Global Maxwellians over All Space and Their Relation to Conserved Quantites of Classical Kinetic Equations

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Introduction

When a classical kinetic equation is posed over the spatial domain $\mathbb{R}^D$, every initial data with finite mass, energy, and second spatial moments formally has $4 + 2D + \frac{D(D-1)}{2}$ conserved quantities associated with it.

The family of global Maxwellians with finite mass over the spatial domain $\mathbb{R}^D$ has $4 + 2D + \frac{D(D-1)}{2}$ parameters.

We show that a unique global Maxwellian can be associated with each such initial data by matching the values of their conserved quantities.

Moreover, the set of all such values is characterized by an inequality on the trace norm of the angular momentum matrix.

Outline

1. Preliminaries
2. Statement of Main Theorem
3. Values of Conserved Quantities
4. Range of Global Maxwellian Conserved Quantities
5. Conclusion

Classical Kinetic Equations

We will consider classical kinetic equations that govern the evolution of a kinetic density $F(v,x,t)$ over the velocity-position space $\mathbb{R}^D \times \mathbb{R}^D$ by the Cauchy problem

$$\partial_t F + v \cdot \nabla_x F = \mathcal{C}(F)\,, \qquad F\Big|_{t=0} = F^{\text{in}}\,, \tag{1}$$

where the initial data $F^{\text{in}}(v,x)$ is nonnegative and satisfies the bounds

$$0 < \iint_{\mathbb{R}^D \times \mathbb{R}^D} \left(1 + |v|^2 + |x|^2\right) F^{\text{in}}\, \mathrm{d}v\, \mathrm{d}x < \infty\,. \tag{2}$$

The upper bound will insure the existence of the zeroth through second moments of F^{in}. The lower bound will insure that F^{in} is positive over a set of positive measure.

Collision Operators

The collision operator $\mathcal{C}$ acts only on the v variable.

We assume that $\mathcal{C}$ is defined over a domain $\mathrm{Dom}(\mathcal{C})$ contained within a cone of nonnegative functions of v with sufficiently rapid decay at infinity.

We assume that the kinetic equation is Galilean invariant, which means that $\mathrm{Dom}(\mathcal{C})$ is invariant under v translations while $\mathcal{C}$ commutes with v translations.

We also assume that $\mathcal{C}$ and $\mathrm{Dom}(\mathcal{C})$ have the following properties related to conservation, dissipation, and equilibria. These properties are shared by most classical collision operators.

Conservation Properties

The quantities 1, v, and $|v|^2$ are assumed to be locally conserved by $\mathcal{C}$. This means that for every $f \in \mathrm{Dom}(\mathcal{C})$

$$\int_{\mathbb{R}^D} \mathcal{C}(f)\,\mathrm{d}v = 0\,, \quad \int_{\mathbb{R}^D} v\,\mathcal{C}(f)\,\mathrm{d}v = 0\,, \quad \int_{\mathbb{R}^D} |v|^2\,\mathcal{C}(f)\,\mathrm{d}v = 0\,.$$

These reflect mass, momentum, and energy conservation by collisions. We assume moreover, that every locally conserved quantity is a linear combination of these three. More specifically, given any $\xi = \xi(v)$, the following statements are equivalent:

(i) $\displaystyle\int_{\mathbb{R}^D} \xi\,\mathcal{C}(f)\,\mathrm{d}v = 0\,,$ for every $f \in \mathrm{Dom}(\mathcal{C})$;

(ii) $\xi \in \mathrm{span}\{1, v_1, v_2, \cdots, v_D, |v|^2\}\,.$

This means that there is no other quantity that is locally conserved by $\mathcal{C}$.

Local Conservation Laws

Any solution F of the kinetic equation formally satisfies the local conservation law

$$\partial_t \int_{\mathbb{R}^D} \xi \, F \, \mathrm{d}v + \nabla_x \cdot \int_{\mathbb{R}^D} v \, \xi \, F \, \mathrm{d}v = 0 \, ,$$

when $\xi(v, x, t)$ is any quantity that satisfies

$$\xi \in \operatorname{span}\{1, v_1, v_2, \cdots, v_D, |v|^2\} \, , \qquad \partial_t \xi + v \cdot \nabla_x \xi = 0 \, .$$

It has been known essentially since Boltzmann, who worked out the case $D = 3$, that the only such quantities ξ are linear combinations of the $4 + 2D + \frac{D(D-1)}{2}$ quantities

$$\begin{gathered} 1 \, , \qquad v \, , \qquad x - vt \, , \\ \tfrac{1}{2}|v|^2 \, , \qquad v \wedge x \qquad v \cdot (x - vt) \, , \qquad \tfrac{1}{2}|x - vt|^2 \, , \end{gathered}$$

where $v \wedge x = v \, x^T - x \, v^T$ is the skew tensor product.

