Remark: We are working in the context of Riemann Integrals.

Problem 1
2.1.1 Solve $u_{tt} = c^2 u_{xx}, u(x, t) = e^x, u_t(x, 0) = \sin x$.

Solution: By d’Alembert’s formula, we have that

$$u(x, t) = \frac{1}{2} [e^{x+ct} + e^{x-ct}] + \frac{1}{2c} \int_{x-ct}^{x+ct} \sin s \, ds$$

$$= e^x \left( \frac{e^{ct} + e^{-ct}}{2} \right) + \frac{1}{2c} \left[ \cos(x - ct) - \cos(x + ct) \right]$$

$$= e^x \cosh ct + \frac{1}{2c} \left[ \cos x \cos ct + \sin x \sin ct - \cos x \cos ct + \sin x \sin ct \right]$$

$$= e^x \cosh ct + \frac{1}{c} \sin x \sin ct.$$

Problem 2
2.1.5 *(The hammer blow)* Let $\phi(x) \equiv 0$ and $\psi(x) = 1$ for $|x| < a$ and $\psi(x) = 0$ for $|x| \geq a$. Sketch the string profile ($u$ versus $x$) at each of the successive instants $t = a/2c, a/c, 3a/2c, 2a/c, \text{and } 5a/c$.

Solution: At $t = a/2c$, we have that

$$u(x, a/2c) = \frac{1}{2c} \int_{x-a/2}^{x+a/2} \psi(s) \, ds$$

$$= \begin{cases} 
0 & \text{for } x \leq -\frac{3}{2}a, \\
\frac{1}{2c} (\frac{3}{2}a + x) & \text{for } -\frac{3}{2}a < x < -\frac{1}{2}a, \\
\frac{a}{2c} & \text{for } -\frac{1}{2}a < x < \frac{1}{2}a, \\
\frac{1}{2c} (\frac{3}{2}a - x) & \text{for } \frac{1}{2}a \leq x < \frac{3}{2}a, \\
0 & \text{for } x \geq \frac{3}{2}a.
\end{cases}$$

At $t = a/c$, we have that

$$u(x, a/c) = \frac{1}{2c} \int_{x-a}^{x+a} \psi(s) \, ds$$

$$= \begin{cases} 
0 & \text{for } x \leq -2a, \\
\frac{1}{2c} (2a + x) & \text{for } -2a < x < 0, \\
\frac{1}{2c} (2a - x) & \text{for } 0 \leq x < 2a, \\
0 & \text{for } x \geq 2a.
\end{cases}$$
At $t = 3a/2c$, we have that
\[
\begin{align*}
\frac{1}{2c} \int_{x-3a/2}^{x+3a/2} \psi(s) \, ds & = \begin{cases} 
0 & \text{for } x \leq -\frac{5}{2}a, \\
\frac{a}{c} \left( \frac{5}{2}a + x \right) & \text{for } -\frac{5}{2}a < x < -\frac{1}{2}a, \\
\frac{1}{2c} (\frac{5}{2}a - x) & \text{for } -\frac{1}{2}a \leq x \leq \frac{1}{2}a, \\
0 & \text{for } x \geq \frac{5}{2}a. 
\end{cases}
\end{align*}
\]

At $t = 2a/c$, we have that
\[
\begin{align*}
\frac{1}{2c} \int_{x-2a}^{x+2a} \psi(s) \, ds & = \begin{cases} 
0 & \text{for } x \leq -3a, \\
\frac{a}{c} (3a + x) & \text{for } -3a < x < -a, \\
\frac{1}{2c} (3a - x) & \text{for } -a \leq x \leq a, \\
0 & \text{for } x \geq 3a. 
\end{cases}
\end{align*}
\]

And finally, at $t = 5a/c$, we have that
\[
\begin{align*}
\frac{1}{2c} \int_{x-5a}^{x+5a} \psi(s) \, ds & = \begin{cases} 
0 & \text{for } x \leq -6a, \\
\frac{a}{c} (6a + x) & \text{for } -6a < x < -4a, \\
\frac{1}{2c} (6a - x) & \text{for } -4a \leq x \leq 4a, \\
0 & \text{for } x \geq 6a. 
\end{cases}
\end{align*}
\]

### Problem 3

2.1.8 A spherical wave is a solution of the three-dimensional wave equation of the form $u(r,t)$, where $r$ is the distance to the origin (the spherical coordinate). The wave equation takes the form
\[
u_{tt} = c^2 \left( u_{rr} + \frac{2}{r} u_r \right) \quad \text{ (“spherical wave equation”).}
\]

(a) Change variables $v = ru$ to get the equation for $v : v_{tt} = c^2 u_{rr}$.

