unitary representation theory of non-compact Lie groups, which today is now a large subject, grew out of the work of Gelfand-Naimark, Bargmann, Mackey, and Harish-Chandra in the late 1940's and the 1950's. One of the key papers that prompted this development was Wigner's paper "On unitary representations of the inhomogeneous Lorentz group" (*Ann. of Math.* (2) **40** (1939), 149–204; reprinted in [32], pp. 334–389). Curiously, this paper was first submitted to the *American Journal of Mathematics*, usually

[4]This stands for "infinite conjugacy classes."

[5]By von Neumann's Double Commutant Theorem, found in §II of "Zur Algebra der Funktionaloperationen und Theorie der normalen Operatoren" (*Math. Ann.* **102** (1929), 370–427; [29], vol. 2, pp. 86–143), this is just the commutant $\pi(G)''$ of $\pi(G)'$. The Double Commutant Theorem implies that a $*$-subalgebra A of $\mathcal{B}(\mathcal{H})$ is a von Neumann algebra if and only if it is equal to its double commutant A''.

[6]Note that $\pi(\phi)$ as we have just defined it can be viewed as a sort of operator-valued Fourier coefficient of ϕ. Its trace $\chi_\pi(\phi)$, when defined, is a sort of scalar-valued Fourier coefficient.

regarded as being somewhat less prestigious than the *Annals*, and was rejected there with the remark that "this work is not interesting for mathematics" ([32], p. 9). In any event, this paper is important for two reasons: it promoted interest in the unitary representations of the actual Lorentz groups $SO_0(2,1)$ and $SO_0(3,1)$, classified soon afterwards by Bargmann [1], and it amounted to the working out of an important special case of what was later formulated as the Mackey Imprimitivity Theorem [14], and thus motivated the modern point of view on the decomposition of representations of group extensions.[7] More precisely, Wigner's paper studies unitary representations of the semidirect product $G = V \rtimes H$, where $V = \mathbb{R}^{3,1}$ is Minkowski space and $H = SO_0(3,1)$ is the Lorentz group acting on V the usual way. Wigner proves that each irreducible unitary representation of G is supported on a single H-orbit in $\widehat{V} \cong V$. Wigner only studies the cases where this orbit is either one of the light cones or else half of a two-sheeted hyperboloid. (The representations supported on the trivial H-orbit $\{0\}$ factor through H, and were only classified later in [1].) In either case, if one thinks of an elementary particle corresponding to such an irreducible representation, and views its wave function ϕ as a vector-valued L^2-function on Minkowski space V, then this analysis shows that the Fourier transform of ϕ is supported on one of the two components of the variety where $\hat{x}_4^2 - \hat{x}_1^2 - \hat{x}_2^2 - \hat{x}_3^2 = m^2$, where m is a constant corresponding to the mass of the particle and the $\hat{x}_j$ are the Fourier transform variables, and thus ϕ itself satisfies the Klein-Gordon equation

$$\Box\phi = m^2\phi, \text{ where } \Box = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2} - \frac{\partial^2}{\partial x_4^2}.$$

We have already discussed (in section 1.5) the work of von Neumann on unitary representations of discrete groups, and the application of von Neumann's theory of direct integral decompositions to the decomposition of unitary representations of arbitrary second countable locally compact groups. However, Haar also worked on representation theory. A little-known paper of Haar, "Über die Gruppencharaktere gewisser unendlichen Gruppen" (*Acta Sci. Math.* (Szeged) **5** (1932), 172–186; reprinted in [11], pp. 172–186) extended the Frobenius theory of group characters from finite groups to what are now usually called *FC-groups*,[8] groups in which every conjugacy class is finite. What Haar called characters in this case turn out to be the same thing as characters in the more modern sense of traces of finite factor representations, as later studied by Elmar Thoma [27]. An interesting fact about FC-groups is that if they are finitely generated, then they are virtually abelian groups, that is, have an abelian subgroup of finite index. It was eventually shown by Thoma [28] that the virtually abelian groups are precisely the class of type I groups, discrete groups whose unitary representations always generate type I von Neumann algebras, and in fact the

[7] The other key case of the Imprimitivity Theorem that was known before was von Neumann's theorem on uniqueness of representations of the Heisenberg commutation relations, discussed above in Section 1.3.

[8] "FC" stands for "finite conjugacy classes."

dimensions of their irreducible representations are bounded. Hence for finitely generated FC-groups, non-commutative harmonic analysis in the sense of Haar is precisely the decomposition of functions into matrix coefficients of irreducible representations, as in the situation of the Peter-Weyl Theorem.

2 Sz.-Nagy and Pukánszky

2.1 Béla Sz.-Nagy

Béla Sz.-Nagy, one of the great operator theorists of the twentieth century, made a few interesting contributions to non-commutative harmonic analysis, even though this was not his primary mathematical interest. Here we will just mention four of them. The first is a strengthening of von Neumann's automatic analyticity theorem for homomorphisms of Lie groups (see Section 1.1): Nagy [21] proves that measurability is enough to guarantee analyticity; one does not need to assume continuity from the start.

