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Introduction

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Abstract. The papers in this volume are the outgrowth of an NSF-CBMS Regional Conference in the Mathematical Sciences, May 18–22, 2009, organized by Robert Doran and Greg Friedman at Texas Christian University. This introduction explains the scientific rationale for the conference and some of the common themes in the papers.

During the week of May 18–22, 2009, Robert Doran and Greg Friedman organized a wonderfully successful NSF-CBMS Regional Research Conference at Texas Christian University. I was the primary lecturer, and my lectures have now been published in [**29**]. However, Doran and Friedman also invited many other mathematicians and physicists to speak on topics related to my lectures. The papers in this volume are the outgrowth of their talks.

The subject of my lectures, and the general theme of the conference, was highly interdisciplinary, and had to do with the confluence of superstring theory, algebraic topology, and C^*-algebras. While with "20/20 hindsight" it seems clear that these subjects fit together in a natural way, the connections between them developed almost by accident.

Part of the history of these connections is explained in the introductions to [**11**] and [**17**]. The authors of [**11**] begin as follows:

> Until recently the interplay between physics and mathematics followed a familiar pattern: physics provides problems and mathematics provides solutions to these problems. Of course at times this relationship has led to the development of new mathematics. . . . But physicists did not traditionally attack problems of pure mathematics.
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> This situation has drastically changed during the last 15 years. Physicists have formulated a number of striking conjectures (such as the existence of mirror symmetry) . . . . The basis of the physicists' intuition is their belief that underlying quantum field theory

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and string theory is a (as yet undiscovered) self-consistent mathematical framework.

Of course this was written over 10 years ago. In the last 10 years, the same principle has been borne out time and again. As far as the subject matter of this volume is concerned, there are a few key developments from the last 35 years that one can point to, that played an essential role:

(1) The Baum-Douglas [**2, 3**] and Kasparov [**19, 20**] approaches to (respectively) topological and analytic K-homology, and the realization that these theories are naturally isomorphic.
(2) The "Second Superstring Revolution" around 1995. Geometric objects, known as D-branes, were shown to play a fundamental role in string theory, and as time went on, it was realized that they naturally carry vector bundles and topological charges (see for example [**23, 31, 22, 32**]), living in K-theory or K-homology (or still more complicated generalized homology theories).
(3) The development of Connes' theory of "noncommutative differential geometry," epitomized by the book [**9**], and the gradual acceptance of noncommutative geometry as a natural tool in quantum field theories.
(4) The invention of "twisted K-theory," and the realization that it has a natural realization in terms of continuous trace C^*-algebras (see [**28, 1, 18**]).

My own interest in combining string duality with topology and noncommutative geometry followed a rather circuitous route. A classical theorem of Grothendieck and Serre [**15**] computed the Brauer group $\operatorname{Br} C(X)$ for X a finite complex, and found that it is isomorphic to the torsion subgroup of $H^3(X, \mathbb{Z})$. In the 1970's, Phil Green [**14**] worked out a more general theory of the Brauer group of $C_0(X)$, for X a locally compact Hausdorff space. Green had the idea to drop all technical conditions on X and to allow continuous-trace algebras with infinite fiber dimension, not just classical Azumaya algebras, so as to get an isomorphism of the Brauer group $\operatorname{Br} C_0(X)$ with all of $H^3(X, \mathbb{Z})$, not just with its torsion subgroup. (When X is a finite complex, it doesn't matter what cohomology theory one uses, but for general locally compact spaces, Čech cohomology is appropriate here.) Now it so happens that Donovan and Karoubi [**12**] had used classical Azumaya algebras to define twisted K-theory with torsion twistings, so Green's idea of using more general continuous-trace algebras to replace Azumaya algebras made possible defining twisted K-theory of X with arbitrary twistings from $H^3(X, \mathbb{Z})$. In [**27**, §6] I pointed this out and explained how to generalize the Atiyah-Hirzebruch spectral sequence to make this twisted K-theory somewhat computable. But for the most part, the idea just sat around for a while since nobody had any immediate use for it.