(b) Solve for $v$ using (3) and thereby solve the spherical wave equation.

(c) Use (8) to solve it with initial conditions $u(r,0) = \phi(r), u_t(r,0) = \psi(r)$, taking both $\phi(r)$ and $\psi(r)$ to be even functions of $r$.

Solution:
(a) Define \( v(r,t) := ru(r,t) \), then we see that \( v_t = ru_t, v_{tt} = ru_{tt}, v_r = u + ru_r, \) and \( v_{rr} = 2u_r + ru_{rr} \). Hence it follows
\[
v_{tt} - c^2 v_{rr} = ru_{tt} - c^2(2u_r + ru_{rr})
\]
\[
= r \left( u_{tt} - c^2 \frac{2}{r} u_r - c^2 u_{rr} \right) = 0.
\]

(b) By equation (3), we get that
\[
v(r,t) = f(r + ct) + g(r - ct)
\]
which means
\[
u(r,t) = \frac{f(r + ct) + g(r - ct)}{r}.
\]

(c) Observe
\[
\begin{cases}
  u_{tt} = c^2 (u_{rr} + \frac{2}{r} u_r) & 0 < r < \infty \text{ and } -\infty < t < \infty \\
u(r,0) = \phi(r), u_t(r,0) = \psi(r)
\end{cases}
\]
gives rise to the problem
\[
\begin{cases}
v_{tt} = c^2 v_{rr} & -\infty < r < \infty \text{ and } -\infty < t < \infty \\
v(r,0) = \tilde{\phi}(r) := r\phi(r), v_t(r,0) = \tilde{\psi}(r) := r\psi(r)
\end{cases}
\]
since \( \tilde{\phi}(r) \) and \( \tilde{\psi}(r) \) are both odd functions. Using d’Alembert’s formula, we get that
\[
v(r,t) = \frac{1}{2} [\tilde{\phi}(r+ct) + \tilde{\phi}(r-ct)] + \frac{1}{2c} \int_{r-ct}^{r+ct} \tilde{\psi}(s) \, ds
\]
\[
= \frac{1}{2} [(r+ct)\phi(r+ct) + (r-ct)\phi(r-ct)] + \frac{1}{2c} \int_{r-ct}^{r+ct} \psi(s) \, ds
\]
then it follows
\[
u(r,t) = \frac{1}{2r} [(r+ct)\phi(r+ct) + (r-ct)\phi(r-ct)] + \frac{1}{2cr} \int_{r-ct}^{r+ct} \psi(s) \, ds
\]
for \( r > 0 \) and \( -\infty < t < \infty \).

**Problem 4**

2.2.6 Prove that, among all possible dimensions, only in three dimensions can one have distortionless spherical wave propagation with attenuation. This means the following. A spherical wave in \( n \)-dimensional space satisfies the PDE
\[
u_{tt} = c^2 \left( u_{rr} + \frac{n-1}{r} u_r \right),
\]
where \( r \) is the spherical coordinate. Consider such a wave that has the special form \( u(r,t) = \alpha(r)f(t - \beta(r)) \), where \( \alpha(r) \) is called the attenuation and \( \beta(r) \) the delay. The question is whether such solutions exist for "arbitrary" functions \( f \).

(a) Plug the special form into the PDE to get an ODE for \( f \).

(b) Set the coefficients of \( f'' \), \( f' \), and \( f \) equal to zero.

(c) Solve the ODEs to see that \( n = 1 \) or \( n = 3 \) (unless \( u \equiv 0 \)).
(d) If \( n = 1 \), show that \( \alpha(r) \) is a constant (so that "there is no attenuation").

Solution:

(a) If \( u(r, t) = \alpha(r)f(t - \beta(r)) \), then we see that

\[
\begin{align*}
  u_r(r, t) &= \alpha'(r)f(t - \beta(r)) - \alpha(r)\beta'(r)f'(t - \beta(r)) \\
  u_{rr}(r, t) &= \alpha''(r)f(t - \beta(r)) - 2\alpha'(r)\beta'(r)f'(t - \beta(r)) \\
&- \alpha(r)\beta''(r)f'(t - \beta(r)) + \alpha(r)[\beta'(r)]^2 f''(t - \beta(r)).
\end{align*}
\]

