Unique decomposition in classifiable theories

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1 Introduction

By a classifiable theory we shall mean a theory which is superstable, without
the dimensional order property, which has prime models over pairs. In order
to define what we mean by unique decomposition, we remind the reader of
several definitions and results. We adopt the usual conventions of stability
theory and work inside a large saturated model of a fixed classifiable theory
$T$; for instance, if we write $M \subseteq N$ for models of $T$, $M$ and $N$ we are thinking
of these models as elementary submodels of this fixed saturated models; so,
in particular, $M$ is an elementary submodel of $N$. Although the results will
not depend on it, we will assume that $T$ is countable to ease notation.

We do adopt one piece of notation which is not completely standard: if $T$
is classifiable, $M_0 \subseteq M_i$ for $i = 1, 2$ are models of $T$ and $M_1$ is independent
from $M_2$ over $M_0$ then we write $M_1 \oplus_{M_0} M_2$ for the prime model over $M_1 \cup M_2$.

Definition 1.1 1. If $M \subseteq N$ are models of $T$ then $M \subseteq_{na} N$ if whenever
$\varphi(x) \in \mathcal{L}(M)$ such that $\varphi(N) \setminus M$ is non-empty and $F \subseteq M$ is any
finite set then $\varphi(M) \setminus \text{acl}(F)$ is non-empty.

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2. We write $M \subseteq \aleph_1 N$ and say that $M$ is a relatively $\aleph_1$-saturated substructure of $N$ if, whenever $A$ and $B$ are countable subsets of $M$ and $N$ respectively, there is $B'$ in $M$ with the same type as $B$ over $A$.

3. If $M_0 \subseteq M \subseteq N$ then $M$ is an $M_0$-component of $N$ if the weight of $\text{tp}(M/M_0)$ is 1 and $M$ is maximal with respect to domination over $M_0$ i.e. if $M \subseteq X \subseteq N$ and $M$ dominates $X$ over $M_0$ then $M = X$.

The following Theorem from [1] explains the importance of components in classifiable theories.

**Theorem 1.2** If $T$ is classifiable and $N$ is a model of $T$ with $M \subseteq_{\text{na}} N$ then $N$ is prime and minimal over any maximal $M$-independent collection of $M$-components of $N$.

Keeping the notation of the previous Theorem, we will call such a maximal collection of $M$-components an $M$-component decomposition of $N$ or simply a component decomposition of $N$ if the intended $M$ is clear from context. The question we wish to address in this paper is, under what general circumstances is the component decomposition unique?

**Definition 1.3** Let $T$ be a classifiable theory.

1. If $M \subseteq_{\text{na}} N$ then we say that $N$ has a unique decomposition over $M$ if whenever $\mathcal{C}_1$ and $\mathcal{C}_2$ are component decompositions of $N$ over $M$ there is a bijection $f : \mathcal{C}_1 \rightarrow \mathcal{C}_2$ such that for $C \in \mathcal{C}_1$, $C$ is isomorphic to $f(C)$ over $M$; we say that these two component decompositions are $M$-isomorphic.

2. A substructure notion for $T$, $\subseteq_*$, is a relation between pairs of models of $T$ such that

   (a) if $M \subseteq_* N$ then $M \subseteq_{\text{na}} N$ and,

   (b) if $M_0 \subseteq_* M_i$ for $i = 1, 2$, $M_1$ and $M_2$ are independent over $M_0$ and $N = M_1 \oplus_{M_0} M_2$ then $M_i \subseteq_* N$ for $i = 1, 2$.

3. We say that a classifiable theory $T$ has unique decompositions with respect to a substructure notion $\subseteq_*$ if whenever $M \subseteq_* N$ are models of $T$ then $N$ has a unique decomposition over $M$. 

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Remark: $\subseteq_{na}$ is a substructure notion for any superstable theory; see the appendices of [2].