A number of years later, Raeburn and I [**24**] happened to study crossed products of continuous trace algebras by smooth actions, and we discovered the following interesting "reciprocity law" [**24**, Theorem 4.12]:

THEOREM 1. *Let $p\colon X \to Z$ be a principal $\mathbb{T}$-bundle, where $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ is the circle group. Also assume X and Z are second-countable, locally compact Hausdorff, with finite homotopy type. Let $H \in H^3(X, \mathbb{Z})$ and let $A = CT(X, H)$ be the corresponding stable continuous-trace algebra with Dixmier-Douady class H. Then the free action of $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ on X lifts* (*in a unique way, up to exterior equivalence*) *to an action of* $\mathbb{R}$ *on* A *inducing the given action of* $\mathbb{R}/\mathbb{Z}$ *on* $\widehat{A} = X$*. The crossed product* $A \rtimes \mathbb{R}$ *is again a stable continuous-trace algebra* $A^{\#} = CT(X^{\#}, H^{\#})$*, with* $p^{\#}\colon X^{\#} \to Z$ *again a principal* $\mathbb{T}$*-bundle. Furthermore, the characteristic classes of* p *and* $p^{\#}$ *are related to the Dixmier-Douady classes* H *and* $H^{\#}$ *by*

$$p_!(H) = [p^{\#}], \quad (p^{\#})_!(H^{\#}) = [p],$$

where $p_!$ *and* $(p^{\#})_!$ *are the Gysin maps of the circle bundles.*

At the time, Raeburn and I regarded this entirely as a curiosity, and we certainly didn't expect any physical applications. A bit later [**28**], I continued my studies of continuous-trace algebras and twisted K-theory, but I still didn't expect any physical applications.

Much to my surprise, I discovered many years later that my studies of continuous-trace algebras and twisted K-theory were starting to show up in the physics literature in papers such as [**7**] and [**4**]. In fact, twisted K-theory seemed to be exactly the mathematical framework needed to studying D-brane charges in string theory. Not only that, but the "reciprocity law" of [**24**] for continuous-trace algebras associated to circle bundles also showed up in physics, as the recipe for topology change and H-flux change in T-duality [**5, 6**]. Since that time, there has been a fruitful continuing interaction between the subjects of string theory, topology, and C^*-algebras, an interaction that led to the organization of the CBMS conference at TCU in 2009.

With this as background, I can now explain how the various papers in this volume fit together. The papers of Baum and of Carey and Wang deal with D-brane charges in K-homology and twisted K-homology, a natural continuation of the combination of items (1), (2), and (4) on the list of key developments above (page 2). Baum's paper deals with the extension to the twisted case of the Baum-Douglas approach to topological K-homology. While Baum does not go into the associated physics, D-branes in type II string theories come with precisely the structures he is discussing, and thus produce "topological charges" in the twisted K-homology of spacetime. The paper of Carey and Wang goes into more detail on the same subject, and also discusses a Riemann-Roch theorem in twisted K-theory. Carey and Wang explain how D-brane charges in twisted K-theory arise in both type II and type I string theories.

The papers of Ando and Sati deal with roughly the same theme as those of Baum and Carey-Wang, but in a somewhat generalized context. Ando explains (from the point of view of a stable homotopy theorist) how to construct twisted generalized cohomology theories in general, and then specializes to the cases of twisted K, twisted elliptic cohomology, and twisted TMF. TMF [**16**], topological modular forms, is a version of elliptic cohomology that seems to play an important role in M-theory, the "master" theory that gives rise (on reduction from 11 dimensions to 10) to the five superstring theories. Sati's paper concentrates on the physics side of the same topic, and explains how the physics of M-branes (which play the same role in M-theory that the D-branes play in string theory) leads to twisted String and Fivebrane structures. (These are higher-dimensional analogues of Spin and Spin^c structures.) Sati also discusses the kinds of orientation conditions that arise for branes in F-theory [**30**], a 12-dimensional theory that is supposed to reduce to M-theory in certain circumstances.

Two of the papers in this volume, by Kang and by Baez and Huerta, deal with Yang-Mills gauge theory and its connection with noncommutative geometry. The basic Yang-Mills action on a spacetime manifold M is (up to a scalar factor) $-\int_M \mathrm{Tr}(F \wedge F)$, where F is the curvature of a connection on a principal G-bundle over M. Here G is some Lie group which depends on the details of the theory; for example, in the "standard model" of particle physics it is $SU(3) \times SU(2) \times U(1)$. Physicists have known for some time [**8**] that in some circumstances one can make this action supersymmetric, by adding in a fermionic term of the form (again, up to a constant factor) $\langle \psi, \partial\!\!\!/ \psi \rangle$, where ψ is a spinor field and $\partial\!\!\!/$ is the Dirac operator. However, this only appears to work in three, four, six and ten dimensions. The paper of Baez and Huerta gives an explanation for this fact in terms of the fact that division algebras over $\mathbb{R}$ only occur in dimensions 1, 2, 4, and 8 (where one has the reals, complexes, quaternions, and octonions, respectively). Kang's paper deals with noncommutative Yang-Mills in the sense of Connes and Rieffel [**10**], where the basic Yang-Mills action becomes $-\mathrm{Tr}(\{\Theta, \Theta\})$, where Θ is the curvature 2-form for a connection on a finitely generated projective module (the natural analogue of a vector bundle) over the smooth subalgebra of some C^*-algebra A. Connes and Rieffel took A to be A_θ, the irrational rotation algebra generated by two unitaries U and V with $UV = e^{2\pi i\theta} VU$. Kang considers the somewhat more complicated case of the "quantum Heisenberg manifold" in the sense of Rieffel [**25**]; this is a deformation quantization of the algebra of functions on a Heisenberg nilmanifold.

