Math 462: HW5 Solutions
Due on August 15, 2014

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**Remark:** We are working in the context of Riemann Integrals.

**Problem 1**

5.1.4 Find the Fourier cosine series of the functions $|\sin x|$ in the interval $(-\pi, \pi)$. Use it to find the sums

$$\sum_{n=1}^{\infty} \frac{1}{4n^2 - 1} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{(-1)^n}{4n^2 - 1}.$$

**Solution:** Since $|\sin x|$ is an even function on $(-\pi, \pi)$, then it has a Fourier cosine series given by

$$|\sin x| \sim \frac{1}{2} A_0 + \sum_{n=1}^{\infty} A_n \cos nx$$

where the $A_n$s can be readily computed by the integral formula

$$A_n = \frac{1}{\pi} \int_{-\pi}^{\pi} |\sin x| \cos nx \, dx$$

$$= \frac{1}{\pi} \int_{0}^{\pi} \sin x \cos nx \, dx \Bigg|_{-\pi}^{0} - \frac{1}{\pi} \int_{-\pi}^{0} \sin x \cos nx \, dx$$

$$= \frac{2}{\pi} \int_{0}^{\pi} \sin x \cos nx \, dx$$

Consider the trigonometric identity

$$\sin[(n+1)x] - \sin[(n-1)x] = \sin x \cos nx,$$

then it follows

$$A_n = \begin{cases} 
-\frac{4}{\pi} \frac{1}{n^2-1} & \text{if } n \text{ is even} \\
0 & \text{if } n \text{ is odd} 
\end{cases}$$

for all $n \in \mathbb{N}$. Moreover, it is trivial to check that $A_0 = \frac{4}{\pi}$. Hence the Fourier cosine series is given by

$$|\sin x| \sim \frac{2}{\pi} - \frac{4}{\pi} \sum_{\text{even } n} \frac{1}{n^2-1} \cos nx = \frac{2}{\pi} - \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{4n^2 - 1} \cos 2nx.$$

Assume the Fourier cosine series converges pointwise to $|\sin x|$ on $(-\pi, \pi)$, then we have that

$$|\sin x| = \frac{2}{\pi} - \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{4n^2 - 1} \cos 2nx.$$

Set $x = 0$, we get that

$$\sum_{n=1}^{\infty} \frac{1}{4n^2 - 1} = \frac{1}{2}.$$

Set $x = \frac{\pi}{2}$, we get that

$$1 = \frac{2}{\pi} - \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{4n^2 - 1} \quad \text{or} \quad \sum_{n=1}^{\infty} \frac{(-1)^n}{4n^2 - 1} = 1 - \frac{\pi}{4}.$$
Problem 2

5.2.11 Find the full Fourier series of $e^x$ on $(-l, l)$ in its real and complex forms.

Solution: The complex form of the full Fourier series of $e^x$ on $(-l, l)$ is given by

$$e^x \sim \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/l}$$

where $c_n$ can be readily computed via the following formula which yields

$$c_n = \frac{1}{2l} \int_{-l}^{l} \exp \left( \frac{1 - in\pi}{l} x \right) \, dx = \frac{e^{-in\pi} - e^{-l+in\pi}}{2(l - in\pi)}.$$  

Hence it follows

$$e^x \sim \sum_{n=-\infty}^{\infty} \frac{e^{-in\pi} - e^{-l+in\pi}}{2(l - in\pi)} e^{in\pi x/l} = \sum_{n=-\infty}^{\infty} (-1)^n \sinh \frac{l + in\pi}{l^2 + n^2\pi^2} e^{in\pi x/l}.$$  

The real form is given by

$$e^x \sim \sum_{n=-\infty}^{\infty} (-1)^n \sinh \frac{l + in\pi}{l^2 + n^2\pi^2} e^{in\pi x/l} = \sum_{n=-\infty}^{\infty} (-1)^n \sinh \frac{l + in\pi}{l^2 + n^2\pi^2} e^{in\pi x/l} = \sum_{n=-\infty}^{\infty} (-1)^n \sinh \frac{l + in\pi}{l^2 + n^2\pi^2} e^{in\pi x/l}.$$  