Example: $T = Th(Z, +)$ does not have unique decompositions with respect to $\subseteq_{na}$. To see this, suppose that $N$ is a saturated model of $T$ of cardinality greater than $2^{\aleph_0}$ and $M$ is any countable submodel. The $M$-components of $N$ are of the form $\langle M, a \rangle$ for any element $a \in N \setminus M$. Let $J$ be a maximal $M$-independent collection of representatives of cosets of the connected component not realized in $M$; note that since $M$ is countable, $J$ is of cardinality $2^{\aleph_0}$. Fix $a \in J$. Let $I$ be a maximal $M$-independent collection of realizations of the connected component in $N$; $I$ has cardinality greater than $2^{\aleph_0}$.

The first component decomposition is the collection of subgroups generated by $M$ and $b$ for any $b \in I \cup J$. The second component decomposition is the collection of subgroups generated by $M$ and $b \in (a + I) \cup J$ where $a + I = \{a + c : c \in I\}$. Simply by considering the size of $I$, one sees that the two component decompositions are not $M$-isomorphic.

On the other hand, in Chapter XIII of [3], the following Theorem is proved.

**Theorem 1.4** If $T$ is classifiable and non-multidimensional then $T$ has unique decompositions with respect to relative $\aleph_1$-saturation.

To make the statement of the following Proposition easier, let us say that for a classifiable theory, if $M \subseteq_{na} N$ then we say that $N$ has unique components over $M$ if whenever $M \subseteq M_i, P \subseteq N, N = M_1 \oplus_M P = M_2 \oplus_M P$ and $M_i$ is an $M$-component of $N$ then $M_1$ is isomorphic to $M_2$ over $M$.

**Proposition 1.5** Suppose that $T$ is a classifiable theory.

1. If $\subseteq_*$ is a substructure notion then $T$ has unique decompositions with respect to $\subseteq_*$ iff whenever $M \subseteq_* N, N$ has unique components over $M$.

2. Suppose that $M \subseteq_{na} N$ and $N$ has unique components over $M$ then $N$ has a unique decomposition over $N$. 


Proof: Suppose that $T$ has unique decompositions with respect to $\subseteq^*$; we adopt the notation of the paragraph preceding the Proposition. Let $N'$ be the prime model over $|P|^+$-many $P$-independent copies of $N$ over $P$. Now $N'$ has two decompositions over $M$: one which involves $|P|^+$-many $M$-independent copies of $M_1$ and one which involves the same number of $M$-independent copies of $M_2$. It follows then, since $T$ has unique decompositions, that $M_1$ is isomorphic to $M_2$ over $M$.

To prove the other direction and the second part of the Proposition, we make use of the following fact:

**Fact 1.6** Suppose that $X$ is a closed subset of some pregeometry and $I$ and $J$ are two bases of $X$. Then there is a bijection $f : I \to J$ such that for any $a \in I$, $J \cup \{a\} \setminus \{f(a)\}$ is a basis for $X$ and for any $b \in J$, $I \cup \{b\} \setminus \{f^{-1}(b)\}$ is a basis for $X$.

So if $C_1$ and $C_2$ are two decompositions of $N$ over $M$ then by applying the fact, there is a bijection $f : C_1 \to C_2$ such that for any $C \in C_1$, $\{C\} \cup C_2 \setminus \{f(C)\}$ is a decomposition of $N$. If $P$ is the prime model over $C_2 \setminus \{f(C)\}$ inside $N$ then since $N = C \oplus_M P = f(C) \oplus_M P$, by assumption, $C$ is isomorphic to $f(C)$ over $M$ which proves that $N$ has a unique decomposition over $M$. \qed

2 The main theorem

Suppose that $M \subseteq N$. We now describe a game between two players. For the first move, Player A fixes a countable subset of $M$, $C$, and then chooses a countable set $A_1 \subseteq N$ and Player B responds by choosing $B_1 \subseteq M$ and a $C$-elementary map $f_1 : A_1 \to B_1$. After the $n$th play of the game, Player I will have chosen a countable set $A_n \subseteq N$ and Player B will have chosen $B_n \subseteq M$ and a $C$-elementary map $f_n : A_n \to B_n$. For the $(n+1)^{st}$ move then Player A chooses a countable subset of $N$, $A_{n+1} \supseteq A_n$ and Player B responds with $B_{n+1}$ and an elementary map $f_{n+1} : A_{n+1} \to B_{n+1}$ extending $f_n$. If Player B can always make a legal move then B wins. If Player B has a winning strategy for this game then we write $M \subseteq^* N$.