The second contribution, on its face, only deals with *commutative* harmonic analysis. In [22], Nagy proved that a one-parameter group $\{T^n\}_{n\in\mathbb{Z}}$ or $\{T_s\}_{s\in\mathbb{R}}$ of invertible linear operators on a Hilbert space $\mathcal{H}$ is similar to a unitary representation (of $\mathbb{Z}$ or $\mathbb{R}$) if and only if it is *uniformly bounded*, i.e., there is a constant $K > 0$ such that $\|T^n\| \leq K$ for all $n \in \mathbb{Z}$ or $\|T_s\| \leq K$ for all $s \in \mathbb{R}$. The connection with unitary representation theory is that essentially the same theorem, with almost the same proof, holds for uniformly bounded representations π of arbitrary amenable locally compact groups G, as was pointed out by Jacques Dixmier [4]. In other words, if π is a strongly continuous homomorphism from G to the invertible linear operators on $\mathcal{H}$, and if $\|\pi(g)\| \leq K$ for all $g \in G$, then

$$\langle \xi, \eta \rangle' = \int_G \langle \pi(g)\xi, \pi(g)\eta \rangle \, dg,$$

with $\int_G$ denoting an invariant mean on G, defines a new inner product on the Hilbert space, equivalent to the original one, with respect to which the representation π is unitary. Incidentally, the requirement of amenability is essential here; the group $SL(2, \mathbb{R})$ was shown in [7] to have uniformly bounded representations that are not unitarizable.

The third concerns the following problem. Suppose $\mathcal{H}$ is a Hilbert space and $\sigma : G \to \mathcal{B}(\mathcal{H})$ is a map from a discrete group to bounded operators on $\mathcal{H}$. What is the necessary and sufficient condition for σ to be the compression of a unitary representation, or in other words, for there to be a Hilbert space $\mathcal{H}' \supseteq \mathcal{H}$ and a unitary representation π of G on $\mathcal{H}'$ such that if $P : \mathcal{H}' \to \mathcal{H}$ is the orthogonal projection, then $\sigma(g) = P\pi(g)P$ for all $g \in G$? Nagy solved this problem in [23]. This can be viewed as an operator-valued analogue of the characterization of matrix coefficients $g \mapsto \langle \pi(g)\xi, \xi \rangle$ of unitary representations as the positive-definite functions, which follows from the Gelfand-Naimark-Segal construction.

The fourth problem studied by Nagy (in joint work with Ciprian Foiaş and László Gehér [8] and in [24]) can be viewed as a postscript to von Neumann's work (cited above in Section 1.3) on uniqueness of representations of the canonical commutation relations (1). Recall that the method of Weyl and von Neumann was to study unitary representations of the Heisenberg Lie group, not representations by unbounded operators of original relations, which are more numerous and which pose difficult analytic questions. However, Nagy was able to give necessary and sufficient conditions for a representation of the CCR to come from a representation of the group: an irreducible pair of closed symmetric operators Q and P on a Hilbert space $\mathcal{H}$ is a "Schrödinger couple" if and only if $QP - PQ = iI$ holds on a subspace $\mathcal{D} \subset \mathcal{D}_{QP-PQ}$ which is large enough so that the restrictions of Q and P to $\mathcal{D}$ are essentially selfadjoint and that at least one of the eight sets $(Q \pm iI)(P \pm iI)\mathcal{D}$, $(P \pm iI)(Q \pm -iI)\mathcal{D}$ is dense in $\mathcal{H}$.

2.2 Lajos Pukánszky

Lajos Pukánszky, who was born in Budapest in 1928, studied with Sz.-Nagy in Szeged. His earliest papers (dating from 1951 through 1960) deal with von Neumann algebras; all his subsequent publications were on the unitary representation theory of Lie groups. While it was von Neumann who first anticipated the importance of the theory of rings of operators (i.e., von Neumann algebras) to non-commutative harmonic analysis, it was Pukánszky who finally achieved a deep synthesis of these two subjects. Again, we have no room here to go into details, some of which may be found in [6] and [5]. The reader interested in Pukánszky's work could also see his posthumous monograph [18]. The subject of this book, indeed of much of Pukánszky's work, concerns the following question. For a type I Lie group G, there is a bijective correspondence between $\widehat{G}$, the set of unitary equivalence classes of irreducible unitary representations π, and the set of "characters" of G, the generalized functions $g \mapsto \operatorname{Tr} \pi(g)$. What is the substitute for this bijection in the case of non-type I Lie groups?

The answer, which is quite remarkable, is that for non-type I Lie groups, one needs to replace $\widehat{G}$ by $\overset{\cap}{G}$, the set of quasi-equivalence classes[9] of "normal" representations. These are factor representations π of G of types I or II, for which the (usually unbounded) trace Tr on the factor $\pi(G)''$ is finite and non-zero on some ideal in $C^*(G)$, the C^*-completion of the convolution algebra $L^1(G)$. On a type I group, there is no difference between $\widehat{G}$ and $\overset{\cap}{G}$. But on non-type I connected Lie groups, Pukánszky showed that $\overset{\cap}{G}$ is big enough to support the canonical (i.e., central) direct integral decomposition of the left regular representation of G on $L^2(G)$. Thus *harmonic analysis of L^2 functions on the group sometimes forces one to consider normal representations that are not irreducible*, and the left regular representation of G on $L^2(G)$ generates a von Neumann algebra

[9]Two factor representations π_1 and π_2 of G are called *quasi-equivalent* if there is an isomorphism $\Phi : \pi_1(G)'' \to \pi_2(G)''$ such that $\pi_2 = \Phi \circ \pi_1$. This is the natural equivalence relation on factor representations. The difference between this and unitary equivalence is that Φ need not be given by conjugation by a unitary operator.

with no type III summand. (This last fact was proved through the joint efforts of Pukánszky and Dixmier.) Not only this, but every primitive ideal of $C^*(G)$ (that is, the kernel of an irreducible representation of this algebra) is the kernel of a unique quasi-equivalence class of normal representations, so that there is a natural bijection between $\overset{\cap}{G}$ and $\operatorname{Prim} C^*(G)$. Finally, in a stunning generalization of Alexander Kirillov's character formula [13], Pukánszky [18] was able to give a geometric parametrization of the normal representations and a formula for their characters, at least for connected solvable Lie groups.

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