Problem 3

5.3.10 (The Gram-Schmidt orthogonalization procedure) If $X_1, X_2, \ldots$ is any sequence (finite or infinite) of linearly independent vectors in any vector space with an inner product, it can be replaced by a sequence of linear combinations that are mutually orthogonal. The idea is that at each step one subtracts off the components parallel to the previous vectors. The procedure is as follows. First, we let $Z_1 = X_1/\|X_1\|$. Second, we define $Y_2 = X_2 - (X_2, Z_1)Z_1$ and $Z_2 = Y_2/\|Y_2\|$. Third, we define $Y_3 = X_3 - (X_3, Z_2)Z_2 - (X_3, Z_1)Z_1$ and $Z_3 = Y_3/\|Y_3\|$, and so on.

(a) Show that all the vectors $Z_1, Z_2, Z_3, \ldots$ are orthogonal to each other.

(b) Apply the procedure to the pair of functions $\cos x + \cos 2x$ and $3 \cos x - 4 \cos 2x$ in the interval $(0, \pi)$ to...
get an orthogonal pair.

Solution:

(a) We shall prove the statement by strong induction. The base case is trivially true. Now, suppose $Z_1, \ldots, Z_{k-1}$ are mutually orthogonal, i.e.

$$(Z_i, Z_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

for $i = 1, \ldots, k-1$ and $j = 1, \ldots, k-1$. Consider

$$Z_k = \frac{X_k - \sum_{i=1}^{k-1} (X_k, Z_i) Z_i}{\| X_k - \sum_{i=1}^{k-1} (X_k, Z_i) Z_i \|}$$

then for a fix $1 \leq l \leq k-1$ we have that

$$(Z_k, Z_l) = \frac{(X_k, Z_l) - \sum_{i=1}^{k-1} (X_k, Z_i)(Z_l, Z_i)}{\| X_k - \sum_{i=1}^{k-1} (X_k, Z_i) Z_i \|}$$

$$= \frac{(X_k, Z_l) - \sum_{i=1}^{k-1} (X_k, Z_i) \delta_{il}}{\| X_k - \sum_{i=1}^{k-1} (X_k, Z_i) Z_i \|}$$

$$= \frac{(X_k, Z_l) - (X_k, Z_l)}{\| X_k - \sum_{i=1}^{k-1} (X_k, Z_i) Z_i \|} = 0.$$

Hence $Z_k$ is orthogonal to all $Z_l$ where $l = 1, \ldots, k-1$. Moreover, we also have that

$$(Z_k, Z_k) = \frac{(X_k - \sum_{i=1}^{k-1} (X_k, Z_i) Z_i, X_k - \sum_{i=1}^{k-1} (X_k, Z_i) Z_i)}{\| X_k - \sum_{i=1}^{k-1} (X_k, Z_i) Z_i \|^2} = 1.$$

(b) Observe

$$\| \cos x + \cos 2x \|_2^2 = \int_0^\pi (\cos x + \cos 2x)^2 \, dx$$

$$= \int_0^\pi \cos^2 x + 2 \cos x \cos 2x + \cos^2 2x \, dx$$

$$= \int_0^\pi \frac{1 + \cos 2x}{2} + \cos 3x + \cos x + \frac{1 + \cos 4x}{2} \, dx$$

$$= \int_0^\pi 1 + \cos x + \frac{1}{2} \cos 2x + \cos 3x + \frac{1}{2} \cos 4x \, dx = \pi,$$

then it follows

$$Z_1 = \frac{\cos x + \cos 2x}{\| \cos x + \cos 2x \|_2} = \frac{\sqrt{\pi}}{\sqrt{\pi}} (\cos x + \cos 2x).$$

Next, observe

$$(X_2, Z_1) = \frac{1}{\sqrt{\pi}} \int_0^\pi (\cos x + \cos 2x)(3 \cos x - 4 \cos 2x) \, dx$$

$$= \frac{1}{\sqrt{\pi}} \int_0^\pi 3 \cos^2 x - \cos x \cos 2x - 4 \cos^2 2x \, dx$$

$$= \frac{1}{2 \sqrt{\pi}} \int_0^\pi -1 - \cos x + 3 \cos 2x - \cos 3x - 4 \cos 4x \, dx = -\frac{\sqrt{\pi}}{2}.$$
which means
\[ Y_2 = X_2 - (X_2, Z_1)Z_1 = \frac{7}{2}(\cos x - \cos 2x). \]