Then it follows

\[
\begin{align*}
  u_{tt} - c^2 (u_{rr} + \frac{n-1}{r} u_r) &= \alpha f'' - c^2 \left( \alpha'' f - 2\alpha' \beta' f' - \alpha\beta'' f' + \alpha[\beta']^2 f'' + \frac{n-1}{r} [\alpha' f - \alpha\beta' f] \right) \\
&= \alpha \left( 1 - c^2 [\beta']^2 \right) f'' + c^2 \left( 2\alpha' \beta' + \alpha\beta'' + \frac{n-1}{r} \alpha\beta' \right) f' \\
&- c^2 \left( \frac{n-1}{r} \alpha' + \alpha'' \right) f = 0.
\end{align*}
\]

(b) Assume \( \alpha(r) \neq 0 \). Setting the coefficients of \( f'' \), \( f' \), and \( f \) equal to zero, we get the following system of differential equations

\[
\begin{align*}
  1 - c^2 [\beta']^2 &= 0 \\
  2\alpha' \beta' + \alpha\beta'' + \frac{n-1}{r} \alpha\beta' &= 0 \\
  \frac{n-1}{r} \alpha' + \alpha'' &= 0.
\end{align*}
\]

(c) By the first equation, we see that

\[
\beta'(r) = \pm \frac{1}{c}
\]

which means either

\[
\beta(r) = \frac{r}{c} + \beta_0 \quad \text{or} \quad \beta(r) = -\frac{r}{c} + \beta_0.
\]

By the first ode, we see that the second ode reduces to

\[
\alpha' + \frac{n-1}{2r} \alpha = 0
\]

which means

\[
\alpha(r) = \begin{cases} 
  C & \text{if } n = 1 \\
  C r^{-(n-1)/2} & \text{if } n > 1
\end{cases}
\]

Next, let us solve the third ode. It’s clear that when \( n = 1 \), then \( C \) is also a solution to the third ode. If \( n > 1 \), then we see that

\[
\frac{n-1}{r} \alpha' + \alpha'' = C \left( -\frac{(n-1)^2}{2} + \frac{(n-1)(n+1)}{4} \right) r^{-(n+3)/2}
\]

\[
= - C \frac{(n-1)(n-3)}{4} r^{-(n+3)/2} = 0
\]

which means \( n = 3 \). Thus, the system has a solution provided \( n = 1 \) or \( n = 3 \).

(d) It’s clear from (c) that when \( n = 1 \), we have that \( \alpha(r) = \text{const.} \)
Problem 5

2.3.3 Consider the diffusion equation \( u_t = u_{xx} \) in the interval \((0, 1)\) with \( u(0, t) = u(1, t) = 0 \) and \( u(x, 0) = 1 - x^2 \). Note that this initial function does not satisfy the boundary condition at the left end, but that the solution will satisfy it for all \( t > 0 \).

(a) Show that \( u(x, t) > 0 \) at all interior points \( 0 < x < 1, 0 < t < \infty \).

(b) For each \( t > 0 \), let \( \mu(t) = \max u(x, t) \) over \( 0 \leq x \leq 1 \). Show that \( \mu(t) \) is a decreasing (i.e., non increasing) function of \( t \).

(c) Draw a rough sketch of what you think the solution looks like (\( u \) versus \( x \)) at a few times.

Solution:

(a) If \( u(x, t) \) solves the above diffusion equation, then \( v(x, t) := -u(x, t) \) also solves the following Cauchy problem

\[
\begin{align*}
v_t - v_{xx} &= 0 \quad \text{on } 0 < x < 1, 0 < t < \infty \\
v(0, t) &= v(1, t) = 0, v(x, 0) = x^2 - 1
\end{align*}
\]

since

\[
v_t - v_{xx} = -(u_t - u_{xx}) = 0.
\]

Fix \( T > 0 \). Consider the rectangle \( R_T = \{(x, t) \mid 0 \leq x \leq 1 \text{ and } 0 \leq t \leq T\} \) and the parabolic boundary,

\[
\Gamma_T = \{(x, t) \mid x = 0 \text{ or } x = 1 \text{ for } 0 \leq t \leq T\} \cup \{(x, t) \mid 0 \leq x \leq 1 \text{ and } t = 0\}.
\]

By the strong maximum principle, we know that the maximum of \( v(x, t) \) in \( R_T \) is only attained on \( \Gamma_T \), i.e. \( v(x, t) < 0 \) for all \( (x, t) \in R_T - \Gamma_T \) since the maximum value of \( v \) on \( \Gamma_T \) is zero. Hence it follows \( u(x, t) > 0 \) for all \( (x, t) \in R_T - \Gamma_T \). Moreover, since this holds for all fixed \( T \), then we see that \( u(x, t) > 0 \) for all \( (x, t) \in \{(x, t) \mid 0 \leq x \leq 1 \text{ and } t = 0\} \).

