Short Answer and True/False: Please write each definition in full detail. For the True or False questions, please circle the appropriate answer (no justification is needed).

Problem 1 (5 points): Suppose that \( x_1(t), \ldots, x_n(t) \) are a set of \( n \) solutions to \( x' = Ax \) where \( A \) is an \( n \times n \) matrix. Define the Wronskian of these \( n \) solutions.

\[
W[x_1, \ldots, x_n](t) = \det(x_1(t) \cdots x_n(t))
\]

Problem 2 (5 points): State the definition of a fundamental matrix for the system \( x' = Ax \) where \( A \) is an \( n \times n \) matrix.

If \( x_1(t), \ldots, x_n(t) \) is a set of fundamental solutions then

\[
\Psi(t) = \begin{pmatrix} x_1(t) & \cdots & x_n(t) \end{pmatrix}
\]

is called the fundamental matrix.
Problem 3 (5 points): Suppose that \((\lambda, \mathbf{v})\) is an eigenpair for the matrix \(A\). Show that \(x(t) = v e^{\lambda t}\) is a solution to \(x' = Ax\).

Suppose that \((\lambda, \mathbf{v})\) is an eigenpair for the matrix \(A\). This means that \(A\mathbf{v} = \lambda \mathbf{v}\). Consider \(x(t) = \mathbf{v} e^{\lambda t}\). Then,
\[
x'(t) = \mathbf{v} \lambda e^{\lambda t}.
\]
Notice that
\[
x'(t) = \mathbf{v} \lambda e^{\lambda t} = \mathbf{v} A \mathbf{v} e^{\lambda t} = A x(t).
\]
The eigenpair \((\lambda, \mathbf{v})\) ensures that \(x(t)\) is a solution to \(x' = Ax\). √

Problem 4 (5 points): Consider the system of differential equations
\[
\begin{align*}
x'_1 &= a_{11} x_1 + a_{12} x_2 \\
x'_2 &= a_{21} x_1 + a_{22} x_2
\end{align*}
\]
where \(a_{ij}\) are constants. Describe the process for finding the matrix exponential \(e^{tA}\) where \(A\) is the coefficient matrix associated with the system.

To find \(e^{tA}\) we do the following:

1. Solve the system using eigenmethods.
2. Form a fundamental matrix from two fundamental solutions \(x_1(t), x_2(t)\) (call this matrix \(\Psi(t)\)).
3. Compute \(e^{tA} = \Psi(t)(\Psi(0))^{-1}\).
Problem 5 (10 points): For each statement below, circle T or F.

\( T \) \( F \) Suppose that \( \alpha \pm \beta i \) are eigenvalues of the coefficient matrix \( A \) corresponding to the system \( \mathbf{x}' = A \mathbf{x} \) where \( \alpha, \beta \neq 0 \). If \( \alpha \) is negative, then the origin is a spiral sink.

\( T \) \( F \) The natural fundamental matrix associated with initial time zero is independent of the choice of \( \psi(t) \) in its definition, where \( \psi(t) \) is a fundamental matrix for the system.

\( T \) \( F \) Suppose that \( (\lambda, \mathbf{v}) \) is an eigenpair for the coefficient matrix \( A \) associated to the system \( \mathbf{x}' = A \mathbf{x} \). If \( \lambda \) is a real root of the characteristic polynomial of multiplicity two, then the two solutions have the form \( \mathbf{x}_1(t) = \mathbf{v} e^{\lambda t} \) and \( \mathbf{x}_2(t) = t \mathbf{v} e^{\lambda t} \).

\( T \) \( F \) Consider the system \( \frac{d\mathbf{x}}{dt} = \begin{pmatrix} 5/4 & 3/4 \\ 3/4 & 5/4 \end{pmatrix} \mathbf{x} \). In this case the origin is a nodal source.

\[
\begin{vmatrix}
5/4 - \lambda & 3/4 \\
3/4 & 5/4 - \lambda
\end{vmatrix} = (5/4 - \lambda)^2 - 3/16 = \lambda^2 - 5/2 \lambda + 25/16 - 9/16
\]

\[
= \lambda^2 - 5/2 \lambda + 1
\]

\[
\Rightarrow \lambda = \frac{5/2 \pm \sqrt{(25/4 - 4)}}{2} = \frac{5/2 \pm \sqrt{9/4}}{2} = \frac{5/2 \pm 3/2}{2} = \frac{5}{4} \pm \frac{3}{4}
\]

\( T \) \( F \) The system of differential equations

\[
x' = x - y^2
\]

\[
y' = x + y
\]

is linear.
Computations: For each problem below, show all of your work.

Problem 6 (15 points): Determine the general solution of \( y'' + 4y = 3 \csc(2t) \) where \( 0 < t < \frac{\pi}{2} \).

Use variation of parameters.