Computing the norm of \( Y_2 \) yields
\[ \|Y_2\|_2^2 = \frac{49}{4} \int_0^\pi (\cos x - \cos 2x)^2 \, dx = \frac{49}{4} \pi \]
then it follows
\[ Z_2 = \frac{Y_2}{\|Y_2\|_2} = \frac{1}{\sqrt{\pi}}(\cos x - \cos 2x). \]

**Problem 4**

5.3.12 Prove Green's first identity: For every pair of functions \( f(x), g(x) \) on \((a,b)\),
\[ \int_a^b f''(x)g(x) \, dx = -\int_a^b f'(x)g'(x) \, dx + f'(x)g(x) \bigg|_a^b. \]

Solution: Assume \( f \in C^2[a,b] \) and \( g \in C^1[a,b] \), then \( f'g \in C^1[a,b] \). In particular, we may apply the Fundamental Theorem of Calculus for Riemann Integral to get
\[ f'(x)g(x) \bigg|_a^b = \int_a^b \frac{d}{ds}(f'(s)g(s))' \, ds = \int_a^b f''(s)g(s) + f'(s)g'(s) \, ds. \]

Hence it follows
\[ \int_a^b f''(x)g(x) \, dx = f'(x)g(x) \bigg|_a^b - \int_a^b f'(x)g'(x) \, dx. \]

**Problem 5**

5.4.5 Let \( \phi(x) = 0 \) for \( 0 < x < 1 \) and \( \phi(x) = 1 \) for \( 1 < x < 3 \).

(a) Find the first four nonzero terms of its Fourier cosine series explicitly.

(b) For each \( x(0 \leq x \leq 3) \), what is the sum of this series?

(c) Does it converge to \( \phi(x) \) in the \( L^2 \) sense? Why?

(d) Put \( x = 0 \) to find the sum
\[ 1 + \frac{1}{2} - \frac{1}{4} - \frac{1}{5} + \frac{1}{7} + \frac{1}{8} - \frac{1}{10} - \frac{1}{11} + \cdots. \]

Solution:

(a) It’s clear that \( A_0 = \frac{4}{3} \). Next, observe
\[ A_n = \frac{2}{3} \int_1^3 \cos \frac{n\pi x}{3} \, dx = -\frac{2}{n\pi} \sin \frac{n\pi}{3} \]
where
\[
\sin \frac{n\pi}{3} = \begin{cases} 
0 & \text{if } n \equiv 0, 3 \mod 6 \\
\frac{\sqrt{3}}{2} & \text{if } n \equiv 1, 2 \mod 6 \\
-\frac{\sqrt{3}}{2} & \text{if } n \equiv 4, 5 \mod 6
\end{cases}
\]

Hence, it follows
\[
A_n = \begin{cases} 
0 & \text{if } n \equiv 0, 3 \mod 6 \\
-\frac{\sqrt{3}}{n\pi} & \text{if } n \equiv 1, 2 \mod 6 \\
\frac{\sqrt{3}}{n\pi} & \text{if } n \equiv 4, 5 \mod 6
\end{cases}
\]

This gives us the following Fourier cosine series
\[
\phi(x) \sim \frac{2}{3} - \frac{\sqrt{3}}{\pi} \cos \frac{\pi x}{3} - \frac{\sqrt{3}}{2\pi} \cos \frac{2\pi x}{3} + \frac{\sqrt{3}}{4\pi} \cos \frac{4\pi x}{3} + \frac{\sqrt{3}}{5\pi} \cos \frac{5\pi x}{3} - \ldots
\]

(b) By Theorem 4(ii), the Fourier cosine series converges pointwise everywhere on \( \mathbb{R} \). Moreover, let us extend to an even periodic function then for each fixed \( x \in [0, 3] \) we have that
\[
\frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \cos \frac{n\pi x}{3} = \frac{1}{2} \left[ \phi_{\text{ext}}(x+) + \phi_{\text{ext}}(x-) \right]
\]

where
\[
\frac{1}{2} \left[ \phi_{\text{ext}}(x+) + \phi_{\text{ext}}(x-) \right] = \begin{cases} 
0 & \text{if } 0 \leq x < 1 \\
\frac{1}{2} & \text{if } x = 1, 3 \\
1 & \text{if } 1 < x < 3
\end{cases}
\]

(c) Since \( \phi(x) \) is \( L^2 \) integrable, then by Theorem 3 we know the Fourier cosine series does indeed converge in the \( L^2 \) sense to \( \phi \).

