Definitions: Please write each definition in complete detail.

Problem 1 (5 points): State the definition of “well-ordered.”

A nonempty set \( S \) of real numbers is well-ordered if every nonempty subset of \( S \) has a least element.

Problem 2 (5 points): Let \( R \) be a relation from \( B \) to \( C \). State the definition of the “domain of \( R \).”

The domain of \( R \) is

\[
\text{dom}(R) = \{ b \in B \mid \text{there exists } c \in C \text{ such that } (b, c) \in R \}
\]
Prove or Disprove: For each problem below, either prove the statement or disprove it by providing a counterexample.

Problem 3 (10 points): The equation $x^5 - x^2 + x - 9 = 0$ has a real solution.

**Proof:** Let $f(x) = x^5 - x^2 + x - 9$. Then $f$ is continuous on all of $\mathbb{R}$ since $f$ is a polynomial. Notice that $f(0) = -9 < 0$ and $f(2) = 32 - 4 + 2 - 9 = 21 > 0$. By the Mean Value Theorem, $\exists c \in (0, 2)$ s.t. $f(c) = 0$. Thus $f$ has a real zero and the equation has a real solution.

Problem 4 (10 points): Suppose $x, y \in \mathbb{R}$. If $x^2 < y^2$, then $x < y$.

**Disprove:**

Consider $x = -\frac{1}{2}$ and $y = -1$. Then $x^2 = \frac{1}{4}$ and $y^2 = 1$. Notice that $x^2 < y^2$, however $y < x$. Hence this is a counterexample.
Short Answer: Please provide a solution to the problem below and explain your answer.

Problem 5 (10 points): Let $R$ be a relation defined on $\mathbb{Z}$ by $x \; R \; y$ if $x \equiv y \pmod{4}$. This relation is an equivalence relation. Determine the distinct equivalence classes.

$x \; R \; y$ if $x \equiv y \pmod{4}$ \quad \begin{align*}
[0] &= \{ x \in \mathbb{Z} \mid x \; R \; 0 \}\} = \{ x \in \mathbb{Z} \mid x \text{ is a mult. of 4} \} \\
&= \{ \ldots, -8, 0, 4, 8, \ldots \} \\
[1] &= \{ x \in \mathbb{Z} \mid x \; R \; 1 \}\} = \{ x \in \mathbb{Z} \mid x-1 \text{ is a mult. of 4} \} \\
&= \{ \ldots, -3, 1, 5, 9, \ldots \} \\
[2] &= \{ x \in \mathbb{Z} \mid x \; R \; 2 \}\} = \{ x \in \mathbb{Z} \mid x-2 \text{ is a mult. of 4} \} \\
&= \{ \ldots, -2, 2, 6, 10, \ldots \} \\
[3] &= \{ x \in \mathbb{Z} \mid x \; R \; 3 \}\} = \{ x \in \mathbb{Z} \mid x-3 \text{ is a mult. of 4} \} \\
&= \{ \ldots, -1, 3, 7, 11, \ldots \}
\end{align*}

Thus there are four distinct eq. classes, namely \([0], [1], [2], [3]\).
Proof: Assume to the contrary that $\sqrt{3}$ is rational. Then $\sqrt{3} = \frac{p}{q}$ for some $p, q \in \mathbb{Z}$ with $q \neq 0$ and $\gcd(p, q) = 1$.

Notice that $3 = p^2/q^2$, which implies that $p^2 = 3q^2$. Hence $3|p^2$. Thus $3|p$. This means we can write $p = 3k$ for some $k \in \mathbb{Z}$. Notice,

$$p^2 = 3q^2 \Rightarrow (3k)^2 = 3q^2 \Rightarrow 9k^2 = 3q^2 \Rightarrow q^2 = 3k^2.$$

Thus $3|q^2$, which means that $3|q$.

This contradicts the assumption that $\gcd(p, q) = 1$.

$\therefore \sqrt{3}$ is irrational.
Problem 7 (20 points): Let $n \in \mathbb{Z}$ be such that $n \geq 2$. Define a relation $R$ on $\mathbb{Z}$ by $a R b$ if $a \equiv b \pmod{n}$. Prove that $R$ is an equivalence relation.

**Proof:** To prove that $R$ is an eq. relation, we have three properties to check.

1. Let $a \in \mathbb{Z}$. Then $a - a = 0 = n(0)$. Since $0 \in \mathbb{Z}$, $a - a$ is a mult. of $n$ and $a \equiv a \pmod{n}$. Therefore $aRa$ and $R$ is reflexive.

2. Let $a, b \in \mathbb{Z}$ and assume $a R b$. Then $a \equiv b \pmod{n}$. This means that $a - b = nk$ for some $k \in \mathbb{Z}$. Notice that,

   $$b - a = -(a - b) = -nk = n(-k).$$

   Since $-k \in \mathbb{Z}$, $b - a$ is a mult. of $n$. Therefore $b \equiv a \pmod{n}$. Hence $bRa$ and $R$ is symmetric.

3. Let $a, b, c \in \mathbb{Z}$ and assume $a R b$ and $b R c$. Then $a \equiv b \pmod{n}$ and $b \equiv c \pmod{n}$. This means that $a - b = nk$ and $b - c = nl$ for some $k, l \in \mathbb{Z}$. Notice,

   $$a - c = a - b + b - c = (a-b) + (b-c) = nk + nl = n(k+l).$$

   Since $(k+l) \in \mathbb{Z}$, $a - c$ is a mult. of $n$ and $a \equiv c \pmod{n}$. Therefore $a Rc$ and $R$ is transitive.

\[ \therefore \] $R$ is an eq. relation.
Problem 8 (20 points): Use induction to prove that $n! > 2^n$ for every integer $n \geq 4$. (Recall that $n! = n \cdot (n-1) \cdot (n-2) \cdots 2 \cdot 1$.)

Proof: This will be a proof by induction.

1. Consider $n=4$. Notice that

$$4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24 > 16 = 2^4.$$  

Thus, the statement is true for the base case.

2. Assume that $k! > 2^k$ for some integer $k \geq 4$. Then,

$$(k+1)! = (k+1)(k)(k-1) \cdots 2 \cdot 1 = (k+1)(k!) > (k+1)2^k.$$  

Since $k \geq 4$, $k+1 > 2$. Hence,

$$(k+1)! > (k+1)2^k > 2 \cdot 2^k = 2^{k+1}.$$  

$\therefore n! > 2^n$ for all integers $n \geq 4$. $\blacksquare$