On Tuesday in class we learned how to find the general solution of higher order linear homogeneous differential equations of the form $y^{(n)} + a_1 y^{(n-1)} + \cdots + a_n y = 0$. Today we consider nonhomogeneous equations, that is $y^{(n)} + a_1 y^{(n-1)} + \cdots + a_n y = g(t)$. Today, we will develop the method of undetermined coefficients.

Problem 1: Suppose that $Y_1(t)$ and $Y_2(t)$ are solutions to the nonhomogeneous equation.

(1) What can you say about $L(Y_1)$ and $L(Y_2)$?

(2) Show that $Y_1(t) - Y_2(t)$ is a solution to the associated homogeneous equation.

** Notice that $L(Y_1) = L(Y_2) = g(t)$. To show that $Y_1(t) - Y_2(t)$ is a solution to the associated homogeneous equation, we must show that $L(Y_1 - Y_2) = 0$. Since the derivative of the sum of two functions is the sum of the derivatives,

$$L(Y_1 - Y_2) = L(Y_1) - L(Y_2) = g(t) - g(t) = 0.$$ 

Problem 2: Consider the following theorem:

**Theorem 1.** The general solution of the nonhomogeneous equation can be written in the form

$$y(t) = Y_c(t) + Y_p(t)$$

where $Y_c(t)$ is the general solution of the associated homogeneous equation and $Y_p(t)$ is some specified solution to the nonhomogeneous equation.

Suppose that $y(t)$ is some arbitrary solution to the nonhomogeneous equation. Use the previous problem to show that $y(t)$ has the form given in the above theorem.

** Suppose that $y(t)$ is a solution to the nonhomogeneous equation. Also note that $Y_p(t)$ is a solution to the nonhomogeneous equation. By problem 1, we know that the difference is a solution to the associated homogeneous equation. Thus $y(t) - Y_p(t)$ is a solution to the associated homogeneous equation. Since $Y_c(t)$ is the general solution of the associated homogeneous equation, we see that

$$y(t) - Y_p(t) = Y_c(t) \quad \Rightarrow \quad y(t) = Y_c(t) + Y_p(t).$$
Problem 3: The $Y_p(t)$ in the previous theorem is called a particular solution. Consider the ODE: $y'' - 3y' - 4y = 3e^{2t}$. Find a particular solution by following the steps below:

1. Recall that the exponential function reproduces itself under differentiation. With that in mind, what is the most plausible form to assume that $Y_p(t)$ takes?

2. Now, let’s assume that $Y_p(t) = Ae^{2t}$. Find $A$.

3. What is a particular solution to the ODE?

** Let’s assume that $Y_p(t) = Ae^{2t}$. To solve for $A$ we find the first and second derivatives of $Y_p(t)$ and plug back into the differential equation. We obtain, $Y_p(t) = -\frac{1}{2}e^{2t}$.

Problem 4: Consider the ODE: $y'' - 3y' - 4y = 2\sin(t)$. Use the intuition from the previous problem to determine the most plausible form that the solution will take and then determine the coefficients to find a particular solution.

** Let’s assume that $Y_p(t) = A\sin(t) + B\cos(t)$. To solve for $A$ and $B$ we find the first and second derivatives of $Y_p(t)$ and plug back into the differential equation. We obtain, $Y_p(t) = -\frac{5}{17}\sin(t) + \frac{3}{17}\cos(t)$.

Problem 5: Consider the ODE: $y'' - 3y' - 4y = 4t^2 - 1$. Determine a particular solution by assuming that $Y_p(t)$ is a polynomial of the same degree as the nonhomogeneous term.

** Let’s assume that $Y_p(t) = At^2 + Bt + C$. To solve for $A$, $B$, and $C$ we find the first and second derivatives of $Y_p(t)$ and plug back into the differential equation. We obtain, $Y_p(t) = -t^2 + \frac{3}{2}t - \frac{11}{8}$.

Problem 6: When $g(t)$ is a product of the types of functions that we have discussed, the same principles still hold. Consider the ODE: $y'' - 3y' - 4y = -8e^t\cos(2t)$.

1. What is the most plausible form to assume that $Y_p(t)$ takes?

2. Using your assumption, determine a particular solution.

** Let’s assume that $Y_p(t) = Ae^t\cos(2t) + Be^t\sin(2t)$. To solve for $A$ and $B$ we find the first and second derivatives of $Y_p(t)$ and plug back into the differential equation. We obtain, $Y_p(t) = \frac{10}{13}e^t\cos(2t) + \frac{2}{13}e^t\sin(2t)$.