For a arbitrary model $N$, an $\subseteq^*$-substructure is in general quite large. However, if $M$ is $\aleph_1$-saturated then $M \subseteq^* N$ for any model $N$ of which it is an elementary submodel. Moreover, we will see later in this section that models
of classifiable, shallow theories have reasonably small $\subseteq^*$-substructures. The main theorem of this section is

**Theorem 2.1** Suppose that $T$ is a countable, classifiable theory and $M \subseteq^* N$. Then $N$ has a unique decomposition over $M$.

**Proof:** By Proposition 1.5, it suffices to prove that if $\overline{M}$ and $\overline{M}'$ are two $P_0$-components of $N$ such that $P_0 \subseteq^* N$ and there is $P$, $P_0 \subseteq P \subseteq N$ such that $N = \overline{M} \oplus P_0 = \overline{M}' \oplus P_0$ then $\overline{M} \cong \overline{M}'$ over $P_0$.

So fix $\overline{M}, \overline{M}', P_0, P$ and $N$ and elements $a \in \overline{M}$ and $a' \in \overline{M}'$ which dominate $\overline{M}$ and $\overline{M}'$ over $P_0$ respectively. Choose $M_0 \subseteq_{na} P_0$ such that $a$ and $a'$ are independent from $P_0$ over $M_0$. Let $M$ and $M'$ be the $M_0$-components dominated over $M_0$ by $a$ and $a'$ inside $\overline{M}$ and $\overline{M}'$ respectively. Since under these circumstances $\overline{M} = M \oplus M_0 P_0$ and $\overline{M}' = M' \oplus M_0 P_0$, it suffices to prove the following technical lemma

**Lemma 2.2** Suppose $N = M \oplus M_0 P = M' \oplus M_0 P$ where $M_0$ is a countable model, $M_0 \subseteq_{na} M, M' \subseteq N$ and $M_0 \subseteq_{na} P_0 \subseteq^* P \subseteq N$. Then $M \oplus M_0 P_0 \cong P_0 M' \oplus M_0 P_0$.

Before we begin the proof, we remind the reader of the following terminology.

**Definition 2.3** Suppose that $M \subseteq_{na} N$ where $N$ is a model of $T$. Then a tree decomposition of $N$ over $M$ consists of a tree $I$, with ordering $\prec$, of height at most $\omega$ and a family of models $M_\eta$ for $\eta \in I$ such that

1. if $\langle \rangle$ is the root of $I$ then $M_\langle \rangle = M$,
2. whenever $\eta \prec \nu$ in $I$, $M_\eta \subseteq_{na} M_\nu \subseteq_{na} N$ and $\text{wt}(M_\nu/M_\eta) = 1$,
3. for any $\eta \in I$, $\{M_\nu : \nu^- = \eta\}$ is independent over $M_\eta$ where $\nu^-$ represents the predecessor of $\nu$, and
4. if $\eta \prec \nu \prec \mu$ in $I$ then $M_\mu/M_\nu \perp M_\eta$.

Such a decomposition is called a countable decomposition if $M_\eta$ is countable for all $\eta \in I$.  

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Any tree of models which satisfies the last two conditions of the above definition is called a normal tree of models. One important consequence of being a normal tree of models is that the indexed family of models is independent with respect to the underlying tree ordering.

The following is proved in [1].

**Theorem 2.4** If $T$ is classifiable and $M \subseteq_{na} N$ then $N$ is prime and minimal over any maximal tree decomposition over $M$.

