1. Give examples of sets $A, B, C$ such that $A \subseteq B \subseteq C$.
   
   \begin{align*}
   A &= \{1, 3\} & B &= \{3, 13, 2\} & C &= \{3, 13, 2, 3\}
   \end{align*}

2. Construct a truth table for $(P \Rightarrow Q) \Rightarrow (\neg P)$.

   \begin{tabular}{|c|c|c|c|c|}
   \hline
   P & Q & P \Rightarrow Q & \neg P & (P \Rightarrow Q) \Rightarrow (\neg P) \\
   \hline
   T & T & T & F & F \\
   T & F & F & F & T \\
   F & T & T & T & T \\
   F & F & T & T & T \\
   \hline
   \end{tabular}

3. Let $x, y \in \mathbb{Z}$. Prove that $x - y$ is even iff $x$ and $y$ are of the same parity.

   \textbf{Pf:} \\
   $(\Rightarrow)$ Suppose $x - y$ is even. Assume to the contrary that $x, y$ are of opposite parity.

   Case 1: $x$ is even, $y$ is odd.

   Then $x = 2k$ and $y = 2l + 1$ for some $k, l \in \mathbb{Z}$.

   Notice that

   \[ x - y = 2k - (2l + 1) = 2k - 2l - 1 = 2(k - l - 1) + 1. \]

   Thus $x - y$ is odd. $\Rightarrow$.
Case 2: \( x \) is odd, \( y \) is even.

Then \( x = 2k + 1 \) and \( y = 2l \) for some \( k, l \in \mathbb{Z} \).

Notice that
\[
x - y = (2k + 1) - 2l = 2k + 1 - 2l = 2(k - l) + 1
\]

Thus, \( x - y \) is odd. \( \rightarrow \)

\[
\therefore x \text{ and } y \text{ are of the same parity.}
\]

(\( \Leftarrow \)) Suppose \( x \) and \( y \) are of the same parity.

Case 1: \( x, y \) even.

Then \( x = 2k \) and \( y = 2l \) for some \( k, l \in \mathbb{Z} \).

Notice that
\[
x - y = 2k - 2l = 2(k - l)
\]

Thus, \( x - y \) is even.

Case 2: \( x, y \) odd.

Then \( x = 2k + 1 \) and \( y = 2l + 1 \) for some \( k, l \in \mathbb{Z} \).

Notice that
\[
x - y = 2k + 1 - (2l + 1) = 2k + 1 - 2l - 1 = 2(k - l)
\]

Thus, \( x - y \) is even.

\[
\therefore x - y \text{ is even.}
\]
4) Prove that for every three real numbers \( x, y, z \), \( |x-z| \leq |x-y| + |y-z| \).

**Pf:** Let \( x, y, z \in \mathbb{R} \). Consider,

\[ |x-z| = |x-y+y-z| \leq |x-y| + |y-z| \]

by the triangle inequality. 

5) Prove that for two sets \( A, B \), \( A = (A \setminus B) \cup (A \cap B) \).

**Pf:** First we will show that 
\( A \subseteq (A \setminus B) \cup (A \cap B) \). Let \( x \in A \).
Consider the set \( B \). Either the point \( x \) belongs to \( B \) or \( \bar{B} \). That is, \( x \in B \) or \( x \in \bar{B} \).

**Case 1:** \( x \in B \)

Then \( x \in A \) and \( x \in B \). Hence \( x \in A \cap B \) and \( x \in (A \setminus B) \cup (A \cap B) \).

**Case 2:** \( x \in \bar{B} \)

Then \( x \in A \) and \( x \in \bar{B} \). Hence \( x \in A \setminus B \) and \( x \in (A \setminus B) \cup (A \cap B) \).

\( \therefore x \in (A \setminus B) \cup (A \cap B) \) and \( A \subseteq (A \setminus B) \cup (A \cap B) \).
Now we will show that \((A \setminus B) \cup (A \cap B) \subseteq A\).

Let \(x \in (A \setminus B) \cup (A \cap B)\). Then \(x\) belongs to at least one of \(A \setminus B\) and \(A \cap B\). In either case, \(x \in A\). Thus \((A \setminus B) \cup (A \cap B) \subseteq A\).

\[\therefore A = (A \setminus B) \cup (A \cap B)\]

(6) Prove that every nonempty set of negative integers has a largest element.

Pf: Let \(S\) be a nonempty set of negative integers. Consider the set
\[-S = \{-x \mid x \in S\}.\]
Then \(-S\) is a nonempty set of positive integers. Since \(\mathbb{N}\) is well-ordered, \(-S\) has a smallest element, call it \(M\). That is, \(M \leq x\) \(\forall x \in -S\) and \(M \in -S\). Then \(-M \in S\) and \(-M \geq x\) \(\forall x \in S\). Hence \(S\) has a largest element.
7. Use induction to prove that $5 \mid (6^n-1)$ for every nonnegative integer $n$.

Pf: This will be a proof by induction.