Global Conservation Laws

By integrating the corresponding local conservation laws over all space, we formally obtain the conserved quantities

$$\iint_{\mathbb{R}^D \times \mathbb{R}^D} \begin{pmatrix} 1 \\ v \\ x - vt \\ \frac{1}{2}|v|^2 \\ v \wedge x \\ v \cdot (x - vt) \\ \frac{1}{2}|x - vt|^2 \end{pmatrix} F(v, x, t) \, \mathrm{d}v \, \mathrm{d}x \, .$$

These quantities are associated respectively with the conservation laws of mass, momentum, initial center of mass, energy, angular momentum, scalar momentum moment, and scalar inertial moment. The last two are not general physical laws, but are general to solutions of classical kinetic equations.

Dissipation Properties and Maxwellians

The operator $\mathcal{C}$ is assumed to satisfy the local dissipation relation

$$\int_{\mathbb{R}^D} \log(f)\,\mathcal{C}(f)\,\mathrm{d}v \leq 0\,, \qquad \text{for every } f \in \mathrm{Dom}(\mathcal{C})\,.$$

Moreover, we assume that for every nonzero $f \in \mathrm{Dom}(\mathcal{C})$ the following statements are equivalent:

(i) $\displaystyle\int_{\mathbb{R}^D} \log(f)\,\mathcal{C}(f)\,\mathrm{d}v = 0\,,$

(ii) $\mathcal{C}(f) = 0\,,$

(iii) $\displaystyle f = \frac{\rho}{(2\pi\theta)^{\frac{D}{2}}} \exp\left(-\frac{|v-u|^2}{2\theta}\right)$ with $(\rho, u, \theta) \in \mathbb{R}_+ \times \mathbb{R}^D \times \mathbb{R}_+\,.$

These assumptions simply abstract some consequences of Boltzmann's celebrated H-theorem for Maxwell-Boltzmann collision operators.

Global Maxwellians

Functions in the form (iii) with ρ, u, and θ functions of (x,t) are called *local Maxwellians*. Local Maxwellians that satisfy the kinetic equation are called *global Maxwellians*. The family of global Maxwellians over the spatial domain $\mathbb{R}^D$ with positive mass, zero net momentum, and center of mass at the origin has the form

$$\mathcal{M} = \frac{m\sqrt{\det(Q)}}{(2\pi)^D} \exp\left(-\frac{1}{2} \begin{pmatrix} v \\ x - vt \end{pmatrix}^T \begin{pmatrix} cI & bI + B \\ bI - B & aI \end{pmatrix} \begin{pmatrix} v \\ x - vt \end{pmatrix} \right),$$

with $m > 0$, $Q = (ac - b^2)I + B^2$, and $(a,b,c,B) \in \Omega$, where

$$\Omega = \left\{ (a,b,c,B) \in \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}^{D \wedge D} \,:\, Q > 0 \right\}.$$

Here $\mathbb{R}_+$ denotes the positive real numbers and $\mathbb{R}^{D \wedge D}$ denotes the skew-symmetric $D \times D$ real matrices.

This form can be derived from the fact that $\log(\mathcal{M})$ must satisfy

$$\log(\mathcal{M}) \in \operatorname{span}\{1, v_1, v_2, \cdots, v_D, |v|^2\}, \qquad \Big(\partial_t + v \cdot \nabla_x\Big) \log(\mathcal{M}) = 0,$$

whereby $\log(\mathcal{M})$ must be a linear combination of the quantities

$$\begin{array}{cccc} & 1, & v, & x - vt, \\ \tfrac{1}{2}|v|^2, & v \wedge x & v \cdot (x - vt), & \tfrac{1}{2}|x - vt|^2. \end{array}$$

The form of $\mathcal{M}$ then comes from the requirements that it have finite mass, zero net momentum, and center of mass at the origin.

The larger family of global Maxwellians with positive mass is obtained from the form $\mathcal{M}$ by introducing translations in v and x, whereby it has a total of $4 + 2D + \frac{D(D-1)}{2}$ parameters.