(d) Set \( x = 0 \) yields
\[
0 = \frac{2}{3} - \frac{\sqrt{3}}{\pi} - \frac{\sqrt{3}}{2\pi} + \frac{\sqrt{3}}{4\pi} - \frac{\sqrt{3}}{5\pi} - \frac{\sqrt{3}}{7\pi} - \frac{\sqrt{3}}{8\pi} + \ldots
\]
or
\[
\frac{2\pi}{3\sqrt{3}} = 1 + \frac{1}{2} - \frac{1}{4} - \frac{1}{5} + \frac{1}{7} + \frac{1}{8} - \frac{1}{10} - \frac{1}{11} + \ldots
\]

Problem 6

5.4.15 Let \( \phi(x) \equiv 1 \) for \( 0 < x < \pi \). Expand
\[
1 = \sum_{n=0}^{\infty} B_n \cos \left( n + \frac{1}{2} \right) x.
\]

(a) Find \( B_n \).

(b) Let \(-2\pi < x < 2\pi\). For which such \( x \) does this series converges? For each such \( x \), what is the sum of the series?

(c) Apply Parseval’s equality to this series. Use it to calculate the sum
\[
1 + \frac{1}{3^2} + \frac{1}{5^2} + \ldots
\]
Solution:

(a) First, consider the following trigonometric identity

\[
\cos \left[ \left( n + \frac{1}{2} \right) x \right] \cos \left[ \left( m + \frac{1}{2} \right) x \right] = \frac{\cos \left[ (n + m + 1) x \right] + \cos \left[ (n - m) x \right]}{2}
\]

then we have that

\[
\int_0^\pi \cos \left[ \left( n + \frac{1}{2} \right) x \right] \cos \left[ \left( m + \frac{1}{2} \right) x \right] \, dx = \begin{cases} 
0 & \text{if } n \neq m \\
\frac{\pi}{2} & \text{if } n = m
\end{cases}
\]

Therefore, the coefficients of the above trigonometric series can be readily computed using the following formula

\[
B_n = \frac{2}{\pi} \int_0^\pi \cos \left[ \left( n + \frac{1}{2} \right) x \right] 
\]

(b) Consider the series

\[
\frac{4}{\pi} \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \cos \left[ \left( n + \frac{1}{2} \right) x \right].
\]

By Theorem 4, we know the above trigonometric series converges pointwise on \((0, \pi)\) to 1. If \(x \in (-\pi, 0)\) then we have

\[
\frac{4}{\pi} \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \cos \left[ \left( n + \frac{1}{2} \right) x \right] = -\frac{4}{\pi} \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \cos \left[ \left( n + \frac{1}{2} \right) |x| \right] = -1.
\]

Lastly, when \(x = 0\), we have the series

\[
\frac{4}{\pi} \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}
\]

which is convergent by Leibniz’s Test (Alternating Series Test). To evaluate the series, we use the fact

\[
\arctan x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}
\]

for all \(x \in [-1, 1]\), which means

\[
\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = \frac{\pi}{4} \quad \text{or} \quad \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = 1
\]

(c) Apply Parseval’s equality to the series yields

\[
\frac{16}{\pi^2} \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} \int_0^\pi \cos^2 \left[ \left( n + \frac{1}{2} \right) x \right] 
\]

or

\[
\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} = \frac{\pi^2}{8}.
\]
Problem 7

5.4.16 Let \( \phi(x) = |x| \) in \((-\pi, \pi)\). If we approximate it by the function

\[
f(x) = \frac{1}{2} a_0 + a_1 \cos x + b_1 \sin x + a_2 \cos 2x + b_2 \sin 2x,
\]

what choice of coefficients will minimize the \( L^2 \) error?

