Short Answer and True/False: Please write each definition and theorem in full detail. For the True or False questions, please circle the appropriate answer (no justification is needed).

Problem 1 (5 points): State what it means for the ODE $M(x, y) + N(x, y)y' = 0$ be exact.

The equation $M(x, y) + N(x, y)y' = 0$ is exact if there exists a function $\psi(x, y)$ s.t.

$$\frac{\partial \psi}{\partial x}(x, y) = M(x, y) \quad \text{and} \quad \frac{\partial \psi}{\partial y}(x, y) = N(x, y).$$

Problem 2 (5 points): State the theorem that tells us what form the general solution of a nonhomogeneous linear differential equation assumes.

The general solution of a nonhomogeneous linear differential equation can be written in the form

$$y(t) = y_c(t) + y_p(t)$$

where $y_c(t)$ is the general solution of the associated homogeneous equation and $y_p(t)$ is some specified solution to the nonhomogeneous equation.
Problem 3 (5 points): For each statement below, circle T or F.

T  F  The system of equations below has a unique solution if and only if the associated matrix has determinant zero.
\[
\begin{align*}
2x + 4y &= 1 \\
3x + 2y &= 2
\end{align*}
\]

T  F  A set of \(n\) solutions of an \(n\)-th order homogeneous linear differential equation is fundamental if its Wronskian is nonzero.

T  F  Differentiating the Key Identity twice with respect to \(z\) allows you to show that \(t^2e^t\) is a solution to the homogeneous equation \(y'' - 3ry'' + 3r^2y' - r^3y = 0\).
\[
\rho(z) = z^3 - 3rz + 3r^2 - r^3 = (z - r)^3
\]

T  F  There exists a separable differential equation that is not exact.

T  F  The general solution of \(y'' - 2y' + y = 0\) is \(y(t) = c_1e^t + c_2te^{-t}\).
\[
\rho(z) = z^2 - 2z + 1 = (z - 1)^2
\]
\[
\Rightarrow y(t) = c_1e^t + c_2te^t
\]
Computations: For each problem below, show all of your work.

Problem 4 (10 points): Consider a tank used in certain hydrodynamics experiments. After one experiment the tank contains 200 liters of a dye solution with a concentration of 1g/liter. To prepare for the next experiment, the tank is to be rinsed with fresh water flowing in at a rate of 2 liters/min, the well-stirred solution flowing out at the same rate.

(1) (6 points) Find the equation for the amount of dye in the tank at time $t$. [Note: You have an initial value, so there should be no arbitrary constants in your answer.]

\[ \frac{dQ}{dt} = (\text{rate in}) - (\text{rate out}) \]

rate in: \[ 0 \text{g/L} \cdot 2 \text{L/min} = 0 \text{ g/min} \]

rate out: \[ \frac{Q(t)}{100} \text{g/L} \cdot 2 \text{L/min} = \frac{Q(t)}{100} \text{ g/min} \]

2 pts: \[ \Rightarrow \frac{dQ}{dt} = -\frac{Q(t)}{100} \]

Separable form: \[ \frac{1}{100} + \frac{Q(t)}{100} \frac{dQ}{dt} = 0 \]

\[ M(t) = \frac{1}{100} \Rightarrow \int M(t) \, dt = \frac{1}{100} \Rightarrow \frac{t}{100} + \ln|Q(t)| = C \]

3 pts: \[ N(t) = \frac{1}{Q(t)} \Rightarrow \int \frac{1}{Q(t)} \, dQ = \ln|Q(t)| \Rightarrow Q(t) = ke^{-t/100} \]

1 pt: Initial condition: \[ Q(0) = 200 \Rightarrow 200 = k \]

\[ \Rightarrow Q(t) = 200e^{-t/100} \]

(2) (4 points) Find the time that will elapse before the concentration of dye in the tank reaches 1% of its original value.