Just to relate the papers of Baez-Huerta and Kang to the rest of the volume, it is perhaps worthwhile to explain how Yang-Mills and super-Yang-Mills are related to string theory. There are two interconnected ties between the two subjects. On the one hand, as we mentioned already, D-branes naturally carry certain Chan-Paton vector bundles; on these there is a natural Yang-Mills action. In addition, there is a duality, known as the AdS/CFT correspondence, between type IIB string theory on $S^5 \times AdS^5$ (AdS^5 is anti-de Sitter space, a 5-dimensional Lorentz manifold with a metric of constant negative curvature) and 4-dimensional super-Yang-Mills on S^4 in the large-N limit [**21**].

The paper of Sharpe deals with Landau-Ginzburg models, a class of models which were originally constructed to model superconductivity, but which have turned out to be extremely useful for superstring theory as well. A Landau-Ginzburg model in string theory describes propagation of strings on a noncompact spacetime (always a complex manifold) with a holomorphic superpotential W, often having a degenerate critical point. One of the results explained in Sharpe's paper is that A-twisted correlation functions in the Landau-Ginzburg model on $X = \mathrm{Tot}(\mathcal{E}^\vee \xrightarrow{\pi} B)$, $\mathcal{E} \to B$ a holomorphic vector bundle, with $W = p \cdot \pi^* s$, p a tautological section of $\pi^* \mathcal{E}^\vee$ and s a holomorphic section of $\mathcal{E}$, should match correlation functions in the nonlinear sigma model on $\{s = 0\}$. Since the complex geometry of the Landau-Ginzburg spacetime is usually quite different from the one which the sigma model lives, sometimes one gets interesting relations in enumerative algebraic geometry which are hard to explain directly.

The papers of Hannabuss-Mathai, Reiffel, Klein-Schochet-Smith, and an Huef-Raeburn-Williams all deal with various aspects of C^*-algebraic noncommutative geometry. Several of them also have ties to quantum physics and to topology. Rieffel's paper gives explicit examples of sequences of matrix algebras with dimensions going to ∞ whose "proximity" in a rather precise but technical sense goes to 0.

This sort of calculation is motivated by the use of "matrix models" to approximate quantum field theories on spaces with complicated geometry.

The paper of Klein-Schochet-Smith computes the rational homotopy type of the group $U(A)$ of unitary elements in the Azumaya algebra A of sections of a bundle of matrix algebras M_n over a compact space X. This turns out to be independent of what Azumaya algebra one chooses (so that one might as well take $A = C(X) \otimes M_n$), basically because the Brauer group of $C(X)$ is torsion, and the authors are only interested in rational information. This paper also computes the map $\pi_j(U(A)) \otimes \mathbb{Q} \to K_j(A) \otimes \mathbb{Q}$; this gives explicit information on the stable range for rationalized topological K-theory of X. The paper of Hannabuss and Mathai deals with Rieffel's theory of strict deformation quantization [**26**] and the theory of noncommutative principal bundles due to Echterhoff, Nest, and Oyono-Oyono [**13**]. The main theorem of this paper is that for every such bundle with a suitable smooth structure $\mathcal{A}^\infty(X)$, there is a principal torus bundle $T \to X$ and a corresponding strict deformation quantization σ of $C^\infty_{\text{fibre}}(Y)$ (the continuous functions on Y that are fibrewise smooth), so that $\mathcal{A}^\infty(X) \cong C^\infty_{\text{fibre}}(Y)_\sigma$.

Finally, the paper by an Huef, Raeburn, and Williams talks about functoriality issues in the theories of C^*-crossed products and fixed-point algebras for proper actions. Issues like this come up when one tries to use C^*-algebraic noncommutative geometry to study the geometry of spacetime in various physical theories.

We hope the diversity of the papers in this volume will give the reader some idea of the breadth and vitality of the current interplay between superstring theory, geometry/topology, and noncommutative geometry.

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