**Proof of technical Lemma 2.2:**

We first produce a tree $I$ with ordering and, for $\eta \in I$, countable models $C_\eta, C'_\eta, D_\eta, D'_\eta$ and $P_\eta$ such that

1. if $\eta \mu$ then $C_\eta \subseteq_{na} C_\mu \subseteq M$, $C'_\eta \subseteq_{na} C'_\mu \subseteq M'$ and $P_\eta \subseteq P$; if $\langle \rangle$ is the root of $I$ then $C_{\langle \rangle} = C'_{\langle \rangle} = P_{\langle \rangle} = M_0$,

2. $\{C_\eta : \eta \in I\}$ and $\{C'_\eta : \eta \in I\}$ are normal trees of models, $M$ is prime over $C_I$ and $M'$ is prime over $C'_I$,

3. $M_\eta$ is the prime model over $C_\eta D_\eta P_\eta$ as well as the prime model over $C'_\eta D'_\eta P_\eta$,

4. $C_\eta$ is domination equivalent to $C'_\eta$ over $M_\eta$,

5. $D_\eta = C_{I_\eta}$ for some countable, downward closed subset of $\{\mu \in I : \mu / \eta\}$ and

6. $D'_\eta \subseteq M'$ and $C'_\eta$ is independent from $D'_\eta$ over $C'_\eta$.

The above data is produced by starting with countable decompositions of $M$ and $M'$ and then working upwards, by induction, to produce the required models. In fact, $\{C_\eta : \eta \in I\}$ and $\{C'_\eta : \eta \in I\}$ can be decompositions of $M$ and $M'$ respectively except that possibly, for an $\eta \in I$ which has no successors, the weight of $tp(C_\eta/C_{\eta^-})$ may not be one due to the presence of non-trivial types; similarly for the $C'_\eta$'s. The $D_\eta$ and $D'_\eta$ play a purely auxiliary role simply to guarantee the existence of $M_\eta$.

Now pick $P^*_\eta \subseteq P_0$ and an $M_0$-elementary map $f_\eta : P_\eta \to P^*_\eta$ inductively such that if $\eta \nu$ then $f_\nu$ extends $f_\eta$. We can do this because $P_0 \subseteq^* P$.

Now pick inductively $C^*_\eta$, $D^*_\eta$ and an elementary map $g_\eta$, extending $f_\eta$ such that $g_\eta$ fixes $C_\eta$ and $D_\eta$ and $g_\eta(C'_\eta) = C^*_\eta$, $g_\eta(D'_\eta) = D^*_\eta$ and if $\nu \eta$. 

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then $g_\nu$ extends $g_\eta$. Let $M^*_\eta$ be the prime model over $C_\eta D_\eta P^*_\eta$ (equivalently the prime model over $C^*_\eta D^*_\eta P^*_\eta$). Let $h_\eta = g_\eta C^*_\eta$.

We will show that $\bigcup\{ h_\eta : \eta \in I \}$ is elementary. Since $\{ C'_\eta : \eta \in I \}$ is a normal tree and $h_\eta$ is elementary for each $\eta$, it suffices to show that for every $\nu \in I$, $\{ C^*_\eta : \eta \setminus \nu \}$ is independent over $C^*_\nu$. To see this, note that $\{ C^*_\eta : \eta \setminus \nu \}$ is independent over $M^*_\nu$. Now for any $\eta$ such that $\eta \setminus \nu$, $C_\eta$ is domination equivalent to $C'_\eta$ over $M_\nu$. Since $g_\eta$ is elementary, $C_\eta$ is domination equivalent to $C^*_\eta$ over $M^*_\nu$. It follows then that $\{ C^*_\eta : \eta \setminus \nu \}$ is independent over $M^*_\nu$. But $C^*_\nu$ is independent from $M^*_\eta$ over $C^*_\eta$ and so we conclude that $\{ C^*_\eta : \eta \setminus \nu \}$ is independent over $C^*_\nu$.