We can bring $\mathcal{M}$ into the local Maxwellian form

$$\mathcal{M} = \frac{\rho(x,t)}{\Big(2\pi\theta(t)\Big)^{\frac{D}{2}}} \exp\left(-\frac{|v - u(x,t)|^2}{2\theta(t)}\right) ,$$

where the temperature $\theta(t)$, bulk velocity $u(x,t)$, and mass density $\rho(x,t)$ are given by

$$\theta(t) = \frac{1}{at^2 - 2bt + c}, \qquad u(x,t) = \theta(t)\Big(axt - bx - Bx\Big),$$

$$\rho(x,t) = m \left(\frac{\theta(t)}{2\pi}\right)^{\frac{D}{2}} \sqrt{\det(Q)} \exp\left(-\frac{\theta(t)}{2} x^T Q x\right) .$$

Because $a, c > 0$ and $ac > b^2$, we see that $at^2 - 2bt + c > 0$ for every t. Because $Q > 0$, we see that $\rho(x,t)$ is integrable over $\mathbb{R}^D$.

The second moments of $\mathcal{M}$ may be computed by evaluating the Gaussian integrals as

$$\iint_{\mathbb{R}^D\times\mathbb{R}^D} \begin{pmatrix} v \\ x - vt \end{pmatrix} \begin{pmatrix} v \\ x - vt \end{pmatrix}^T \mathcal{M} \,\mathrm{d}v\,\mathrm{d}x = m \begin{pmatrix} cI & bI + B \\ bI - B & aI \end{pmatrix}^{-1}$$
$$= m \begin{pmatrix} aQ^{-1} & -bQ^{-1} - Q^{-1}B \\ -bQ^{-1} + Q^{-1}B & cQ^{-1} \end{pmatrix}.$$

In particular, the values of the quadratic conserved quantities are given by

$$\iint_{\mathbb{R}^D\times\mathbb{R}^D} |v|^2 \mathcal{M}\,\mathrm{d}v\,\mathrm{d}x = m\ \mathrm{tr}\Big(Q^{-1}\Big)a\,,$$
$$\iint_{\mathbb{R}^D\times\mathbb{R}^D} v\cdot(x - vt)\,\mathcal{M}\,\mathrm{d}v\,\mathrm{d}x = -m\ \mathrm{tr}\Big(Q^{-1}\Big)b\,,$$
$$\iint_{\mathbb{R}^D\times\mathbb{R}^D} |x - vt|^2 \mathcal{M}\,\mathrm{d}v\,\mathrm{d}x = m\ \mathrm{tr}\Big(Q^{-1}\Big)c\,,$$
$$\iint_{\mathbb{R}^D\times\mathbb{R}^D} v\wedge x\,\mathcal{M}\,\mathrm{d}v\,\mathrm{d}x = -2mQ^{-1}B\,.$$

Main Theorem

We show that every Cauchy problem (1) with initial data $F^{\mathsf{in}}(v,x)$ that satisfies the bounds (2) can be associated with a unique global Maxwellian determined by the values of the conserved quantities computed from F^{in}. This is a statement only about the initial data, not the Cauchy problem. So we will consider nonnegative integrable functions F that satisfy the bounds

$$0 < \iint_{\mathbb{R}^D\times\mathbb{R}^D} \left(1 + |v|^2 + |x|^2\right) F \, \mathrm{d}v \, \mathrm{d}x < \infty \,. \tag{3}$$

By choosing an appropriate rescaling and Galilean frame, we may assume without loss of generality that

$$\begin{gathered} \iint_{\mathbb{R}^D\times\mathbb{R}^D} F \, \mathrm{d}v \, \mathrm{d}x = 1 \,, \\ \iint_{\mathbb{R}^D\times\mathbb{R}^D} v \, F \, \mathrm{d}v \, \mathrm{d}x = \iint_{\mathbb{R}^D\times\mathbb{R}^D} x \, F \, \mathrm{d}v \, \mathrm{d}x = 0 \,. \end{gathered} \tag{4}$$

Our main result is the following.

Theorem. 1 *Let $F(v,x)$ be a nonnegative integrable function that satisfies the bounds (3) and the normalizations (4). Let a_*, b_*, c_*, and B_* be given by*

$$\iint_{\mathbb{R}^D\times\mathbb{R}^D} |v|^2 F \,\mathsf{d}v\,\mathsf{d}x = a_*\,, \qquad \iint_{\mathbb{R}^D\times\mathbb{R}^D} v\cdot x\, F \,\mathsf{d}v\,\mathsf{d}x = b_*\,,$$
$$\iint_{\mathbb{R}^D\times\mathbb{R}^D} |x|^2 F \,\mathsf{d}v\,\mathsf{d}x = c_*\,, \qquad \iint_{\mathbb{R}^D\times\mathbb{R}^D} v\wedge x\, F \,\mathsf{d}v\,\mathsf{d}x = B_*\,. \tag{5}$$