(1) Suppose $n = 0$. Then $6^0 - 1 = 6^0 - 1 = 5$.
Hence $5 \mid 5$ since $5 = 5 \cdot 1$ and the base case is true.

(2) Assume $5 \mid (6^k - 1)$ for some nonnegative integer $k$. We want to show that $5 \mid (6^{k+1} - 1)$. Since $5 \mid (6^k - 1)$, $\exists m \in \mathbb{Z}$ s.t. $6^k - 1 = 5m$. Consider,

$$6^{k+1} - 1 = 6(6^k) - 1 = 6(5m + 1) - 1 = 30m + 6 - 1 = 30m + 5 = 5(6m + 1).$$

Since $(6m + 1) \in \mathbb{Z}$, $5 \mid (6^{k+1} - 1)$.

:\: $5 \mid (6^n - 1)$ for every nonnegative integer $n$. \qed
8. Let the relation $R$ be defined on the set of positive integers by $aRb$ if $a/b$ is a power of 3. Prove that $R$ is an eq. relation.

Pf:

1. Reflexive

Let $a \in \mathbb{N}$. Then $a/a = 1 = 3^0$. Hence $aRa$ and $R$ is reflexive.

2. Symmetric

Let $a, b \in \mathbb{N}$. Suppose that $aRb$. Then $a/b = 3^k$ for some $k \in \mathbb{Z}$. Consider $b/a = \frac{1}{a/b} = \frac{1}{3^k} = 3^{-k}$.

Since $-k \in \mathbb{Z}$, $bRa$ and $R$ is symmetric.

3. Transitive

Let $a, b, c \in \mathbb{N}$. Suppose that $aRb$ and $bRc$. Then $a/b = 3^k$ and $b/c = 3^l$ for some $k, l \in \mathbb{Z}$. Consider $a/c = \frac{a}{c} \cdot \frac{b}{b} = \frac{a/b}{c/b} = \frac{3^k}{3^{-l}} = 3^{k+l}$.

Since $(k+l) \in \mathbb{Z}$, $aRc$ and $R$ is transitive.

$\therefore$ $R$ is an eq. relation.
9) A function \( f: \mathbb{Z} \to \mathbb{Z} \) is defined by \( f(n) = 5n + 2 \). Determine if \( f \) is (a) 1-1, (b) onto.

(a) Let \( n, m \in \mathbb{Z} \) and suppose \( f(n) = f(m) \). Then,
\[
5n + 2 = 5m + 2
\]
\[
\Rightarrow 5n = 5m
\]
\[
\Rightarrow n = m.
\]
Thus \( n = m \) and \( f \) is 1-1.

(b) The function \( f \) is not onto. There is no \( n \in \mathbb{Z} \) s.t. \( f(n) = 0 \).

10) Prove or Disprove: The intersection of an uncountable set of \( \mathbb{R} \) with a countable subset of \( \mathbb{R} \) is countable.

Pf: Let \( A \subseteq \mathbb{R} \) be uncountable and \( B \subseteq \mathbb{R} \) be countable. Notice that \( A \cap B \subseteq B \). Since subsets of countable sets are countable, \( A \cap B \) is countable.
11. A dyadic rational is a rational number of the form \( \frac{m}{2^n} \) where \( m \in \mathbb{Z} \) and \( n \in \mathbb{N} \). Prove that the dyadic rationals are dense in \( \mathbb{R} \).

**Proof:** Consider the interval \((x, y)\) where \( x < y \). Then \( y - x > 0 \). Let \( n \in \mathbb{N} \) be large enough so that \( \frac{1}{2^n} < y - x \). We can do this since \( \left( \frac{1}{2^n} \right) \) converges to zero. Then \( 1 < 2^ny - 2^nx \) and \( 2^nx + 1 < 2^ny \).

Let \( M \in \mathbb{Z} \) be such that \( M \leq 2^nx + 1 < M+1 \). Then \( M - 1 \leq 2^nx < M \). This implies that \( x < \frac{M}{2^n} \). Also, \( \frac{M}{2^n} \leq x + \frac{1}{2^n} \) since \( M \leq 2^nx + 1 \).

Since \( x + \frac{1}{2^n} < y \) we have:

\[
x < \frac{M}{2^n} \leq x + \frac{1}{2^n} < y.
\]

Thus \( \frac{M}{2^n} \in (x, y) \).

\[ \therefore \text{The dyadic rationals are dense.} \]
Suppose that the number \( a \) has the property that \( \forall n \in \mathbb{N}, \ a \leq \frac{1}{n} \). Prove that \( a \leq 0 \).

\textbf{Pf}: Assume to the contrary that \( a > 0 \). Recall that \( \left( \frac{1}{n} \right) \) converges to 0.

Let \( \varepsilon = \frac{a}{2} \). Let \( N \in \mathbb{N} \) be s.t. \( \forall n \geq N, \ \left| \frac{1}{n} - 0 \right| < \varepsilon \). Thus,

\[ \frac{1}{n} = \left| \frac{1}{n} \right| < \varepsilon = \frac{a}{2} < a. \]

This is a contradiction.

\[ \therefore \ a \leq 0. \]