Let $M^*$ be prime over $C^*_I$ and contained in $\tilde{N} = M \oplus M_0 P_0$. It is clear that $M^*$ is independent from $P_0$ over $M_0$ and so it suffices to show that if $C^*_I \subseteq M^* \subseteq \hat{M} \subseteq \tilde{N}$ and $\hat{M}$ is independent from $P_0$ over $M_0$ then $M^* = \hat{M}$. If not, by standard arguments, one can find a $b \in \tilde{N}$ and $\eta \in I$ such that $tp(b/C^*_I)$ is orthogonal to $C^*_\eta$- or $\eta$ is the root of $I$, and $b$ is independent from $M^*$ over $C^*_\eta$.

Let’s handle the case when $\eta$ is the root of $I$ first. In this case, $b$ is independent from $C^*_\eta = \{ C^*_\mu : \mu \setminus \eta \}$ over $M_0$. Moreover, $bC^*_\eta$ is independent from $P_0$ over $M_0$. Since $C^*_\eta$ is domination equivalent to $\{ C^*_\mu : \mu \setminus \eta \}$ over $M_0$ and the latter dominates $M$ over $M_0$, it follows that $b$ is independent from $MP_0$ over $M_0$ which implies that $b \in M_0$ which is a contradiction.

So now we are assuming that $\eta$ is not the root of $I$. By construction, $C^*_\eta$ is independent from $M^*_\eta$ over $C^*_\eta$. We know that $C_\eta$ is independent from $\{ C^*_\mu : \mu \setminus \eta \} P_0$ over $M^*_\eta$. Now since $C_\eta$ is domination equivalent to $C^*_\eta$ over $M^*_\eta$, we get by transitivity that $C^*_\eta$ is independent from $\{ C^*_\mu : \mu \setminus \eta \} P_0$ over $C^*_\eta$. Let $N_1$ be contained in $N$ and prime over $\{ C^*_\mu : \mu \setminus \eta \} \cup M^*_\eta$ and $P_0$ and $N_2$ be contained in $N$ and prime over $\{ C^*_\mu : \mu \setminus \eta \} \cup C_\eta \cup P_0$. Let $N_\eta$ be prime over $M^*_\eta P_0$.

Suppose for a moment that $tp(c/C^*_\eta)$ is any type orthogonal to $C^*_\eta$. From above we get that $c$ is independent from $\{ C^*_\mu : \mu \setminus \eta \} P_0$ over $C^*_\eta$ and so, in particular,

\[
c \downarrow_{C^*_\eta} N_\eta \quad \text{and} \quad c \downarrow_{N_\eta} N_2
\]

We apply this to obtain that $bC^*_\eta$ is independent from $N_2$ over $N_\eta$ and $bC^*_\eta$ is independent from $N_\eta$ over $C^*_\eta$ where $\overline{C^*_\eta} = \{ C^*_\mu : \mu \setminus \eta \}$. Since $b$ and $\overline{C^*_\eta}$ are independent over $C^*_\eta$, it follows that $\overline{C^*_\eta}$ is independent from $bN_2$ over...
$N_\eta$. $C^*_\eta$ and $N_1$ are domination equivalent over $N_\eta$ so it follows that $b$ is independent from $\hat{N}$ over $N_2$; $b$ is independent from $N_2$ over $C^*_\eta$ so $b \in C^*_\eta$ which is a contradiction.

**Corollary 2.5** Any countable, classifiable theory $T$ has unique decompositions with respect to $\subseteq^*$.

The following corollary is interesting and appears to be new.

**Corollary 2.6** If $T$ is superstable without the dimensional order property then the class of $\aleph_1$-saturated models of $T$ has unique decompositions; that is, if $M \subseteq N$ are both $\aleph_1$-saturated models of $T$ then $N$ has a unique decomposition over $M$.

**Proof:** By our earlier remark, any $\aleph_1$-saturated model is a $\subseteq^*$-substructure of any model of which it is a substructure; by the main Theorem, the result follows immediately.