1. General sol. to homogeneous part:
   \[ p(t) = t^2 + 4 = 0 \implies \text{zeros are} \quad r = \pm 2i \]
   \[
   \Rightarrow y_c(t) = c_1 \cos(2t) + c_2 \sin(2t)
   \]

2. Vary the parameters:
   \[
   y(t) = u_1(t) \cos(2t) + u_2(t) \sin(2t)
   \]
   \[
   y'(t) = u_1'(t) \cos(2t) - 2u_1(t) \sin(2t) + u_2'(t) \sin(2t) + 2u_2(t) \cos(2t)
   \]
   \[
   = -2u_1(t) \sin(2t) + 2u_2(t) \cos(2t)
   \]
   Assume \( u_1'(t) \cos(2t) + u_2'(t) \sin(2t) = 0 \).
   \[
   y''(t) = -2u_1(t) \sin(2t) - 4u_1(t) \cos(2t) + 2u_2(t) \cos(2t) - 4u_2(t) \sin(2t)
   \]
   \[
   = 3 \csc(2t)
   \]

Plug back in:
   \[
   (-2u_1(t) \sin(2t) - 4u_1(t) \cos(2t) + 2u_2(t) \cos(2t) - 4u_2(t) \sin(2t))
   \]
   \[
   + 4u_1(t) \cos(2t) + 4u_2(t) \sin(2t) = 3 \csc(2t)
   \]

\[
\Rightarrow -2u_1(t) \sin(2t) + 2u_2(t) \cos(2t) = 3 \csc(2t)
\]

Solve for \( u_1, u_2 \):

\[
\frac{-\sin(2t)}{\cos(2t)} u_2' = -2 \left( \frac{-\sin(2t)}{\cos(2t)} u_2' \right) \sin(2t) + 2u_2 \cos(2t) = 3 \csc(2t)
\]

\[
\Rightarrow u_2' \left( \frac{2 \sin^2(2t)}{\cos(2t)} + 2 \cos(2t) \right) = 3 \csc(2t)
\]

\[
\Rightarrow u_2' \left( \frac{2}{\cos(2t)} \right) = 3 \csc(2t) \quad \Rightarrow u_2'(t) = \frac{3}{2} \cot(2t)
\]
\[ u_1'(t) = -\frac{\sin(2t)}{\cos(2t)} \left( \frac{3}{2} \frac{\cos(2t)}{\sin(2t)} \right) = -3/2 \]

\[ u_1(t) = \int u_1'(t) \, dt = -3/2 \int dt = -3/2 t + C_1 \]

\[ u_2(t) = \int u_2'(t) \, dt = \int 3/2 \cot(2t) \, dt = 3/4 \ln |\sin(2t)| + C_2 \]

\[ y(t) = (-3/2 t + C_1) \cos(2t) + (3/4 \ln |\sin(2t)| + C_2) \sin(2t) \]

General Solution:

\[ y(t) = (C_1 \cos(2t) + C_2 \sin(2t)) - 3/2 t \cos(2t) + 3/4 \ln |\sin(2t)| \sin(2t) \]
Problem 7 (10 points): Suppose that $x_1(t)$ and $x_2(t)$ are solutions to a system of equations, where $x_1(t) = (e^{4t}, e^{5t})^T$ and $x_2(t) = (-e^t, 2e^t)^T$.

(1) (3 points) Show that $x_1$ and $x_2$ comprise a set of fundamental solutions.

$$W[x_1, x_2](t) = \begin{vmatrix} e^{4t} & -e^t \\ e^{5t} & 2e^t \end{vmatrix} = 2e^{5t} + e^{5t} = 3e^{5t} \neq 0$$

$\Rightarrow$ fundamental solutions

(2) (2 points) Determine the fundamental matrix.

$$\Psi = \begin{pmatrix} e^{4t} & -e^t \\ e^{5t} & 2e^t \end{pmatrix}$$

(3) (5 points) Find a first order system such that $x_1$ and $x_2$ are a set of fundamental solutions.

$$A = \Psi(t) \Psi^{-1}(t)$$

(1) $\Psi'(t) = \begin{pmatrix} 4e^{4t} & -e^t \\ 4e^{5t} & -2e^t \end{pmatrix}$

(2) $\Psi^{-1}(t) = \frac{1}{3e^{5t}} \begin{pmatrix} 2e^t & e^t \\ -e^{4t} & e^{4t} \end{pmatrix}$

$A = \begin{pmatrix} 4e^{4t} & -e^t \\ 4e^{5t} & 2e^t \end{pmatrix} \begin{pmatrix} \frac{8}{3} & \gamma_3 \\ -\gamma_3 & \frac{4}{3} - \gamma_3 \end{pmatrix} = \begin{pmatrix} \frac{8}{3} + \gamma_3 & \frac{4}{3} - \gamma_3 \\ \frac{8}{3} - 2\gamma_3 & \frac{4}{3} + 2\gamma_3 \end{pmatrix}$

$$= \begin{pmatrix} 3 & 1 \\ 2 & 2 \end{pmatrix}$$

$\therefore$ The system is

$$x' = \begin{pmatrix} 3 & 1 \\ 2 & 2 \end{pmatrix} x$$
Problem 8 (15 points): Find the general solution of
\[
\frac{dx}{dt} = \begin{pmatrix} a & b \\ b & a \end{pmatrix} x
\]
where \(a, b > 0\).