Then $(a_*,b_*,c_*,B_*)\in\Omega_*$, where $\Omega_*\in\mathbb{R}\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}^{D\wedge D}$ is an open cone defined by

$$\Omega_* = \left\{(a_*,b_*,c_*,B_*)\in\mathbb{R}_+\times\mathbb{R}\times\mathbb{R}_+\times\mathbb{R}^{D\wedge D}\;:\;\tfrac{1}{2}\,\mathrm{tr}(|B_*|) < \sqrt{a_*c_* - b_*^2}\right\}.$$

Here $\mathrm{tr}(\,\cdot\,)$ denotes the trace of a matrix while $|B_*|$ denotes the unique nonnegative definite matrix that satisfies $|B_*|^2 = B_*^T B_* = -B_*^2$.

Conversely, if $(a_*, b_*, c_*, B_*) \in \Omega_*$ *then there is a unique global Maxwellian* $\mathcal{M}$ *with* $m = 1$ *and* $(a, b, c, B) \in \Omega$ *such that the quadratic converved quantities associated with* $\mathcal{M}$ *have values* (a_*, b_*, c_*, B_*) *given by*

$$
\begin{aligned}
a_* &= \operatorname{tr}\Big(Q^{-1}\Big)a\,, \qquad & b_* &= -\operatorname{tr}\Big(Q^{-1}\Big)b\,, \\
c_* &= \operatorname{tr}\Big(Q^{-1}\Big)c\,, \qquad & B_* &= -2Q^{-1}B\,,
\end{aligned}
\tag{6}
$$

where $Q = (ac - b^2)I + B^2$*.*

Remark. The fact that $\operatorname{tr}(|B_*|)$ is the trace norm B_* will be used.

Remark. The first part of this theorem states that Ω_* contains the set of values that can be realized by the conserved quantities given by (5). The second part asserts that every point in Ω_* can be so realized. Therefore Ω_* characterizes all such values. This is a moment realizability result. The trace norm appearing in the characterization of Ω_* makes it unusual.

Remark. The second part of this theorem is equivalent to the assertion that for every $(a_*, b_*, c_*, B_*) \in \Omega_*$ there exists a unique minimizer of

$$H(F) = \iint_{\mathbb{R}^D \times \mathbb{R}^D} F \log(F) - F \, \mathrm{d}v \, \mathrm{d}x$$

over the set of nonnegative integrable functions that satisfy the bounds (3), the normalizations (4), and the constraints (5). Indeed, when we apply the method of Lagrange multipliers to this constrained minimization problem, the problem of finding a solution to the resulting Euler-Lagrange equations reduces to showing that the algebraic system (6) has a unique solution $(a, b, c, B) \in \Omega$. The unique minimizer will then be the global Maxwellian given by $m = 1$ and (a, b, c, B).

If $(a_*, b_*, c_*, B_*) \notin \Omega_*$ then the resulting Euler-Lagrange equations will not have a solution from among the admissible functions because otherwise it would violate the first part of the theorem.

Remark. The sets Ω_* and Ω respectively are characterized as subsets of $\mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}^{D \wedge D}$ by the inequalities

$$\tfrac{1}{2}\,\mathrm{tr}(|B|) < \sqrt{ac - b^2}\,, \qquad \|B\| < \sqrt{ac - b^2}\,,$$

where $\|B\|$ is the ℓ^2 matrix norm. Because B is skew-symmetric, $\|B\|$ equals the spectral radius of B, and the nonzero eigenvalues of B come in conjugate pairs. Whenever B has at most two nonzero eigenvalues then $\tfrac{1}{2}\,\mathrm{tr}(|B|) = \|B\|$.

If $D = 1$, $D = 2$, or $D = 3$ then $\Omega_* = \Omega$ because in those cases B can have at most two nonzero eigenvalues. (In fact, $B = 0$ when $D = 1$.)

If $D > 3$ then Ω_* will be a proper subset of Ω because $\|B\| < \tfrac{1}{2}\,\mathrm{tr}(|B|)$ whenever B has more than two nonzero eigenvalues.

Values of Conserved Quantities

Here we prove the first part of the Main Theorem, which states that the values taken by the quadratic conserved quantities must lie in Ω_*.