**Definition 2.7** Suppose that $A = \{M_i : i \in I\}$ and $B = \{N_j : j \in J\}$ are increasing families of models which are independent with respect to trees $I$ and $J$ respectively. We say that $A$ and $B$ are isomorphic as labelled trees via a system of maps $\{f_i : i \in I\}$ if there is an order isomorphism $\nu : I \to J$ so that $f_i : M_i \to N_{\nu(i)}$ is an isomorphism and $f_i |_{M_j} = f_j$ whenever $j \leq i$.

**Definition 2.8** Suppose that $N$ has a countable decomposition $P = \{N_\eta : \eta \in I\}$.

1. We say that $N$ is homogeneous with respect to $P$ if whenever $J$ and $J'$ are countable downward closed subsets of $I$ so that $\{N_\eta : \eta \in J\}$ is isomorphic to $\{N_\eta : \eta \in J'\}$ via a collection of elementary maps $F$ then for any countable downward closed $K$, $J \subseteq K \subseteq I$ then there is a downward closed $K'$, $J' \subseteq K' \subseteq I$ so that $\{N_\eta : \eta \in K\}$ is isomorphic to $\{N_\eta : \eta \in K'\}$ as labelled trees via a collection of elementary maps which extends $F$.

2. If $M$ is prime over $N_J$ for some $J \subseteq I$ then $M \subseteq^*_{\aleph_1} N$ if whenever there is countable, downward closed $J' \subseteq J$ and countable $I'$, $J' \subseteq I' \subseteq I$ there is a countable $I''$, $J' \subseteq I'' \subseteq I$ such that $\{N_\eta : \eta \in I''\}$ is isomorphic to $\{N_\eta : \eta \in I''\}$ as labelled trees via a collection of elementary maps which contains the identity maps on $N_\eta$ for all $\eta \in J'$.
Proposition 2.9  
1. If $N$ is any model with decomposition $\mathcal{P}$ then there is a model $M$, $|M| \leq 2^{\aleph_0}$, such that $M \subseteq^* \mathcal{P} N$.

2. If $N$ is homogeneous with respect to $\mathcal{P}$ and $M \subseteq^* \mathcal{P} N$ then $M \subseteq^* N$.

Proof: The first is a routine union of chains argument and the second is straightforward remembering that in $N$, any countable set is constructible over a countable, downward closed part of $\mathcal{P}$. $\Box$

In [2], the following is critical.

Theorem 2.10 Suppose $T$ is a countable, classifiable theory, $M \subseteq^* N_i$ and $\text{wt}(N_i/M) = 1$ for $i = 1, 2$, $P$ is homogeneous with respect to $\mathcal{P}$ and $M \subseteq^* \mathcal{P} P$. Then if $N_1 \oplus_M P \cong_P N_2 \oplus_M P$ then $N_1 \cong_M N_2$.

Proof: The proof is identical to the proof to Lemma 2.2. $\Box$

Theorem 2.11 Suppose that $T$ is a countable, classifiable theory of depth $d$. Then any model $N$ has a $\subseteq^*$-substructure of size at most $d$; in fact, if $d$ is an infinite successor ordinal then there is a $\subseteq^*$-substructure of size $d - 1$.

Proof: Fix any countable decomposition of $N$, $\mathcal{P} = \langle M_\eta : \eta \in I \rangle$. We now label the nodes of $I$ by induction on depth; remember that $I$ is well-founded. For a node $\eta$ of depth 0, label it by the isomorphism type of the chain of models $\langle M_\eta n : n \leq l(\eta) \rangle$. Note that there are at most 1 many such labels; we call these labels of depth zero. For the sake of induction, say that $M_\eta$ is the witness for its label for any $\eta$ of depth zero.

