Quiz 4 Solutions, Math 246, Professor David Levermore
Tuesday, 1 March 2011

(1) What is the interval of definition for the solution to the initial-value problem
\[
\frac{d^3 z}{dt^3} + \frac{t^2}{2 + t} z = \frac{\sin(t)}{t - 1}, \quad z(-1) = z'(-1) = z''(-1) = 3.
\]

**Solution.** The equation is already in normal form. The coefficient is defined and continuous everywhere except \( t = -2 \). The forcing is defined and continuous everywhere except \( t = 1 \). The initial time is \( t = -1 \). The interval of definition is therefore \((-2, 1)\).

(2) Compute the Wronskian \( W[Y_1, Y_2](t) \) of the functions \( Y_1(t) = e^t \) and \( Y_2(t) = te^t \). (Evaluate the determinant and simplify.)

**Solution.** Because \( Y_1'(t) = e^t \) and \( Y_2'(t) = e^t + te^t \), the Wronskian is
\[
W[Y_1, Y_2](t) = \det \begin{pmatrix} Y_1(t) & Y_2(t) \\ Y_1'(t) & Y_2'(t) \end{pmatrix} = \det \begin{pmatrix} e^t & te^t \\ e^t & e^t + te^t \end{pmatrix}
\]
\[
= e^t(e^t + te^t) - e^t(te^t) = e^{2t} + te^{2t} - te^{2t} = e^{2t}.
\]

(3) Given that \( e^t \) and \( te^t \) make up a fundamental set of solutions to \( y'' - 2y' + y = 0 \), find the solution \( Y(t) \) to the general initial-value problem
\[
\frac{d^2 y}{dt^2} - 2\frac{dy}{dt} + y = 0, \quad y(0) = y_0, \quad y'(0) = y_1.
\]

**Solution.** Because \( e^t \) and \( te^t \) is a fundamental set of solutions to \( y'' - 2y' + y = 0 \), a general solution has the form \( Y(t) = c_1 e^t + c_2 te^t \). Then \( Y'(t) = c_1 e^t + c_2 (e^t + te^t) \). To satisfy the initial conditions one needs
\[
y_0 = Y(0) = c_1, \quad y_1 = Y'(0) = c_1 + c_2.
\]
You solve these equations to obtain \( c_1 = y_0 \) and \( c_2 = y_1 - y_0 \). The solution to the general initial-value problem is therefore
\[
Y(t) = y_0 e^t + (y_1 - y_0)te^t.
\]

**Remark.** Because this solution to the general initial-value problem can be written
\[
Y(t) = (e^t - te^t)y_0 + te^ty_1,
\]
the associated natural fundamental set of solutions is given by
\[
N_0(t) = e^t - te^t, \quad N_1(t) = te^t.
\]