1% of original value = 0.01 \times 1 = 0.01 = \frac{1}{100}

\[ \frac{Q(t)}{200} = \frac{1}{100} \Rightarrow Q(t) = 2 \]

\[ \Rightarrow 200e^{-t/100} = 2 \]

\[ e^{-t/100} = \frac{1}{100} \]

\[ -\frac{t}{100} = \ln\left(\frac{1}{100}\right) \]

\[ t = 100 \ln(100) \text{ min.} \]
Problem 5 (15 points): For each problem below, determine if the differential equation is exact or not. If it is exact, then please solve it.

(1) \((3x^2 + x \ln(y) + 4) + (y \ln(x) + \frac{y^2}{x})y' = 0\)

\[ M(x,y) = 3x^2 + x \ln(y) + 4 \]
\[ N(x,y) = y \ln(x) + \frac{y^2}{x} \]
\[ \frac{\partial M}{\partial y} (x,y) = \frac{x}{y} \]
\[ \frac{\partial N}{\partial x} (x,y) = \frac{y}{x} \]

\[ \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \implies \text{NOT EXACT} \]

(2) \((3x^2 + xy^2) + (x^2y + y^2)y' = 0\)

\[ M(x,y) = 3x^2 + xy^2 \]
\[ N(x,y) = x^2y + y^2 \]
\[ \frac{\partial M}{\partial y} (x,y) = 2xy \]
\[ \frac{\partial N}{\partial x} (x,y) = 2xy \]

\[ \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \implies \text{EXACT} \]

\[ \Psi(x,y) = \int M(x,y) dx = \int (3x^2 + xy^2) dx = x^3 + \frac{x^2y^2}{2} + h(y) \]

\[ \frac{\partial \Psi}{\partial y} (x,y) = x^2y + h'(y) = N(x,y) \]

\[ x^2y + h'(y) = x^2y + y^2 \]
\[ h'(y) = y^2 \]

\[ h(y) = \frac{y^3}{3} \]

\[ x^3 + \frac{x^2y^2}{2} + \frac{y^3}{3} = C \]
Problem 6 (10 points): Consider the ODE
\[ y'' - 4y' + 4y = 0. \]
Show that \( \{e^{2t}, te^{2t}\} \) is a fundamental set of solutions. Then, find the general solution.

\[
W[e^{2t}, te^{2t}] = \begin{vmatrix} e^{2t} & te^{2t} \\ 2e^{2t} & 2te^{2t} + e^{2t} \end{vmatrix} =
\]
\[
= e^{2t} \left(2te^{2t} + e^{2t}\right) - 2te^{4t} =
\]
\[
= 2te^{4t} + e^{4t} - 2te^{4t} =
\]
\[
e^{4t} 
eq 0.
\]

Since the Wronskian is nonzero, \( \{e^{2t}, te^{2t}\} \) is a set of fundamental solutions.

The general solution is

\[
y(t) = c_1 e^{2t} + c_2 te^{2t}.
\]
Problem 7 (15 points): Consider the differential equation
\[ y'' - 2ry' + r^2y = 0 \]
where \( r \) is a real number.

(1) (4 points) Find the characteristic polynomial and determine the type of roots (i.e. simple real; repeated real; simple complex; repeated complex).

\[ 2 \text{pts. } p(z) = z^2 - 2rz + r^2 = (z - r)^2 \]

roots of \( p(z) \): \( z = r \) (mult. 2)

\[ 2 \text{pts. } z = r \text{ is a repeated real root} \]

(2) (8 points) The Key Identity tells us that \( e^{zt} \) is a solution. Differentiate the Key Identity with respect to \( z \) and use that information to show that \( t e^{zt} \) is a solution as well. [Hint: Check that \( L(te^{zt}) = 0 \).]

Key Identity: \[ L(e^{zt}) = p(z)e^{zt} \]

\[ 5 \text{pts. } \Rightarrow L(te^{zt}) = p'(z)e^{zt} + tp(z)e^{zt} \]

Note that \( p(r) = 0 \). Also, \( p'(z) = 2z - 2r \) and so \( p'(r) = 0 \).

\[ 3 \text{pts. } \Rightarrow L(te^{zt}) = p'(r)e^{zt} + tp(r)e^{zt} = 0 \]

\( \therefore \) \( t e^{zt} \) is a solution to the homogeneous equation.