Now suppose we have labelled all nodes of depth less than $\alpha$ and have assigned witnesses for every such label. Fix a node $\eta$ of depth $\alpha$. Define a function $f$ from the set of labels of depth less than $\alpha$ to $\omega \cup \{\omega, \omega_1\}$ where, for a label $L$, $f(L)$ is the number of immediate successors of $\eta$ with label $L$ if this number is countable and $\omega_1$ otherwise. $f$ will be the label for $\eta$ and will be a label of depth $\alpha$. To obtain a witness for the label $f$, take $M_\eta$ together the union of all the witnesses of labels $L$ which appear countably often immediately above $\eta$ and $\aleph_1$-many witnesses for labels $L$ which appear uncountably often immediately above $\eta$. It is easy to check by induction that the witness for the label $f$ will have size at most $\alpha$ if $\alpha$ is finite and not zero, or if $\alpha$ is an infinite limit ordinal. Otherwise, the witness will have size at most $\alpha - 1$. 9
In the end, let \( P' = \langle M_\eta : \eta \in I_0 \rangle \) be the witness for the label of the root node and let \( M \) be prime over \( P' \). It is straightforward to show that \( M \subseteq N \).

**Remark:** In fact, in the last Theorem, the \( M \) we found is slightly more than an \( L_{\infty, \omega_1} \)-substructure.

### 3 An example

This section is devoted to the proof of the following

**Proposition 3.1** There is a countable, classifiable theory \( T \) of depth 3 which fails to have unique decompositions with respect to relative \( \aleph_1 \)-saturation.

We shall describe the theory and a standard model of the theory simultaneously. To begin with, there is an index set \( P \) which we will treat as one sort and a cover of this set \( Q \) which we will treat as a separate sort. Let \( \pi \) represent a surjective map from \( Q \) to \( P \). Moreover, there will be a free action of \( P \) on each fibre of \( \pi \).

In the standard model of the theory in question, we do the following: let \( G \) be the free group on some infinite set of generators \( I \). Let \( P = I \cup I^{-1} \) where \( I^{-1} \) is the set of inverses of the elements in \( I \) in \( G \). Let \( Q = P \times G \) and let \( \pi \) be the first projection from \( Q \) to \( P \). For the action of \( P \) on \( Q \), let \( \tau \) be defined by \( \tau(p, (p', g)) := (p', pg) \).

Now there will be two additional sorts, \( \hat{P} \) and \( \hat{Q} \) which are covers of \( P \) and \( Q \) respectively; \( \rho_P \) and \( \rho_Q \) will be surjective maps from \( \hat{P} \) to \( P \) and \( \hat{Q} \) to \( Q \) respectively. Furthermore, elements of \( \hat{P} \) and \( \hat{Q} \) will be “coloured” by elements of \( 2^{\aleph_0} \). Finally there will be an action \( \cdot \) of \( \hat{P} \) on \( \hat{Q} \).

In the standard model, these sorts are realized as follows: \( \hat{P} = P \times 2^{\aleph_0} \) and \( \hat{Q} = Q \times 2^{\aleph_0} \). \( \rho_P \) and \( \rho_Q \) are the first projections onto \( P \) and \( Q \) respectively. We define \( \cdot \) by \( (p, \eta) \cdot (q, \mu) := (\tau(p, q), \eta + \mu) \) where the addition in the second component is occurring co-ordinatewise modulo 2 in \( 2^{\aleph_0} \). We note then that the two actions are compatible in the sense that

\[
\rho_Q(\hat{p} \cdot \hat{q}) = \tau(\rho_P(\hat{p}), \rho_Q(\hat{q}))
\]

holds in the standard model.
To obtain the “colours” on \( \hat{P} \) and \( \hat{Q} \) we introduce predicates \( U_\eta \) and \( V_\eta \) for \( \eta \in 2^{<\omega} \). In the standard model, for \( \eta \in 2^{<\omega} \), \( U_\eta = \{(p, \mu) \in \hat{P} : \eta \subseteq \mu\} \) and \( V_\eta = \{(q, \mu) \in \hat{Q} : \eta \subseteq \mu\} \). For any model of the theory so far described, we see that we can define the colour of an element of \( \hat{P} \) or \( \hat{Q} \) as follows: if \( x \in \hat{P} \) then \( c_P(x) = \mu \) iff \( x \in U_\mu \) for all \( n \in \omega \). Similarly, if \( x \in \hat{Q} \) then \( c_Q(x) = \mu \) iff \( x \in V_\mu \) for all \( n \in \omega \). We record the following relationship between the colours that holds in the standard model (and any model of its theory):

\[
c_Q(\hat{p} \cdot \hat{q}) = c_P(\hat{p}) + c_Q(\hat{q})
\]