1. **Eigenvalues:**
\[
\begin{vmatrix} a-\lambda & b \\ b & a-\lambda \end{vmatrix} = (a-\lambda)^2 - b^2 = \lambda^2 - 2a\lambda + a^2 - b^2
\]
\[
\Rightarrow \lambda = \frac{2a \pm \sqrt{4a^2 - 4a^2 + 4b^2}}{2} = a \pm b
\]
\[
\Rightarrow \text{Eigenvalues: } \lambda_1 = a + b, \ \lambda_2 = a - b
\]

2. **Eigenvectors:**
Find \( \vec{v}_1 \):
\[
A - \lambda_2 I = \begin{pmatrix} b & b \\ b & b \end{pmatrix} \Rightarrow \vec{v}_1 = (1)
\]
Find \( \vec{v}_2 \):
\[
A - \lambda_1 I = \begin{pmatrix} -b & b \\ b & -b \end{pmatrix} \Rightarrow \vec{v}_2 = (1)
\]

3. **General Solution:**
\[
x(t) = C_1 e^{(a+b)t} \vec{v}_1 + C_2 e^{(a-b)t} \vec{v}_2
\]
\[
= \begin{pmatrix} C_1 e^{(a+b)t} + C_2 e^{(a-b)t} \\ C_1 e^{(a+b)t} - C_2 e^{(a-b)t} \end{pmatrix}
\]
Problem 9 (15 points): Find the real valued general solution of

\[
\frac{dx}{dt} = \begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix} x.
\]

[Hint: \( e^{i\theta} = \cos(\theta) + i\sin(\theta) \).]

\( (i) \) Eigenvalues:

\[
\begin{vmatrix} 2-\lambda & -5 \\ 1 & -2-\lambda \end{vmatrix} = (2-\lambda)(-2-\lambda) + 5 = \lambda^2 - 2\lambda + 2\lambda - 4 - 5 = \lambda^2 + 1
\]

\( \Rightarrow \) Eigenvalues: \( \lambda_1 = i \), \( \lambda_2 = -i \)

\( (ii) \) Eigenvectors:

Find \( \vec{v}_1 \):

\[
A - \lambda_2 I d = \begin{pmatrix} 2+i & -5 \\ 1 & -2+i \end{pmatrix} \Rightarrow \vec{v}_1 = \begin{pmatrix} 2+i \\ 1 \end{pmatrix}
\]

\( \Rightarrow \vec{v}_2 = \begin{pmatrix} 2-i \\ 1 \end{pmatrix} \)

\( (iii) \) Real / Complex Part of \( x_1(t) \):

\[
x_1(t) = e^{\lambda_1} \vec{v}_1 = e^{it} \begin{pmatrix} 2+i \\ 1 \end{pmatrix} = (\cos(t) + i\sin(t)) \begin{pmatrix} 2+i \\ 1 \end{pmatrix}
\]

\[
= \begin{pmatrix} 2\cos(t) - \sin(t) \\ \cos(t) \end{pmatrix} + i \begin{pmatrix} \cos(t) + 2\sin(t) \\ \sin(t) \end{pmatrix}
\]

\( (iv) \) General Solution:

\[
y(t) = c_1 x_1(t) + c_2 x_2(t) = \begin{pmatrix} (2c_1 + c_2)\cos(t) + (-c_1 + 2c_2)\sin(t) \\ c_1\cos(t) + (2c_2)\sin(t) \end{pmatrix}
\]
Problem 10 (15 points): Suppose that \((-3, (1, -4)^T)\) and \((2, (1, 1)^T)\) are eigenpairs for the matrix \(A\).

1. (3 points) Determine the general solution of \(x' = Ax\).

\[x_1(t) = e^{-3t} \begin{pmatrix} 1 \\ 4 \end{pmatrix}, \quad x_2(t) = e^{2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}\]

\[x(t) = \begin{pmatrix} c_1 e^{-3t} + c_2 e^{2t} \\ -4c_1 e^{-3t} + c_2 e^{2t} \end{pmatrix}\]

2. (8 points) Sketch several trajectories of the solution in the \(x_1x_2\)-plane.

**Eigenvalues:**

1. line through \((1, -4)^T\) \(\rightarrow x_1\)

2. line through \((1, 1)^T\) \(\rightarrow x_2\)

3. (4 points) Determine the behavior near the origin.

\[\text{Saddle point}\]