(b) Assume the maximum of \( u(x, t) \) at time \( t \) occurs at \( X(t) \). By part (a), we know that \( X(t) \in (0, 1) \) since \( u(x, t) > 0 \) for \( (x, t) \in R_T - \Gamma_T \). Fix \( T > 0 \) and define \( \nu(x) = u(x, T) \), then it follows

\[
\frac{\partial}{\partial x} u(x, t) \bigg|_{(x, t) = (X(T), T)} = \frac{d}{dx} \nu(x) \bigg|_{x = X(T)} = 0
\]

and

\[
\frac{\partial^2}{\partial x^2} u(x, t) \bigg|_{(x, t) = (X(T), T)} = \frac{d^2}{dx^2} \nu(x) \bigg|_{x = X(T)} \leq 0
\]

since \( X(T) \) is a maximum on \( \{(x, T) \mid 0 \leq x \leq 1\} \). Now, define \( \mu(t) = u(X(t), t) \) and assuming \( X(t) \) is differentiable, we get that

\[
\frac{d}{dt} u(t) = u_t(X(t), t) + u_x(X(t), t)X'(t)
= u_{xx}(X(t), t) + u_x(X(t), t)X'(t)
= u_{xx}(X(t), t) \leq 0.
\]

Thus, we see that \( \mu(t) \) must be a decreasing function.

(c) Left to the reader.
Problem 6

2.3.4 Consider the diffusion equation $u_t = u_{xx}$ in $\{0 < x < 1, 0 < t < \infty\}$ with $u(0, t) = u(1, t) = 0$ and $u(x, 0) = 4x(1 - x)$.

(a) Show that $0 < u(x, t) < 1$ for all $t > 0$ and $0 < x < 1$.

(b) Show that $u(x, t) = u(1 - x, t)$ for all $t \geq 0$ and $0 \leq x \leq 1$.

(c) Use the energy method to show that $\int_0^1 u^2 \, dx$ is strictly decreasing function of $t$.

Solution:

(a) Fix $T > 0$. By the strong maximum principle, we know that the maximum of $u(x, t)$ in $R_T$ is only attained on $\Gamma_T$, i.e. $u(x, t) < 1$ for all $(x, t) \in R_T - \Gamma_T$ since the maximum value of $u$ on $\Gamma_T$ is 1 which occurs at the point $(1/2, 0)$.

Likewise, applying the strong maximum principle to $v(x, t) := -u(x, t)$, we see that $v(x, t) > 0$ on $R_T - \Gamma_T$, i.e. $u(x, t) > 0$ on $R_T - \Gamma_T$. Thus, we have that $0 < u(x, t) < 1$ on $R_T - \Gamma_T$. Since this holds for all $T$, then we see that $0 < u(x, t) < 1$ for all $(x, t) \in \{(x, t) \mid 0 \leq x \leq 1$ and $0 \leq t < \infty\}$.

(b) Define $v(x, t) = u(1 - x, t)$. Observe $v_t(x, t) = u_t(1 - x, t), v_x(x, t) = -u_x(1 - x, t)$, and $u_{xx}(1 - x, t)$ which means

$$v_t - v_{xx} = u_t(1 - x, t) - u_{xx}(1 - x, t) = 0$$

since $0 < 1 - x < 1$. Now, it’s clear that $v(x, t)$ solves the same Dirichlet problem as $u(x, t)$

$$\begin{cases} v_t - v_{xx} = 0 & \text{on } 0 < x < 1, 0 < t < \infty \\ v(0, t) = v(1, t) = 0, & v(x, 0) = 4x(1 - x). \end{cases}$$

By the uniqueness theorem, we have that $u(x, t) = v(x, t) = u(1 - x, t)$ for $t \geq 0$ and $0 \leq x \leq 1$.

(c) Observe

$$\frac{d}{dt} \int_0^1 u(x, t)^2 \, dx = -2 \int_0^1 u_x(x, t)^2 \, dx.$$

If $u_x(x, t) \equiv 0$, then we see that $u_{xx} \equiv 0$ which means $u_t \equiv 0$. Thus, it follows $u(x, t) = \text{const}$, which clearly does not satisfy the initial condition. Hence $u_x(x, t) \neq 0$ which means

$$\int_0^1 u_x(x, t)^2 \, dx > 0$$

i.e., we have that

$$\frac{d}{dt} \int_0^1 u(x, t)^2 \, dx = -2 \int_0^1 u_x(x, t)^2 \, dx < 0.$$