Note also that in a saturated model of the theory described so far the maps \( c_P \) and \( c_Q \) are onto \( 2^{\aleph_0} \).

The theory \( T \) will be the theory of the standard model describe above with sorts \( P, Q, \hat{P} \) and \( \hat{Q} \) together with the functions \( \pi, \rho_P, \rho_Q, \tau \) and \( \cdot \) and predicates \( U_\eta \) and \( V_\eta \) for every \( \eta \in 2^{<\omega} \). It is left to the reader to verify that this theory is classifiable and of depth 3.

We will construct models of \( T, N \) and a weight one extension \( N' \) such that for any model \( N_0 \subseteq N \) of cardinality less than \( 2 \) there are models \( M_1 \) and \( M_2 \) extending \( N_0 \), independent from \( N \) over \( N_0 \) such that \( N' \) is prime (in fact algebraic) over \( M_i \cup N \) for \( i = 1, 2 \) but such that \( M_1 \) and \( M_2 \) are not isomorphic over \( N_0 \). This will be enough to show the failure of unique decompositions with respect to relative \( \aleph_1 \)-saturation.

We start by fixing some notation and terminology. For any model \( M \) of \( T \) and any \( p \in P(M) \), we call the set \( \{c_P(\hat{p}) : \hat{p} \in \hat{P}(M), \rho_P(\hat{p}) = p\} \), the colours of \( p \). Similarly, for any \( q \in Q(M) \), we call the set \( \{c_Q(\hat{q}) : \hat{q} \in \hat{Q}(M), \rho_Q(\hat{q}) = q\} \), the colours of \( q \). Let \( X_0 = 2^{<\omega} \) and let \( B \) be a basis of \( 2^{\aleph_0}/X_0 \) considered as a vector space over \( F_2 \). It is easy to construct \( N \) so that \( P(N) \) contains \( \{c_X : X \subseteq B\} \) and that the colours of \( c_X \) is exactly the subspace of \( 2^{\aleph_0} \) generated by \( X_0 \) together with \( \bigcup X \). For \( Q(N) \), we simply fix a \( b_X \) in the \( \pi \)-fibre above \( c_X \) and it is easy to arrange that the colours of \( b_X \) are \( X_0 \). The rest of \( N \) is filled out by virtue of the action of \( P \) on \( Q \) via \( \tau \) and by the action of \( \hat{P} \) on \( \hat{Q} \).

For \( N' \) we add a single element \( a \) to \( P(N) \) and arrange that in \( N' \) the colours of \( a \) are \( X_0 \). Now fix an element \( b \) in the \( \pi \)-fibre above \( a \) and arrange that in \( N' \), the colours of \( b \) are \( X_0 \). Now let \( N' \) be the closure of \( N,a \) and \( b \) under the actions \( \tau \) and \( \cdot \).
Now fix any elementary submodel $N_0$ of $N$ of cardinality less than $2$. For some $X \subseteq B$, $c_X \not\in N_0$. Let $M_1$ be the closure under the actions of $N_0, a$ and $b$ and let $M_2$ be the closure under the actions of $N_0, a$ and $b' = \tau(c_X, b)$. It is easy to see that either of $M_1$ or $M_2$ together with $N$ generates $N'$. However, the colours of $b'$ are not the colours of any point in $M_1$ and so $M_1$ is not isomorphic to $M_2$ over $N_0$.

References