Theorem. 2 *Let $F(v, x)$ be a nonnegative function that satisfies the bounds (3) and the nomalizations (4). Let a_*, b_*, c_*, and B_* be given by (5).*

Then $(a_*, b_*, c_*, B_*) \in \Omega_*$ where Ω_* is an open cone in $\mathbb{R}\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}^{D\wedge D}$.

Remark. The fact that every $(a_*, b_*, c_*, B_*) \in \Omega_*$ gives the values of the conserved quantities for some F will be established by Theorem 6.

A key role in our proof of Theorem 2 will be played by the following lemma.

Lemma. 3 *Let* $\mathbb{P} = \{(a,b,c) \in \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}_+ \ : \ ac - b^2 > 0\}$. *Let* $\mathbb{P}^c$ *denote the closure of* $\mathbb{P}$ *in* $\mathbb{R}^3$. *Let* $\psi : \mathbb{P}^c \to [0,\infty)$ *be given by* $\psi(a,b,c) = \sqrt{ac - b^2}$.

Then ψ *is continuous and concave over* $\mathbb{P}^c$, *and smooth over* $\mathbb{P}$. *Moreover, for every* $(a_*, b_*, c_*) \in \mathbb{P}$ *and every* $(a,b,c) \in \mathbb{P}^c$ *one has the inequality*

$$\psi(a,b,c) \leq \psi(a_*,b_*,c_*) + \frac{1}{\psi(a_*,b_*,c_*)} \begin{pmatrix} \frac{1}{2}c_* \\ -b_* \\ \frac{1}{2}a_* \end{pmatrix}^T \begin{pmatrix} a - a_* \\ b - b_* \\ c - c_* \end{pmatrix}, \tag{7}$$

with equality if and only if (a,b,c) *is proportional to* (a_*,b_*,c_*).

An immediate consequence of Lemma 3 is the following, which establishes one of the assertions of Theorem 2.

Corollary 4 *The set Ω_* is an open cone in $\mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{D \wedge D}$.*

Proof. By the triangle inequality for the trace norm, the defining inequality for Ω_*, and the concavity of $(a, b, c) \mapsto \sqrt{ac - b^2}$, we see that

$$\begin{aligned} \tfrac{1}{2}\,\mathrm{tr}(|B_1 + B_2|) &\leq \tfrac{1}{2}\,\mathrm{tr}(|B_1|) + \tfrac{1}{2}\,\mathrm{tr}(|B_2|) \\ &< \sqrt{a_1 c_1 - b_1^2} + \sqrt{a_2 c_2 - b_2^2} \\ &\leq \sqrt{(a_1 + a_2)(c_1 + c_2) - (b_1 + b_2)^2}\,. \end{aligned}$$

Hence, $(a_1 + a_2, b_1 + b_2, c_1 + c_2, B_1 + B_2) \in \Omega_*$. Therefore Ω_* is a cone. The fact Ω_* is open is a consequence of the continuity of the mapping $(a_*, b_*, c_*, B_*) \mapsto \sqrt{a_* c_* - b_*^2} - \frac{1}{2}\,\mathrm{tr}(|B_*|)$.

We are now ready to give the proof of Theorem 2.

Proof. The fact that Ω_* is an open cone in $\mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}^{D\wedge D}$ was established by the last corollary. Let (a_*, b_*, c_*, B_*) be given by (5). These values are defined by the upper bound of (3). Because $|v|^2$ and $|x|^2$ are positive almost everywhere while $v \wedge x$ is skew-symmetric, it follows from (5) and the lower bound of (3) that $(a_*, b_*, c_*, B_*) \in \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}^{D\wedge D}$. All that is left to show is that $\frac{1}{2}\,\mathrm{tr}(|B_*|) < \sqrt{a_* c_* - b_*^2}$.

For every $(\xi, \zeta) \in \mathbb{R}^2$ we have

$$\begin{pmatrix} \xi \\ \zeta \end{pmatrix}^T \begin{pmatrix} a_* & b_* \\ b_* & c_* \end{pmatrix} \begin{pmatrix} \xi \\ \zeta \end{pmatrix} = \iint_{\mathbb{R}^D \times \mathbb{R}^D} |\xi v + \zeta x|^2 \, F \, \mathrm{d}v \, \mathrm{d}x \,.$$

If (ξ, ζ) is nonzero then $|\xi v + \zeta x|^2 > 0$ almost everywhere, which implies the right-hand side above is positive. Therefore the matrix on the left-hand side above is positive definite, whereby $a_* c_* - b_*^2 > 0$.