(3) (3 points) Find the general solution.

\[ y(t) = c_1 e^{rt} + c_2 t e^{rt} \]
Problem 8 (10 points): Consider the ODE: \((D^2 - 2D + 5)^2 y = 0\) where \(D = \frac{dy}{dt}\).

(1) (3 points) Find the characteristic polynomial.

\[ p(z) = (z^2 - 2z + 5)^2 \]

(2) (2 points) Determine the roots of the characteristic polynomial.

First we find the roots of \(z^2 - 2z + 5\):

\[
\begin{align*}
z &= 2 \pm \sqrt{4 - 20} \\
&= 2 \pm \sqrt{-16} \\
&= 2 \pm 4i \Rightarrow 1 \mp 2i
\end{align*}
\]

Thus the roots of \(p(z)\) are:

1 + 2i (mult. 2)
1 - 2i (mult. 2)

(3) (2 points) Determine the type of the roots that you found in part (2) (ie. simple real; repeated real; simple complex; repeated complex).

Repeated Complex

(4) (3 points) Find the general solution.

\[
y(t) = C_1 e^t \cos(2t) + C_2 e^t \sin(2t) + C_3 t e^t \cos(2t) + C_4 t e^t \sin(2t)
\]
Problem 9 (15 points): Consider the ODE: $y'' - 3y' + 2y = e^t$.

(1) (5 points) Find the general solution to associated homogeneous equation.

$$p(z) = z^2 - 3z + 2 = (z - 1)(z - 2)$$

Roots of $p(z): \quad r_1 = 1, \quad r_2 = 2$

Therefore, general solution: $\quad y_c(t) = c_1e^t + c_2e^{2t}$

(2) (7 points) Find a particular solution to the nonhomogeneous equation.

Assume $y_p(t) = Ae^t$.

$$y_p(t) = Ae^t$$

$$y_p'(t) = Ae^t$$

$$y_p''(t) = Ae^t + Ae^t = 2Ae^t$$

\[
\begin{cases}
Ae^t + 2Ae^t - 3Ae^t - 3Ae^t + 2Ae^t = e^t \\
(A - 3A + 2A)e^t + (2A - 3A)e^t = e^t \\
-Ae^t = e^t \\
A = -1
\end{cases}
\]

$\therefore \quad y_p(t) = -te^t$

(3) (3 points) Determine the general solution to the nonhomogeneous differential equation.

$y(t) = c_1e^t + (2e^{2t} - te^t)$
Problem 10 (10 points): Consider the ODE: \( y'' - 3y' + 2y = \frac{1}{e^t + e^{-t}}. \)

(1) (6 points) Find the green function associated to this differential equation.

\[ p(z) = z^2 - 3z + 2 = (z-1)(z-2) \]

\[ \Rightarrow y_c(t) = C_1 e^t + C_2 e^{2t} \]

The green function is the solution of

\[ d^2g - 3dg + 2g = 0 \quad g(0) = 0, \quad g'(0) = 1, \]

Set \( g(t) = C_1 e^t + C_2 e^{2t}. \) Thus, \( g'(t) = C_1 e^t + 2C_2 e^{2t}. \)

\[ g(0) = 0 = C_1 + C_2 \quad \Rightarrow C_1 = -1 \]

\[ g'(0) = 1 = C_1 + 2C_2 \quad \Rightarrow C_2 = \frac{1}{2} \]

\[ g(t) = -e^t + e^{2t} \]

(2) (4 points) Find a particular solution of the nonhomogeneous equation using the green function from part (1). [You do not need to evaluate the integrals.]

\[ y_p(t) = \int_0^t g(t-s) \cdot \frac{1}{e^s + e^{-s}} \, ds \]

\[ = \int_0^t (-e^{t-s} + e^{2(t-s)}) \cdot \frac{1}{e^s + e^{-s}} \, ds \]

\[ = \left[ \int_0^t \frac{e^{2(t-s)}}{e^s + e^{-s}} \, ds \right] - \left[ \int_0^t \frac{e^{t-s}}{e^s + e^{-s}} \, ds \right]. \]