Solution: Since \( \phi(x) = |x| \) in \((-\pi, \pi)\) is an even function, then \( \phi(x) \) has a Fourier cosine series representation

\[
\phi(x) \sim \frac{1}{2} A_0 + \sum_{n=1}^{\infty} A_n \cos nx.
\]

Moreover, since \( \phi(x) \) is continuous on \((-\pi, \pi)\) and \( \phi'(x) \) is piecewise continuous on \((-\pi, \pi)\), then by Theorem 4 the Fourier series converges pointwise to \( \phi(x) \) on \((-\pi, \pi)\), i.e.

\[
\phi(x) = \frac{1}{2} A_0 + \sum_{n=1}^{\infty} A_n \cos nx.
\]

Let us rewrite \( f \) as

\[
f(x) = \sum_{n=0}^{\infty} a_n \cos nx + b_n \sin nx
\]

where \( a_n, b_n = 0 \) for \( n \geq 3 \). Now, apply Parseval’s equality yields

\[
\| \phi - f \|_2^2 = \left\| \frac{1}{2} (A_0 - a_0) - b_1 \sin x - b_2 \sin 2x + \sum_{n=1}^{\infty} (A_n - a_n) \cos nx \right\|_2^2
\]

\[
= |b_1|^2 \| \sin x \|_2^2 + |b_2|^2 \| \sin 2x \|_2^2 + \frac{1}{2} |A_0 - a_0|^2 + \sum_{n=1}^{\infty} |A_n - a_n|^2 \| \cos nx \|_2^2
\]

which is minimized provided \( b_1 = b_2 = 0, a_0 = A_0, a_1 = A_1 \) and \( a_2 = A_2 \).

Problem 8

5.5.1 Sketch the graph of the Dirichlet kernel

\[
K_N(\theta) = \frac{\sin \left( N + \frac{1}{2} \right) \theta}{\sin \frac{1}{2} \theta}
\]

in the case \( N = 10 \). Use a computer graphics program if you wish.

Solution: [left to the reader]

Problem 9

6.3.1 Suppose that \( u \) is a harmonic function in the disk \( D = \{ r < 2 \} \) and that \( u = 3 \sin 2\theta + 1 \) for \( r = 2 \).

Without finding the solution, answer the following questions

(a) Find the maximum value of \( u \) in \( \overline{D} \).

Problem 9 continued on next page...
(b) Calculate the value of $u$ at the origin.

**Solution:**

(a) By the maximum principle of harmonic function, we know that

$$
\max_{x \in D} u(x) = \max_{x \in \partial D} u(x) = \max_{\theta \in [0,2\pi)} (3\sin 2\theta + 1).
$$

However, it’s clear that the maximum occurs when $\theta = \frac{\pi}{4}$, which means

$$
\max_{x \in D} u(x) = 4.
$$

(b) By the mean-value property of harmonic function, we have that

$$
u(0) = \frac{1}{2\pi} \int_0^{2\pi} (3\sin 2\theta + 1) \, d\theta = -\frac{3}{4\pi} \cos 2\theta + \frac{1}{2\pi} \frac{2\pi}{0} = 1.
$$

**Problem 10**

6.3.2 Solve $u_{xx} + u_{yy} = 0$ in the disk $\{r < a\}$ with the boundary condition

$$u = 1 + 3 \sin \theta \quad \text{on} \quad r = a.
$$

**Solution:** Using the method of separation of variables, we obtain the following trigonometric series

$$
u(r, \theta) = \frac{1}{2} A_0 + \sum_{n=1}^{\infty} r^n (A_n \cos n\theta + B_n \sin n\theta).
$$

Applying the boundary condition yields

$$
u(a, \theta) = \frac{1}{2} A_0 + \sum_{n=1}^{\infty} a^n (A_n \cos n\theta + B_n \sin n\theta) = 1 + 3 \sin \theta.
$$

Then it follows $A_0 = 2$, $A_n = 0$ for all $n \in \mathbb{N}$ and

$$B_n = \begin{cases} 
3a^{-1} & \text{if } n = 1 \\
0 & \text{if otherwise}
\end{cases}.
$$

Thus, we have that

$$
u(r, \theta) = 1 + 3a^{-1}r \sin \theta
$$

solves the above Dirichlet problem.