Let $\mathbb{P}$, $\mathbb{P}^c$, and ψ be as in Lemma 3. We have just seen that $(a_*, b_*, c_*) \in \mathbb{P}$. For every $(v, x) \in \mathbb{R}^D \times \mathbb{R}^D$ we have $(|v|^2, v \cdot x, |x|^2) \in \mathbb{P}^c$ by the Cauchy inequality. Because the set of $(v, x) \in \mathbb{R}^D \times \mathbb{R}^D$ where $(|v|^2, v \cdot x, |x|^2)$ is proportional to (a_*, b_*, c_*) has measure zero, we see by (7) of Lemma 3 that for almost every $(v, x) \in \mathbb{R}^D \times \mathbb{R}^D$ we have

$$\psi\Big(|v|^2, v \cdot x, |x|^2\Big) < \psi(a_*, b_*, c_*) + \frac{1}{\psi(a_*, b_*, c_*)} \begin{pmatrix} \frac{1}{2}c_* \\ -b_* \\ \frac{1}{2}a_* \end{pmatrix}^T \begin{pmatrix} |v|^2 - a_* \\ v \cdot x - b_* \\ |x|^2 - c_* \end{pmatrix} .$$

Upon integrating this inequality and using the fact F is positive on a set of positive measure, we obtain the strict form of the Jensen inequality

$$\iint_{\mathbb{R}^D \times \mathbb{R}^D} \psi\Big(|v|^2, v \cdot x, |x|^2\Big) \, F \, \mathrm{d}v \, \mathrm{d}x < \psi(a_*, b_*, c_*) \, .$$

Next, we claim that for every $(v, x) \in \mathbb{R}^D \times \mathbb{R}^D$

$$|v \wedge x| = \sqrt{|v|^2|x|^2 - (v \cdot x)^2}\, P_{\{v,x\}}\,,$$

where $P_{\{v,x\}}$ is the orthogonal projection onto $\mathrm{span}\{v, x\}$.

Because $\mathrm{tr}(P_{\{v,x\}}) = 2$ when $|v|^2|x|^2 - (v \cdot x)^2 > 0$, we obtain the trace formula

$$\mathrm{tr}(|v \wedge x|) = 2\sqrt{|v|^2|x|^2 - (v \cdot x)^2} \qquad \text{for every } (v, x) \in \mathbb{R}^D \times \mathbb{R}^D\,.$$

Finally, from the triangle inequality for the trace norm, the trace formula, and the strict Jensen inequality, we get

$$\begin{aligned} \tfrac{1}{2}\,\mathrm{tr}(|B_*|) &\leq \iint_{\mathbb{R}^D \times \mathbb{R}^D} \tfrac{1}{2}\,\mathrm{tr}(|v \wedge x|)\, F \,\mathrm{d}v\,\mathrm{d}x \\ &= \iint_{\mathbb{R}^D \times \mathbb{R}^D} \sqrt{|v|^2|x|^2 - (v \cdot x)^2}\, F \,\mathrm{d}v\,\mathrm{d}x < \sqrt{a_* c_* - b_*^2}\,. \end{aligned}$$

Therefore $(a_*, b_*, c_*, B_*) \in \Omega_*$.

Range of Global Maxwellian Conserved Quantities

We consider the mapping from the global Maxwellians of unit mass, zero net momentum, and center of mass at the origin to the values of their conserved quantities. These global Maxwellians $\mathcal{M}$ with $m = 1$ are parametrized by $(a, b, c, B) \in \Omega$. The values of the quadratic conserved quantities for the global Maxwellian $\mathcal{M}$ associated with $(a, b, c, B) \in \Omega$ are given by $\Psi(a, b, c, B) = (a_*, b_*, c_*, B_*)$, where

$$\begin{aligned} a_* &= \operatorname{tr}\Big(Q^{-1}\Big) a\,, \qquad & b_* &= -\operatorname{tr}\Big(Q^{-1}\Big) b\,, \\ c_* &= \operatorname{tr}\Big(Q^{-1}\Big) c\,, \qquad & B_* &= -2Q^{-1} B\,, \end{aligned} \tag{8}$$

where $Q = (ac - b^2)I + B^2$. We have seen that Ψ maps Ω into Ω_*. The mapping is clearly real analytic over Ω. We now show that it is one-to-one and onto, which establishes the second part of the Main Theorem.

