Solutions to Homework Problems on the Green Function Method II
Spring 2011, Math 246, Professor David Levermore

1. Compute the Green function \( G(t, s) \) for the differential operator \( L(t) \) defined by
   \[
   L(t)y = D^2y - 2tDy + (t^2 - 1)y,
   \]
given that \( e^{\frac{1}{2}t^2} \) and \( te^{\frac{1}{2}t^2} \) solve the homogeneous equation \( L(t)y = 0 \). Use the result to solve the initial-value problem
   \[
y'' - 2ty' + (t^2 - 1)y = t^2 e^{\frac{1}{2}t^2}, \quad y(0) = y'(0) = 0.
   \]

Solution: The Green function is given by
   \[
   G(t, s) = \frac{\det\left(\begin{array}{cc} e^{\frac{1}{2}s^2} & s e^{\frac{1}{2}s^2} \\ e^{\frac{1}{2}t^2} & t e^{\frac{1}{2}t^2} \end{array}\right)}{\det\left(\begin{array}{cc} e^{\frac{1}{2}s^2} & s e^{\frac{1}{2}s^2} \\ s e^{\frac{1}{2}s^2} & (s^2 + 1) e^{\frac{1}{2}s^2} \end{array}\right)} = (t - s)e^{\frac{1}{2}t^2} e^{-\frac{1}{2}s^2}.
   \]
Because the equation is in normal form the forcing is \( t^2 e^{\frac{1}{2}t^2} \). The solution is therefore
   \[
y(t) = \int_0^t G(t, s) 2s e^{\frac{1}{2}s^2} \, ds = e^{\frac{1}{2}t^2} \int_0^t (t - s)s^2 \, ds = \frac{1}{12} t^4 e^{\frac{1}{2}t^2}.
   \]

2. Compute the Green function \( G(t, s) \) for the differential operator \( L(t) \) defined by
   \[
   L(t)y = t D^2y + (t - 1)Dy - y,
   \]
given that \( t - 1 \) and \( e^{-t} \) solve the homogeneous equation \( L(t)y = 0 \). Use the result to solve the initial-value problem
   \[
t y'' + (t - 1)y' - y = 2t^3, \quad y(1) = y'(1) = 0.
   \]

Solution: The Green function is given by
   \[
   G(t, s) = \frac{\det\left(\begin{array}{cc} s - 1 & e^{-s} \\ t - 1 & e^{-t} \end{array}\right)}{\det\left(\begin{array}{cc} s - 1 & e^{-s} \\ 1 & -e^{-s} \end{array}\right)} = \frac{(t - 1)e^{-s} - e^{-t}(s - 1)}{s e^{-s}}.
   \]
Because the normal form of the equation is
   \[
y'' + \left(1 - \frac{1}{t}\right)y' - \frac{1}{t} y = 2t^2,
   \]
the forcing is \( 2t^2 \). The solution is therefore
   \[
y(t) = \int_1^t G(t, s) 2s^2 \, ds = 2 \int_1^t (t - 1)s - e^{-t}(s^2 - s)e^s \, ds = t^3 - 3t^2 + 5t + 5 + 2e^{1-t}.
   \]