Determinants:

**2x2 matrices:**

- A 2x2 matrix has the form $M = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}$

  where $m_{11}, m_{12}, m_{21}, m_{22}$ are numbers.

**Def:** The 2x2 matrix $M$ above has determinant $\det(M) = |M| = m_{11}m_{22} - m_{21}m_{12}$.

- Denoted by $\det(M) = |M| = m_{11}m_{22} - m_{21}m_{12}$

**Ex:** Find the determinant of the following matrices.

- $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$
- $B = \begin{pmatrix} -1 & 5 \\ 6 & -2 \end{pmatrix}$

**$\det(A)$**

- $(1)(4) - (3)(2) = 4 - 6 = -2$

**$\det(B)$**

- $(-1)(-2) - (6)(5) = 2 - 30 = -28$
3x3 matrices:

- A 3x3 matrix has the form
  \[ M = \begin{pmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{pmatrix} \]
  where m_{ij} are numbers.

- Def: The 3x3 matrix M above has the determinant
  \[ m_{11}(m_{22}m_{33} - m_{23}m_{32}) - m_{12}(m_{21}m_{33} - m_{23}m_{31}) + m_{13}(m_{21}m_{32} - m_{22}m_{31}) \]
  \( \text{Denoted by } \det(M) = |M| \)

- Ex: Find the determinant of the following matrices:
  \[ A = \begin{pmatrix} 1 & 0 & 2 \\ 3 & 1 & 0 \\ 0 & 1 & 2 \end{pmatrix} \quad B = \begin{pmatrix} -1 & 1 & 1 \\ 0 & 2 & 1 \\ -2 & 1 & -1 \end{pmatrix} \]
  \[ \det(A) = 1[1(2) - 0(0)] - 0[3(2) - 0(0)] + 2[3(1) - 0(1)] = 2 + 6 = 8 \]
  \[ \det(B) = -1[2(-1) - 1(1)] - 1[0(-1) - (-2)(1)] + 1[0(1) - (-2)(2)] = -1(-2) - 1(0) + 1(4) = 5 \]
Consider the set of \( n \) equations below where \( x_1, x_2, \ldots, x_n \) are unknowns.

\[
\begin{align*}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
&\vdots \\
a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n
\end{align*}
\] (*)

The system (*) has a unique solution for every \( b_1, b_2, \ldots, b_n \) if and only if

\[ \det(A) \neq 0 \] where \( A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \)

Now consider,

\[
\begin{align*}
a_{11}x_1 + \cdots + a_{1n}x_n &= 0 \\
&\vdots \\
a_{n1}x_1 + \cdots + a_{nn}x_n &= 0
\end{align*}
\] (+)

( all bi's are zero)
The system (4) has a nonzero solution if and only if \( \det(A) = 0 \).

(if you would like to see proofs for both theorems, see the Course Notes online.)