Theorem. 5 *The mapping* $\Psi : \Omega \to \Omega_*$ *given by (8) is one-to-one.*

Proof. We will use the fact that the mapping $\mathbf{A} \mapsto \mathtt{log}(\mathtt{det}(\mathbf{A}))$ is strictly concave over the set of symmetric positive definite matrices. This fact can be seen from the calculation

$$\begin{aligned}\left(\dot{\mathbf{A}} \cdot \partial_{\mathbf{A}}\right)^2 \mathtt{log}(\mathtt{det}(\mathbf{A})) &= \left(\dot{\mathbf{A}} \cdot \partial_{\mathbf{A}}\right) \operatorname{tr}\left(\mathbf{A}^{-1}\dot{\mathbf{A}}\right) \\ &= -\operatorname{tr}\left(\mathbf{A}^{-1}\dot{\mathbf{A}}\mathbf{A}^{-1}\dot{\mathbf{A}}\right) < 0 \quad \text{for every } \dot{\mathbf{A}} \neq 0\,.\end{aligned}$$

We then define

$$\phi(a,b,c,B) = \mathtt{log}\left(\mathtt{det}\begin{pmatrix} cI & bI + B \\ bI - B & aI \end{pmatrix}\right) = \mathtt{log}(\mathtt{det}(Q))\ .$$

The mapping $(a,b,c,B) \mapsto \phi(a,b,c,B)$ is a restriction of the mapping $\mathbf{A} \mapsto \mathtt{log}(\mathtt{det}(\mathbf{A}))$ to a linear subspace, so is also strictly concave.

The strict concavity of $\phi(a,b,c,B)$ implies that the Legendre mapping

$$(a,b,c,B) \mapsto \begin{pmatrix} \partial_a \phi(a,b,c,B) \\ \partial_b \phi(a,b,c,B) \\ \partial_c \phi(a,b,c,B) \\ \nabla_B \phi(a,b,c,B) \end{pmatrix} \quad \text{is one-to-one}\,.$$

Directional derivatives of $\phi(a,b,c,B)$ may be computed by the formula $\dot{Q} \cdot \partial_Q \log(\det(Q)) = \mathrm{tr}\Big(Q^{-1}\dot{Q}\Big)$. Direct calculations yield

$$\partial_a \phi(a,b,c,B) = \mathrm{tr}\Big(Q^{-1}\Big)\, c\,, \qquad \partial_b \phi(a,b,c,B) = -2\,\mathrm{tr}\Big(Q^{-1}\Big)\, b\,,$$

$$\partial_c \phi(a,b,c,B) = \mathrm{tr}\Big(Q^{-1}\Big)\, a\,, \qquad \nabla_B \phi(a,b,c,B) = -2Q^{-1}B\,.$$

But then $\Psi : \Omega \to \Omega_*$ is one-to-one because

$$\Psi(a,b,c,B) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & I \end{pmatrix} \begin{pmatrix} \partial_a \phi(a,b,c,B) \\ \partial_b \phi(a,b,c,B) \\ \partial_c \phi(a,b,c,B) \\ \nabla_B \phi(a,b,c,B) \end{pmatrix}\,.$$

We now show that the range of Ψ is Ω_*.

Theorem. 6 *The mapping $\Psi : \Omega \to \Omega_*$ given by (8) is onto.*

Proof. Let $(a_*, b_*, c_*, B_*) \in \Omega_*$. Let $d_* = \sqrt{a_* c_* - b_*^2} > 0$. Because $B_* \in \mathbb{R}^{D \wedge D}$, we can list its nonzero eigenvalues with multiplicity as $\{\pm i\beta_k\}_{k=1}^n$, where $\beta_k > 0$. The definition of Ω_* then shows that

$$\sum_{k=1}^{n} \beta_k = \frac{1}{2} \operatorname{tr}(|B_*|) < d_* \,. \tag{9}$$

It is clear that if (8) holds for some $(a, b, c, B) \in \Omega$ then (a, b, c, B) must have the form

$$a = \frac{\mu_* a_*}{d_*^2} \,, \qquad b = -\frac{\mu_* b_*}{d_*^2} \,, \qquad c = \frac{\mu_* c_*}{d_*^2} \,, \qquad B = -\frac{\mu_*}{d_*} N_* \,, \tag{10}$$

where the scalar $\mu_* > 0$ and the skew-symmetric matrix N_* are to be determined.

Because

$$Q = (ac - b^2)I + B^2 = \frac{\mu_*^2}{d_*^2}\left(I + N_*{}^2\right),$$

we see that $Q > 0$ if and only if $I + N_*{}^2 > 0$. Moreover, we see that

$$Q^{-1} = \frac{d_*^2}{\mu_*^2}\left(I + N_*{}^2\right)^{-1},$$

whereby

$$\begin{aligned}
\operatorname{tr}\!\left(Q^{-1}\right)a &= \frac{a_*}{\mu_*}\operatorname{tr}\!\left(\left(I + N_*{}^2\right)^{-1}\right),\\
-\operatorname{tr}\!\left(Q^{-1}\right)b &= \frac{b_*}{\mu_*}\operatorname{tr}\!\left(\left(I + N_*{}^2\right)^{-1}\right),\\
\operatorname{tr}\!\left(Q^{-1}\right)c &= \frac{c_*}{\mu_*}\operatorname{tr}\!\left(\left(I + N_*{}^2\right)^{-1}\right),\\
-2Q^{-1}B &= 2\,\frac{d_*}{\mu_*}\left(I + N_*{}^2\right)^{-1}N_*\,.
\end{aligned}$$

Equations (8) will be satisfied if and only if μ_* and N_* satisfy

$$1 = \frac{1}{\mu_*}\,\mathrm{tr}\Big(\big(I + N_*{}^2\big)^{-1}\Big)\,, \qquad B_* = 2\,\frac{d_*}{\mu_*}\,\big(I + N_*{}^2\big)^{-1}\,N_*\,. \tag{11}$$

The second equation above can be solved for N_* in terms of μ_*. There is a unique solution that satisfies $I + N_*{}^2 > 0$, which is given by

$$N_* = \mu_*\left(d_* I + \big(d_*^2 I - \mu_*^2 B_*^2\big)^{\frac{1}{2}}\right)^{-1} B_*\,. \tag{12}$$

The nonzero eigenvalues $\pm i\beta_k$ of B_* are related to nonzero eigenvalues $\pm i\nu_k$ of N_* by

$$\frac{\mu_*\beta_k}{2d_*} = \frac{\nu_k}{1-\nu_k^2}\,, \qquad \nu_k = \frac{\mu_*\beta_k}{d_* + \sqrt{d_*^2 + \mu_*^2\beta_k^2}}\,. \tag{13}$$

Notice that $\nu_k^2 < 1$, which is required by the condition that $I + N_*{}^2 > 0$.

We can use (12) to express the first equation in (11) as $1 = \tau(\mu_*)$ where

$$\tau(\mu_*) = \frac{1}{\mu_*}\,\mathrm{tr}\Big(\big(I + N_*^2\big)^{-1}\Big)$$

If we let D_0 be the dimension of the null space of N_* then by using (13) we find

$$\begin{aligned}\tau(\mu_*) &= \frac{D_0}{\mu_*} + \frac{1}{\mu_*}\sum_{k=1}^{n}\frac{2}{1-\nu_k^2} = \frac{D_0}{\mu_*} + \frac{1}{d_*}\sum_{k=1}^{n}\frac{\beta_k}{\nu_k} \\ &= \frac{D_0}{\mu_*} + \frac{1}{d_*}\sum_{k=1}^{n}\frac{d_* + \sqrt{d_*^2 + \mu_*^2\beta_k^2}}{\mu_*}\,.\end{aligned}$$

It is clear that $\tau(\mu_*)$ is a decreasing function over $\mu_* \in (0, \infty)$ whose range is (τ_*, ∞) where

$$\tau_* = \inf_{\mu_* > 0} \left\{ \tau(\mu_*) \right\} = \lim_{\mu_* \to \infty} \tau(\mu_*) = \frac{1}{d_*} \sum_{k=1}^{n} \beta_k = \frac{1}{2d_*} \operatorname{tr}(|B_*|) \, .$$

Hence, the equation $1 = \tau(\mu_*)$ has a unique positive solution if $\tau_* < 1$, and has no positive solution otherwise. But the condition $\tau_* < 1$ is met by (9) because $(a_*, b_*, c_*, B_*) \in \Omega_*$, so the equation $1 = \tau(\mu_*)$ has a unique positive solution μ_*. The (a, b, c, B) constructed from (10) with this μ_* and the N_* given by (12) is in Ω because $I + N_*^2 > 0$.

Remark. We could also argue that $\Psi : \Omega \to \Omega_*$ is one-to-one by arguing that μ_* and N_* in our construction are unique. However, that approach would obscure the simplicity of the result because it is more complicated than the approach taken in the proof of Theorem 5.

Conclusion

1. What does the $\mathcal{M}$ associated with F^{in} have to do with F? When might they share the same large-time asymptotics? Claude Bardos, Irene Gamba and I have preliminary results in a very restrictive setting.

2. Are the $\mathcal{M}$ stable? If so, can one bound the rate of convergence?

3. Can similar questions can be addressed for Navier-Stokes systems?

Thank You!